THE AMERICAN

MATHEMATICAL MONTHLY

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 94

1987

PUBLISHED BY THE ASSOCIATION
MONTPELIER, VT, AND WASHINGTON, DC

THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 1 January 1987

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(ISSN 0002-9890)

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Subscription correspondence, changes of address, and inquiries about nondelivered or defective issues: Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

THE EDITOR'S CORNER
A Greeting; and a View of Riemann's Hypothesis

HERBERT S. WILF

This issue marks another changing-of-the-guard for the Monthly. Paul Halmos' act will be a tough one to follow, but the new team of Associate Editors (see inside front cover) and I will do our best. As always, the Monthly is looking for fine expository writing on mathematical topics. We want reviews of evolving fields of research, new insights into old mathematical truths, elegant ways of dealing with complex ideas in the classroom, and other celebrations of the beauty of our subject.

From time to time I'd like to offer some thoughts of my own, and I'll do that in 'The Editor's Corner.' I don't want to be Cornered into a fixed commitment, so the column will appear only irregularly. In this first effort I offer some thoughts on Riemann's Hypothesis (RH).

Most professional mathematicians are aware that RH exists and that it is an extremely important unsolved question in pure mathematics, particularly in number theory. I want to share with you a very simple way of stating the problem; one that could be explained to a bright class of tenth graders. Then I'll have a little to say about some of the ramifications of RH, and finally there will be some news about exciting recent developments in the field.

First, how might we explain the problem to high school students? Here's one way. Beginning with the set of all positive integers, let's discard those that are divisible by the *square* of any integer larger than 1.

Thus, we throw out 4, 8, 9, 16, 18, 20, 24, ..., etc.

We are left with the list of the *squarefree* positive integers,

$$1,2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,\ldots .$$

If we factor any one of these squarefree integers into primes, no prime will be repeated in the factorization, which is to say that each of these integers is a product of *distinct* primes. Some squarefree integers are a product of an *even* number of distinct primes and some squarefree integers are a product of an *odd* number of distinct primes.

Let's say that an integer is *red* if it is a product of an even number of distinct primes, and that it is *blue* if it is a product of an odd number of distinct primes. Hence 14 is red and 30 is blue (18 is colorless, since it isn't even squarefree).

For instance, the squarefree integers $\leqslant 30$ are

$$1,2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,26,29,30.$$

Of these, there are 8 red numbers,

$$1,6,10,14,15,21,22,26$$

and 11 blue numbers.

$$2,3,5,7,11,13,17,19,23,29,30.$$

Hence, among the first 30 positive integers, there are 3 more blue numbers than red. Riemann's hypothesis says roughly that in every interval $[1, n]$ there are not very different quantities of red and of blue numbers. Here it is precisely, not in the words that Riemann used, but in a form that is fully equivalent to the original, while being, hopefully, explainable to that high school class:

RIEMANN'S HYPOTHESIS. *Fix* $\varepsilon > 0$. *Then there exists* N *such that for all* $n > N$ *the number of blue numbers in* $[1, n]$ *does not differ from the number of red numbers in* $[1, n]$ *by more than* $n^{1/2+\varepsilon}$.

So between 1 and n we would expect the numbers of red and blue integers to be in rough agreement, with a disparity of at most 'about' $\sqrt{n}$.

That was a fairly transparent statement of RH. What it lost was the importance of the problem. If RH really concerned only the fact that these two colors of numbers are pretty well intermixed with each other then it would have been an interesting question but not an overridingly important one. The importance of the proposition derives from its consequences, which are legion, but which we can't really discuss at length here (see **[2]**).

A number of consequences of RH arise as follows. There are many important *approximate* formulas in the theory of numbers. They are important because they are rather accurate approximations to certain interesting functions, whose behavior is so complicated that exact formulas for them would be less useful than approximations. One such is the famous *prime number theorem*. It states that the number of prime numbers that lie between 1 and x is 'close to' $x/\log x$.

How close is 'close to'? We don't know very much about that. We can say that the error of the approximation grows more slowly than $x/\log x$ itself, and we have been able to say even a little more than that. But we are unable to say, for instance, that the error grows no faster than $x^{.75}$. We expect, in fact, that the approximation $x/\log x$ is extremely good: we think that (well anyway, those who believe RH think that) the error grows only like $x^{.5+\varepsilon}$. It turns out that that proposition is implied by RH. Likewise, there are a maddeningly large number of attractive and important error estimates that are 'wired' to RH, in that if we could prove RH then all of those estimates would be simultaneously proved. That is where much of the significance of the proposition comes from.

To get back to the two colors of numbers that we discussed above, in 1897 Mertens* **[4]** decided that the experimental data might warrant an even stronger hypothesis. He observed that for all values of n for which he could do the calculations, it seemed that the difference between the numbers of red and blue integers $\leqslant n$ was never larger than $\sqrt{n}$. He formulated what later came to be called the

MERTENS HYPOTHESIS. *For every* $n \geqslant 1$, *the disparity between the numbers of red and blue integers in the interval* $[1, n]$ *never exceeds* $\sqrt{n}$.

---

*Franz Mertens (1840–1927), born in Poznan, educated in Berlin. See the historical note **[8]**.

If we compare Mertens' hypothesis (MH) with that of Riemann we observe that Mertens was claiming that we can choose $\varepsilon = 0$ and $N = 1$, which if true would be a great simplification (after all, everybody loves to replace ε's by 0's).

Next I'd like to tell you about the results of a pretty calculation that was done by G. Neubauer [5], a number of years ago, on the MH. In that calculation, every value of n up to 10^8 was looked at, along with many values of n larger than 10^8, up to 10^{10}. For each of these n's the disparity between the reds and the blues was calculated.

What Neubauer did was to tabulate what he referred to as $M_1(n)/\sqrt{n}$. Here $M_1(n)$ is the number of red integers in the interval $[1, n]$ minus the number of blue integers there.

So, if the Mertens conjecture were true then the tabulated values of this ratio would always lie between ± 1. In fact, Neubauer was testing an even stronger conjecture than Mertens', namely, that the ratio would never exceed .5 in magnitude for $n > 200$.

Whatever the motivation was, the graphs that are shown in that article should be looked at by all interested readers of this one. One sees there an irregularly oscillating function, that waves back and forth between $\pm .48$ or so, and resembles nothing so much as the Dow Jones averages during a period of unsettled conditions.

Suddenly, at $n = 7{,}700{,}000{,}000$ or thereabouts* the author found what he was looking for: the graph abruptly jumped up over .5, and a counterexample was found to the strengthened Mertens conjecture (= twice strengthened RH).

Such behavior came as no surprise to hardened analytic number theorists. Several times previously they had seen various simplified conjectures about the behavior of delicate number-theoretic functions, conjectures that appeared to be true for all computable values of n, shown to be false for various uncomputable values of n. In some cases we know that the conjectures are false, but we don't know a single value of n for which they fail (see below).

To resume the story, though, after Neubauer's work we still didn't have a counterexample to the original Mertens conjecture, but its plausibility was certainly a bit dented. Nonetheless, the extreme difficulty of settling questions related to RH led most to imagine that a resolution of the MH might be a long way off.

In 1985 the MH was proved false by Odlyzko and te Riele [6].

They proved, in fact, that for infinitely many values of n, the excess of red over blue numbers in the interval $[1, n]$ is larger than $1.06\sqrt{n}$.

They also showed that for infinitely many values of n the excess of blue over red numbers in the interval $[1, n]$ is larger than $1.009\sqrt{n}$.

Their proof does not identify a single value of n for which either of the above conclusions holds, while at the same time showing that infinitely many n's of both kinds exist!

Subsequently, Janos Pintz [7] showed that the Mertens conjecture surely fails for some $x < 3.21 \times 10^{64}$.

*Later work [1] identified the breakthrough to be at $n = 7{,}725{,}038{,}629$.

The proof of Odlyzko and te Riele uses some very recently developed methods. It used, for instance, the *lattice basis reduction algorithm* of Lenstra, Lenstra, and Lovász [3]. Essentially this is an algorithm that finds a short vector in an integer lattice that is described by a given basis, and finds it rapidly. It is probably fair to say that the algorithm was invented primarily to find polynomial time methods for solving computational problems that had previously been beyond reach, such as the problem of completely factoring a given polynomial over the rationals.

But in the paper of Odlyzko and te Riele we find that algorithm being used not for a computational purpose but for a theoretical one, namely to find a good solution to a system of simultaneous diophantine inequalities. The central point is the following. Let $\gamma_j, u_j (j = 1, n)$ be given reals. Then the problem of finding a number y for which all of the quantities $|\gamma_j y - u_j|$ $(j = 1, n)$ are simultaneously small can be transformed into the question of finding a short vector in an integer lattice. The object of interest in their proof turned out to be the *longest* vector in the transformed basis that the algorithm finds.

A story, of course, should not be concluded without a moral or two.

(1) Hold onto those ε's. To set them $= 0$ is to invite trouble.
(2) Don't believe anything just for some trivial reason like knowing that it's true for all values of n up to 7,000,000,000.
(3) Even though a conjecture may be false, your computer may not be able to get within a country mile of finding out.
(4) Algorithms whose development was motivated by questions in theoretical computer science may help to *prove* important theorems in pure mathematics.
(5) Even though your computer hasn't found a counterexample yet, and shows no signs of being near to one, press on with the next billion cases anyway, for who knows what you might find there?

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The Number of Three-Dimensional Convex Polyhedra

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Abstract

A convex polyhedron, or polytope, is the bounded intersection of closed half-spaces. The problems of determining the number of three dimensional convex polyhedra as a function of the number of faces or edges or both have been around for over 150 years. Except for Steinitz's conversion of polyhedra to "planar maps", little was done on the problem until the work on "rooted" planar maps in the 1960's. Recently the original (unrooted) questions have been answered asymptotically. We will retrace the steps that led to this result.

1. Introduction

Convex polyhedra (also called polytopes) are the analogues to convex polygons in higher dimensions. We can define a convex polyhedron as a bounded intersection of closed half-spaces. Alternatively, we could define a convex polyhedron to be the convex hull of a finite set of points. We will be concerned exclusively with three-dimensional polyhedra: those that lie in 3-space but do not lie in a plane. In the future, "polyhedra" will always mean "three-dimensional convex polyhedra." Cubes, tetrahedra, and prisms are all examples of polyhedra. In an obvious way, polyhedra have vertices, edges, and faces. (These can be described formally, but we need not do so here.) We will use the notation P_0, P_1, and P_2 for the numbers of vertices, edges, and faces, respectively, of a polyhedron P. Euler's famous theorem (1752) states that

$$P_0 - P_1 + P_2 = 2.$$

Suppose that v, e, and f are positive integers with $v - e + f = 2$. Does there exist a polyhedron P with $P_0 = v$, $P_1 = e$, and $P_2 = f$? The answer is No in general; however, it is easy to give necessary and sufficient conditions:

THEOREM 1. *There is a convex polyhedron P with*

$$P_0 = v, \qquad P_1 = e, \quad \text{and} \quad P_2 = f$$

if and only if

$$v - e + f = 2, \quad \text{and} \quad 4 \leqslant v \leqslant 2e/3, \quad \text{and } 4 \leqslant f \leqslant 2e/3.$$

Edward A. Bender: Working in the area where matrix theory and number theory meet, I received my Ph.D. under Olga Taussky at Caltech in 1966. I was a Peirce Instructor at Harvard and a member of the research staff at the Communications Research Division of the Institute for Defense Analyses before I joined the University of California in 1974. My major professional interest at present is asymptotic enumeration. I flirt with other areas of "concrete" mathematics, population biology, and computer science.

*Research sponsored by Department of Computer Science, University of Georgia at Athens.

Proof. The first condition is Euler's Theorem. Since the "smallest" polyhedron is a tetrahedron, the lower bounds on v and f are necessary. An edge corresponds to an unordered pair of vertices, called its ends. Every vertex must be the end of at least three edges, and each edge has two ends. If t is the total number of ends, then $3v \leqslant t = 2e$ and so $v \leqslant 2e/3$. Similarly, since each face is bordered by at least three edges and each edge lies on two faces, $f \leqslant 2e/3$.

Here's a sketch of the converse. If the triple (v, e, f) satisfies the conditions, then there are nonnegative integers x and y such that

$$(v, e, f) - x(1,3,2) - y(2,3,1)$$

equals one of (6,10,6), (6,9,5), (5,9,6), and (4,6,4). (Prove it by induction on e.) Each of the four triples just listed can be realized by a polyhedron, which we'll call irreducible. Adding (2,3,1) can be realized by slightly adjusting the edges meeting at a vertex and replacing the vertex by a triangular face. Adding (1,3,2) can be realized similarly by introducing two triangular faces in place of two adjacent edges of some face. (See Figure 1.) By iterating, one of the four irreducible polyhedra can be built up to realize (v, e, f). □

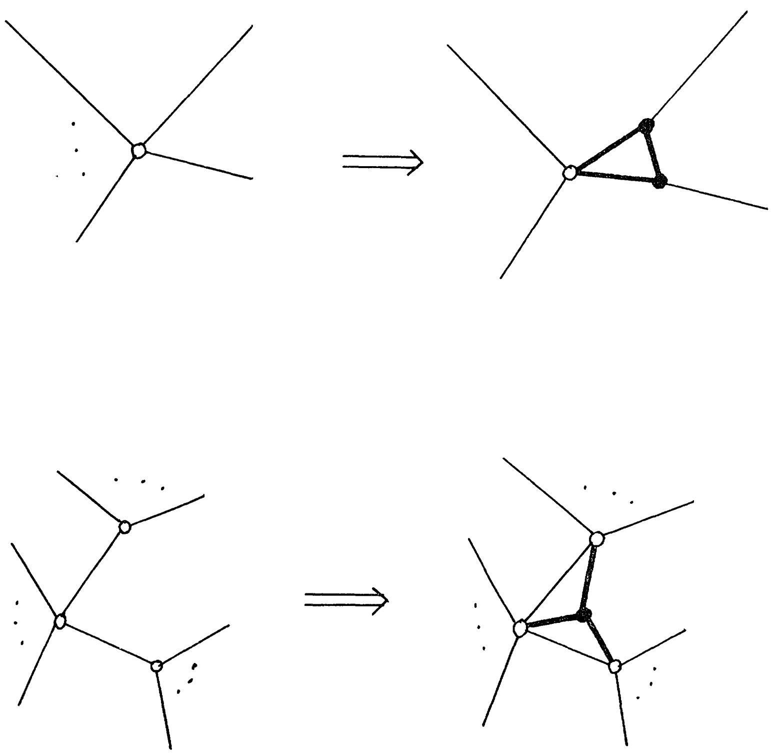

FIG. 1. Building up to a specified (v, e, f). Additions are shown heavy.

Since the existence question was easily settled, we proceed to the next type of question:

Q. How many "distinct" convex polyhedra P have $P_0 = v$ and $P_2 = f$?

There is no need to specify P_1 since it equals $v + f - 2$. By restricting only one of the P_k's, we are led to ask for $k = 0, 1$ and 2.

Q_k. How many "distinct" convex polyhedra P have $P_k = n$?

To answer the questions, we need to say what we mean by distinct polyhedra. Given two polyhedra P and Q, there may be a one-to-one mapping m of the faces of P to the faces of Q that preserves incidence; i.e., if F_1 and F_2 are faces of P that intersect in exactly one edge (resp., vertex), then $m(F_1)$ and $m(F_2)$ intersect in exactly one edge (resp., vertex), and conversely. If no such mapping exists, P and Q are called *combinatorially distinct*. This will be what we mean by "distinct."

Steiner posed Q_2 in 1832 and Kirkman stated in 1878 that he saw no hope of answering Q with the present power of mathematics. Shepard (1968) asked for a close approximation to the answer to Q_0.

The *dual*, P^*, of a polyhedron P is constructed by placing a vertex in each face of P^* and joining two such vertices by an edge if and only if the corresponding faces of P share an edge. The faces of P^* correspond to the vertices of P. One can show that $P^{**} = P$. Thus duality is a bijection, Q_0 and Q_2 have the same answer, and the answer to Q remains unchanged if the values of v and f are switched.

For specific values of v and f (resp., k and n), the corresponding Q (resp., Q_k) can be answered in a finite length of time since an algorithm exists for constructing all polyhedra with given parameters. This method is not an acceptable answer. What is an *acceptable answer*? One possible definition, suggested by the theory of algorithms, is: a way of calculating the number, which requires an amount of time that is a polynomial in v and f (or n). It is quite likely that no answer in this sense exists. How can we relax the definition of an answer? One way is to allow more time for calculating the formula. If this is relaxed too much, one can use the algorithm alluded to earlier for generating all polyhedra.

Another way to adjust the notion of answer is by requiring only a good approximate formula that can be computed quickly. What is a *good approximation*? We will require that the percentage error in the approximation go to zero as v and f (or n) get large. Such answers, which give information about how numbers behave as the parameters get large, are called *asymptotic* formulas. All the questions Q, Q_0, Q_1, Q_2 above have now been answered asymptotically.

In this paper we will retrace the path to the answers. The first step was taken by Steinitz (1922), who converted the questions to problems about counting graphs in the plane. Nothing further was done until Tutte developed methods for planar enumeration in the 1960s. As a result, Mullin and Schellenberg (1968) obtained a "generating function" for "rooted" polyhedra with given numbers of vertices and faces. This led to an explicit but messy formula for rooted polyhedra. Bender and Richmond (1984) used the generating function to obtain an asymptotic formula that

is valid for part of the range of v and f. Bender and Wormald (1985) combined this with various estimates to show that an asymptotic answer to Q or Q_k for rooted polyhedra gives an answer for polyhedra. The entire range was covered by Bender and Wormald (to appear) as a result of work on the paper you are now reading.

If all proofs had been included, this article would have been a small monograph. Therefore, I have replaced most proofs with broad sketches. If you are interested in the details, consult the original articles.

Federico (1975) was the source of the historical information. For a variety of questions concerning polyhedra, see Shepard (1968).

2. The graph-theory problem

A *graph* consists of a set of vertices with edges joining some pairs of the vertices. The vertices and edges are not labeled. If there is a path from every vertex to every other vertex along the edges, then the graph is called *connected*. A *loop* is an edge with both endpoints the same. If the ends of e_1 are the same as the ends of e_2, we say that e_1 and e_2 constitute a *multiple edge*. A (*planar*) *map* is a connected graph drawn on the plane so that no edges cross. The maximal regions containing no edges are called *faces*. The unbounded face is called the *external face*. Two maps are considered the same if one can be converted into the other by stretching, contracting and/or reflecting the plane.

A map is called *k-connected* if there *does not exist* a positive integer $j < k$ and a partition of the edges into two sets E_1 and E_2 such that each set contains at least j edges and the edges in $E_1 \cap E_2$ contain at most j distinct endpoints. Here are some simple useful observations on k-connectedness.

O1. "Connected" is the same as "1-connected."

O2. A map with at least 2 edges is 2-connected if and only if it contains no loops and no vertex is encountered more than once as we walk around a face boundary.

O3. A 2-connected map with at least 4 edges is 3-connected if and only if it contains no multiple edges and every pair of faces that have two vertices in common, say v and w, also have the edge (v, w) in common.

Exercise 1. Prove the observations. To prove O2, note that if v is encountered twice then its removal disconnects the graph. To prove O3, note that removal of v and w splits the boundary of each face containing both v and w. If (v, w) is not a common edge of these faces, this disconnects the graph. You may find it easier to see what is happening if you draw the map so that one of those faces is external.

A polyhedron may be converted to a map as follows. Select a face F. Remove all of the polyhedron except the edges and vertices. Place a plane parallel to F on the opposite side of the polyhedron from F. Place a light outside the polyhedron near the center of F. If the light is placed carefully, none of the shadows that the edges cast on the plane will cross each other and the shadow of the boundary of F will bound the external face of a map formed by the shadows. The faces of the

polyhedron correspond to the faces of the map. See Figure 2. This picture is called a *Schlegel diagram* for the polyhedron.

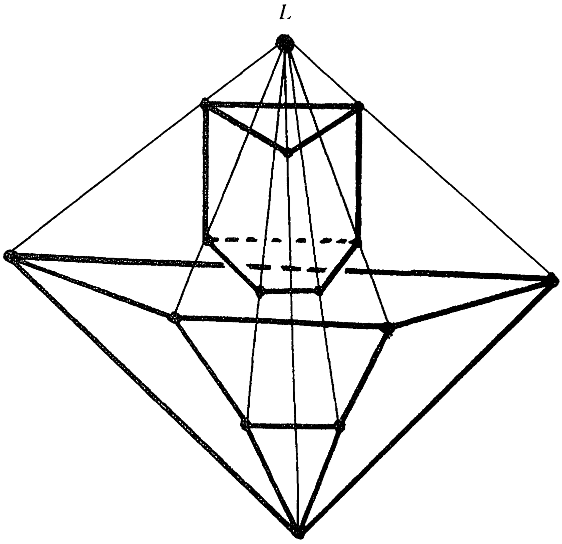

FIG. 2. A polyhedron projected down onto a plane by the light L gives a Schlegel diagram.

THEOREM 2. (Steinitz) *A map is the Schlegel diagram of a convex polyhedron if and only if it has at least* 4 *vertices and is* 3-*connected.*

Exercise 2. Prove the necessity by using O3.

The converse is difficult. For a proof, see Chapter 13 of Grünbaum. The lack of a corresponding result in higher dimensions seriously hampers attempts to count convex polyhedra in those dimensions. □

Unfortunately, there are generally many Schlegel diagrams for each polyhedron. This leads us to the notion of *rooted* maps and polyhedra. A polyhedron is rooted by choosing an edge (called the *root edge*), one vertex on the edge (called the *root vertex*) and one face adjacent to the root edge (called the *root face*). A 2-connected map is rooted if an edge on the external face (also called the root face) is distinguished. The *root-face degree* of a map or polyhedron is the number of edges on the root face.

COROLLARY 2.1. *There is a one-to-one correspondence between rooted convex polyhedra and rooted 3-connected maps with at least 4-vertices.*

Proof. The direction of a root edge will be such that the root vertex is the tail of the root edge. Arrange the polyhedron so that when the root face is viewed from the

outside and traversed in the direction of the root edge, the traversal is clockwise. (This may require reflecting the polyhedron.) Now place the light near the center of the root face. This gives a one-to-one correspondence. □

In the next section we will discuss the generating function for 3-connected maps. In Sections 4 and 5 we will see how that leads to asymptotics for rooted Schlegel diagrams and, hence, for rooted polyhedra. If all the ways of rooting a polyhedron P were distinct, it would have $P_1 \cdot 2 \cdot 2 = 4P_1$ rooted versions. In Section 6 we will see that for most polyhedra all rootings are distinct. This provides us with asymptotic answers to the questions Q and Q_k.

3. The exact number of rooted maps

Generating functions. If a_n is a sequence defined for $n \geqslant 0$, then the *generating function* for the sequence is

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We will adopt the convention of using a lower-case letter for a sequence element and the corresponding upper-case letter for the corresponding generating function. These ideas extend to multiply indexed sequences; for example, the generating function for the triply indexed sequence $b_{i,j,k}$ is

$$B(x, y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{i,j,k} x^i y^j z^k.$$

All the infinite series that we use converge when the variables are sufficiently small.

Suppose that we are given a generating function for some sequence, say, $a_{i,j}$. By Taylor's Theorem for functions of two variables, the coefficients of the power series for $A(x, y)$ are uniquely determined and so must be the sequence $a_{i,j}$. Thus, if we somehow explicitly expand $A(x, y)$ in a power series, we will obtain a formula for the sequence $a_{i,j}$.

There are a variety of rooted maps that one might enumerate. Each of these problems is approached by describing a method for constructing maps out of other maps. When this description is translated into a statement involving generating functions, the result is a functional relationship among the generating functions. If the construction involves only the type of map we are counting, then the functional equation involves only the generating function we are interested in. Thus it can be solved, at least in principle, for that generating function.

2-connected maps

Brown and Tutte (1964) enumerated 2-connected rooted maps. Since their result is central in later calculations, we'll look at their method.

Suppose M is a rooted 2-connected map that is not a single edge. The root edge of M belongs to two faces, the exterior face and some interior face, which are

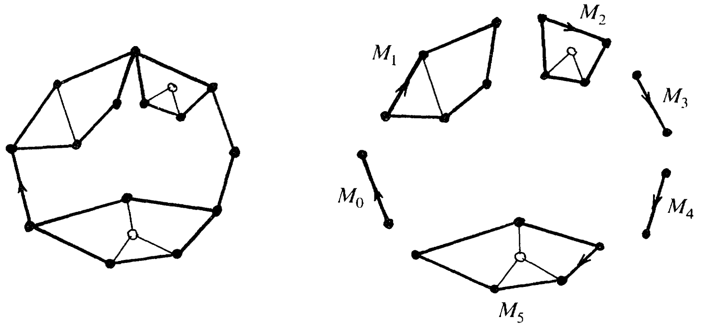

FIG. 3. Decomposing a 2-connected map.

shown by heavy lines in Figure 3. By splitting the map into pieces at all vertices v such that v lies on both faces, we obtain a sequence of maps $M_0, \ldots, M_t$, where M_0 is the root edge. By O2, each M_t can be seen to be 2-connected. Each can be rooted by rooting the edge of M_i first encountered when following the external face in a clockwise direction starting from the root edge. This decomposition is reversible provided we specify which edge on the root face of each M_i is the last edge encountered on the root face of M in our clockwise traversal. Therefore, we have a unique method for building up a 2-connected map from 2-connected maps with fewer faces.

In order to describe this numerically, we need to keep track of the number of vertices, the number of faces, and the degree of the root face. The last quantity is needed because an M_t with root-face degree k has $k - 1$ possible choices for the last vertex on the root face of M.

Let $f_{i,j,k}$ be the number of rooted 2-connected maps with $i + 1$ vertices, $j + 1$ faces, and root-degree k, except that the map consisting of a single edge is not counted. Since the root-face degree does not interest us, we want $F(x, y, 1)$, the generating function for $f_{i,j} = \Sigma_k f_{i,j,k}$. It can be shown that the above construction is equivalent to the equation

$$F(x, y, z) = yz \sum_{t=1}^{\infty} \left(\sum_{i,j,k} f_{i,j,k} x^i y^j (z + z^2 + \cdots + z^{k-1}) + xz \right)^t \tag{3.1}$$

After a little manipulation we get

$$F(x, y, z) = yz \sum_{t=0}^{\infty} \left((zF(x, y, 1) - F(x, y, z))/(1 - z) + xz \right)^t - yz.$$

After summing the geometric series, rearranging, and writing F for $F(x, y, z)$ and F_1 for $F(x, y, 1)$, we obtain

$$F^2 + \left((1 - z)(1 - xz) + yz - zF_1\right)F - yz^2(x - xz + F_1) = 0. \tag{3.2}$$

Since (3.1) is a direct translation of a construction which builds up maps out of maps with fewer faces, (3.2) and the initial conditions $f_{i,0,k} = 0$ must determine $F(x, y, z)$ uniquely. Since both F and F_1 appear in (3.2), it is not clear how to extract F or F_1 from (3.2) (setting $z = 1$ simply leads to the equation $0 = 0$). One approach is to use educated guessing, as done by Brown and Tutte. There is a more systematic approach (Brown, 1968), but it can lead to a morass of algebra: Complete the square in (3.2) to obtain an equation of the form

$$(F + \text{stuff})^2 = G(x, y, z, F_1). \tag{3.3}$$

Let $z = Z(x, y)$ stand for the value of z for which the left side of (3.3) vanishes. Since the left side of (3.3) is a square, its derivative with respect to z also vanishes at Z. Applying this to the right side of (3.3) we obtain the two equations

$$G(x, y, Z, F_1) = 0 \quad \text{and} \quad G_z(x, y, Z, F_1) = 0$$

in the two unknowns Z and F_1. These are rational equations, and they can be "solved" for F_1.

THEOREM 3. (Brown and Tutte) *The number $f_{i,j}$ of rooted 2-connected maps with $i + 1$ vertices and $j + 1$ faces is given by*

$$F(x, y) = uv(1 - u - v),$$

where u and v are given implicitly by

$$x = u(1 - v)^2, \qquad y = v(1 - u)^2, \quad \textit{and} \quad u(0,0) = v(0,0) = 0.$$

The explicit values are

$$\frac{(2i + j - 2)!(2j + i - 2)!}{i!j!(2i - 1)!(2j - 1)!}.$$

Proof. Since $F(x, y) = F(x, y, 1)$, we could solve $G = G_z = 0$ for F_1. This is done by parameterizing x and y as stated in the theorem. The explicit value for $f_{i,j}$ is obtained by a technique known as *Lagrange inversion*. That technique expresses the coefficients of any function $H(u(x, y), v(x, y))$ in terms of derivatives of H and of $u(1 - v)^2$ and $v(1 - u)^2$ with respect to u and v. See Section 5 for a discussion. □

Quadrangulations

A *quadrangulation* is a map such that each internal face is a quadrilateral.

There is a connection between rooted 2-connected maps and rooted quadrangulations with quadrilateral external faces, discovered by Brown (1965). Let M be a rooted map with vertex set V and face set F. Place a vertex in the middle of each face to form a new set of vertices V^*. Define a set of edges by connecting v and v^* if and only if v is a vertex of the face corresponding to v^*. This gives a map Q. Let r be the root vertex of M, $e = (r, v)$ the root edge of M and v^* the vertex corresponding to the external face of M. The root vertex of Q is v and the root edge

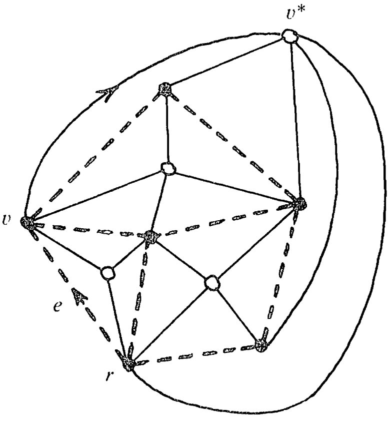

FIG. 4. Heavy dashed lines are the edges of the original 2-connected map. Solid lines are the edges of the corresponding quadrangulation.

is (v, v^*). The edges joining v^* are drawn so that no edges of Q lie outside of the region bounded by (v, v^*), (v^*, r) and e. See Figure 4.

THEOREM 4. *The above correspondence is a bijection between rooted 2-connected graphs with more than one edge and rooted quadrangulations with quadrilateral external faces. Furthermore, the original graph is 3-connected if and only if each 4-cycle of edges in the quadrangulation is a face.*

Proof. The bijection is due to Brown (1965, Sec. 7) and the last part of the theorem is due to Mullin and Schellenberg (1968, Sec. 5).

Exercise 3. Construct proofs using O2 and O3. □

Call a quadrangulation that corresponds to a 3-connected map *simple*.

Exercise 4. Show that the vertices of a quadrangulation can be partitioned uniquely into two sets, called red and green, such that edges connect only vertices of different colors, the red vertices correspond to the vertices of the corresponding 2-connected map and the green vertices to the map's faces.

Note that the number of quadrangulations with root degree 4, $i + 1$ red vertices and $j + 1$ green vertices equals $f_{i,j}$, the number of 2-connected rooted maps with $i + 1$ vertices and $j + 1$ faces. Let $p_{i,j}$ be the number of those that are simple and have at least 8 vertices. By Theorem 4 and Corollary 2.1, the function $P(x, y)$ counts rooted polyhedra. (The condition on vertices in $p_{i,j}$ eliminates small 3-connected maps that do not correspond to polyhedra.)

A quadrangulation has a *diagonal* if there are external vertices v and w and an internal vertex x so that (v, x) and (w, x) are edges. Let $n_{i,j}$ count the number of quadrangulations counted by $f_{i,j}$ that have no diagonals and let $N(x, y)$ be the corresponding generating function.

Every quadrangulation with root degree 4, more than 4 vertices and no diagonals can be built from a simple quadrangulation having at least 6 vertices. This is done by replacing the internal faces of the simple quadrangulation with arbitrary quadrangulations of root degree 4. Mullin and Schellenberg (1968, Sec. 6) show that this construction is uniquely determined and corresponds to the generating function equation

$$(xy/F)P(F/y, F/x) = N(x, y) - xy. \tag{3.4}$$

The quadrangulations counted by $f_{i,j}$ can be broken into three disjoint classes:

(i) no diagonals, counted by $N(x, y)$;
(ii) diagonal at the root, counted by, say, $R(x, y)$;
(iii) diagonal not at the root, counted by, say, $D(x, y)$.

Thus,

$$F(x, y) = N(x, y) + R(x, y) + D(x, y).$$

By the type of construction shown in Figure 5 it follows that

$$R(x, y) = (N(x, y) + D(x, y))F(x, y)/x.$$

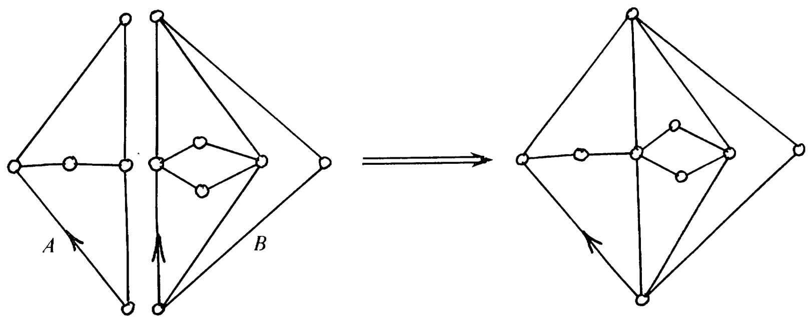

FIG. 5. Building up $R(x, y)$. Quadrangulation A has no root diagonal but B may have any number.

Interchanging the roles of red and green vertices (see Exercise 4) gives

$$D(x, y) = (N(x, y) + R(x, y))F(x, y)/y.$$

By performing some algebraic manipulations on the last three equations, one can show that

$$N = \left((1 + F/y)^{-1} + (1 + F/x)^{-1} - 1\right)F. \tag{3.5}$$

By setting $X = F/y$ and $Y = F/x$ and combining (3.4) and (3.5):

$$P(X, Y) = \left((1 + X)^{-1} + (1 + Y)^{-1} - 1\right)XY - F(x, y).$$

Exercise 5. Using this with Theorem 3 and setting $r = u/(1-u-v)$ and $s = v/(1-u-v)$, show

THEOREM 5. (Mullin and Schellenberg) *The generating function for the number of distinct rooted convex polyhedra with* $i+1$ *vertices and* $j+1$ *faces is given by*

$$P(X,Y) = \left((1+X)^{-1} + (1+Y)^{-1} - 1\right)XY - F,$$

where

$$F = rs/(r+s+1)^3$$

and r *and* s *are given implicitly by*

$$r = X(s+1)^2, \qquad s = Y(r+1)^2, \qquad r(0,0) = s(0,0) = 0.$$

Mullin and Schellenberg applied Lagrange inversion to obtain a formula for $p_{i,j}$. Unfortunately, it is a double summation with alternating signs, so it seems hard to see how $p_{i,j}$ behaves except by computing specific values. I'll say more about Lagrange inversion and an exact formula in Section 5.

Note that the generating function for $P(X, Y)$ is symmetric in X and Y. This result also follows immediately from duality without ever seeing the generating functions. Various other generating functions follow easily from $P(X, Y)$. Here are two examples. The coefficient of Y^k in $P(1, Y)$ is the number of rooted convex polyhedra with k faces. By Euler's theorem, the coefficient of $x^i y^n$ in $P(xy, y)$ is the number of convex polyhedra with $i+1$ vertices and n edges.

4. The asymptotic number of rooted polyhedra

We need some notation for writing asymptotics.

$$f(n) \sim g(n) \text{ means that } f(n)/g(n) \to 1 \text{ as } n \to \infty;$$
$$f(n) = O(g(n)) \text{ means that } f(n)/g(n) \to 0 \text{ as } n \to \infty.$$

By convention, $f(n)/g(n) = 1$ when $f(n) = g(n) = 0$. For functions of two (or more) variables, the terminology is more involved. Let R be a region in the xy-plane containing integer points (x, y) with $\min(x, y)$ arbitrarily large. We say that

$$f(m,n) \sim g(m,n) \text{ uniformly in } R$$

if

$$\sup|f(m,n)/g(m,n) - 1| \to 0 \text{ as } k \to \infty,$$

where the supremum is taken over all (m, n) in R with $\min(m, n) \geqslant k$. We define $f(m, n) = o(g(m, n))$ uniformly in R in a similar fashion.

There are several possible approaches to obtaining asymptotics for rooted maps. The most straightforward is to work with a simple formula like that for $f_{i,j}$ in Theorem 3 together with Stirling's formula,

$$n! \sim (2\pi n)^{1/2}(n/e)^n. \tag{4.1}$$

This idea can also be adapted to certain types of sums, but seldom to those with alternating signs unless the initial terms dominate the sum. In cases like $P(X, Y)$, one tries to work directly with the generating function. For an introduction to asymptotics in combinatorics, see Bender (1974).

We now turn to $P(X, Y)$. Think of it as a function of two complex variables X and Y. Such functions have places at which they misbehave, called "singularities." A little bit of knowledge about the nature of the singularities closest to the origin is often sufficient to provide information about the coefficients of the power series for the function. It would take too much space to define singularities and discuss the connection between their nature and the coefficients of the power series. In this way Bender and Richmond (1984) obtained messy asymptotic formulas from $P(X, Y)$, $P(1, Y)$, $P(xy, y)$, etc. The result for $p_{i,j}$ was valid for $i \to \infty$ provided $1/2 + c < i/j < 2 - c$. This result leaves a gap at each end because i/j is constrained to stay away from $1/2$ and 2 while v/f could approach either $1/2$ or 2 as f gets large. The extreme ends were filled in by

THEOREM 6. (Tutte, 1962) *Let t_i be the number of rooted convex polyhedra with $i + 1$ vertices and all faces triangular. Then*

$$t_i = \frac{2(4i-7)!}{(3i-4)!(i-1)!} \sim \frac{3}{16(6\pi i^5)^{1/2}}\left(\frac{256}{27}\right)^{i-1}.$$

The same formula holds if t_i counts rooted convex polyhedra with $i + 1$ faces and all vertices of degree 3.

5. Reconsideration

While writing this paper, I simplified the messy asymptotic formula for $p_{i,j}$ mentioned earlier. Using this result and Theorem 6 as a guide, I conjectured

THEOREM 7. (Bender and Wormald, to appear) *Uniformly as $\min(i, j) \to \infty$*

$$p_{i,j} \sim \frac{1}{3^5 ij}\binom{2i}{j+3}\binom{2j}{i+3}.$$

Proof. No high powered tools are needed. By applying Lagrange inversion (see below) to Theorem 5 in a slightly different way than Mullin and Schellenberg did, a singly indexed summation is obtained. To make asymptotic calculations easier, the sum is transformed twice by using the Pascal triangle identity

$$\binom{c+1}{k} = \binom{c}{k} + \binom{c}{k-1}$$

and rearranging terms.

What is Lagrange inversion? Suppose we want the coefficient of x^n in $f(g(x))$ where the power series for $f(y)$ is known but that for $g(x)$ is not known. Instead, we only know the power series for the inverse function $g^{-1}(z)$. Lagrange inversion

tells us how to compute the answer. There are various generalizations to functions of several variables. See (S. A. Joni, 1977) for a discussion and also a proof of the following.

THEOREM 8. (I. J. Good) *Suppose that* $f(z_1, \ldots, z_k)$ *and* $H_i(z_1, \ldots, z_k)$, $1 \leqslant i \leqslant k$ *are power series such that the* H_i*'s have nonzero constant terms. Let* $h_i = z_i H_i$. *Then there exist unique power series* $g_j(x_1, \ldots, x_k)$ *satisfying the set of equations* $h_i(g_1, \ldots, g_k) = x_i$. *Also, the coefficient of* $x_1^{n_1} \cdots x_k^{n_k}$ *in* $f(g_1, \ldots, g_k)$ *equals the coefficient of* $z_1^{n_1} \cdots z_k^{n_k}$ *in*

$$\det(\partial h_i/\partial z_j) f/\left(H_1^{n_1+1} \cdots H_k^{n_k+1}\right).$$

This can be applied to Theorem 5 with

$$(z_1, z_2) = (r, s), \qquad (x_1, x_2) = (X, Y), \qquad (n_1, n_2) = (i, j),$$
$$f = -rs/(r+s+1)^3, \qquad h_1 = r/(s+1)^2, \qquad h_2 = s/(r+1)^2.$$

Exercise 6. Show that for $i, j > 1$, $p_{i,j}$ is the coefficient of $r^i s^j$ in

$$(r+s+1)^{-3}(s+1)^{2i-3}(r+1)^{2j-3}(3rs - r - s - 1).$$

If we write

$$(r+s+1)^{-k} = \sum_{t,u} \binom{-k}{t}\binom{t}{u} r^u s^{t-u},$$

then Mullin and Schellenberg's formula is obtained. If we write

$$\begin{aligned}(r+s+1)^{-k} &= (r+s)^{-k}(1 + r/(1+s))^{-k} \\ &= \sum_j \binom{-k}{j} r^j (1+s)^{-k-j},\end{aligned} \tag{5.1}$$

then Bender and Wormald's formula is obtained. You may wish to carry out these calculations. If so, write

$$(1+r+s)^{-3}(3rs - r - s - 1) = 3rs(r+s+1)^{-3} - (r+s+1)^{-2}$$

and use (5.1) with $k = 2$ and $k = 3$.

6. The asymptotic number of polyhedra

In this section we will discuss the proof and application of the following theorem.

THEOREM 9. (Bender and Wormald, 1985) *Let* $u_{i,j}$ *be the number of unrooted convex polyhedra with* $i+1$ *vertices and* $j+1$ *faces. There are constants* A *and* $0 < c < 1$ *such that for all* i *and* j,

$$1 \leqslant 4(i+j)u_{i,j}/p_{i,j} < 1 + c^i,$$

where $0/0$ *is interpreted as* 1.

The theorem says that $u_{i,j}$ approaches $p_{i,j}/4(i+j)$ very quickly. Thus Theorems 7 and 9 answer question Q. Answering Q_k involves estimating

$$\frac{1}{4\cdot 3^5}\sum\frac{1}{ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3},$$

where the sum ranges over appropriate values of i and j. This can be done by standard methods as discussed in Bender (1974) or by using results in Bender and Richmond (1984). The answers are

$$Q\sim\frac{1}{972ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3}=A(i,j),\text{ say;}$$

$$Q_0\sim\left(\pi n(4+\sqrt{7})/4\sqrt{7}\right)^{1/2}A\left(n-1,(n-1)(3+\sqrt{7})/4\right);$$

$$Q_1\sim\frac{\sqrt{\pi n}}{4}A(n/2,n/2);$$

$$Q_2=Q_0,$$

where fractional factorials in binomial coefficients are approximated by Stirling's formula (4.1). Tutte (1963) conjectured and Richmond and Wormald (1982) proved the asymptotic formula for Q_1.

How is Theorem 9 proved? As noted at the end of Section 2, there are $4(i+j)$ distinct ways to root a convex polyhedron with $i+j$ edges and no symmetries. If there are symmetries, the number of rootings is less. That accounts for the left-hand inequality in Theorem 9.

The right-hand inequality is based on estimates for rooted polyhedra with symmetries. There are three different types of symmetries possible for a polyhedron. One type preserves the orientation of the polyhedron and is essentially a rotation. The other two types reverse the orientation and are distinguished by whether or not the symmetry maps any vertex or edge into itself. If it has such an invariant, it can essentially be viewed as a reflection in a plane; otherwise, a reflection in a point.

For each type of symmetry, if you are given (i) a connected piece cut out of the polyhedron whose images under the symmetry cover the entire polyhedron and (ii) the nature of the symmetry, then you can reconstruct the entire polyhedron. The piece cut out for you may involve edges and faces that have been cut in half. The cut may also run through some vertices. Connect all those vertices to a single new vertex v and extend the cut edges to v. If the original cut is chosen carefully, then the resulting figure will be 3-connected. Since the number of vertices and edges in the new 3-connected graph is less than that in the original polyhedron, Theorem 7 can usually be used to obtain a crude but adequate upper bound. Various adaptations of this idea are needed to handle all the cases that arise. If you wish details, see the original paper.

Special cases of Theorem 9 were proved by Tutte (1980) and Richmond and Wormald (1982).

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Combinatorial and Functional Identities in One-Parameter Matrices

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1. Introduction. A one-parameter matrix is one whose entries depend on a parameter in such a way that matrix multiplication corresponds to performing an operation, e.g., addition or multiplication, on the parameter. More formally, if a

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mapping M taking a semigroup S, with operation $*$, into an algebra of square matrices satisfies the *functional identity*

$$M(\alpha)M(\beta) = M(\alpha * \beta), \quad \alpha, \beta \in S, \tag{1}$$

we should refer to the image of S as a one-parameter matrix semigroup. However, adopting the popular abuse of notation that identifies a function with its image at a generic point of its domain, it is more convenient to simply call $M(\alpha)$ *a one-parameter matrix*.

An elementary example of a one-parameter matrix is

$$M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

representing rotations, by an angle α, of the plane about its origin [**2**], [**7**], and satisfying

$$M(\alpha)M(\beta) = M(\alpha + \beta) \tag{2}$$

for any real (or complex) α and β. If we write equation (2) in full,

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix},$$

we recognize it immediately as the sin/cos addition theorem in a *matrix* form. We therefore refer to it as the *Matrix* sin/cos *Addition Theorem*. The common interpretation of planar rotations as $e^{i\alpha}$ and as angular *translations* hint at the role of the exponential functions and the translations of the present article.

Other examples of one-parameter matrices may be found in [**10**], [**14**], [**15**]. One of particular interest here, [**15**], involves a one-parameter matrix whose *defining identity* (1) embodies the Binomial Theorem. For 2×2 matrices it has the form

$$B(r) = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

satisfying

$$B(r)B(s) = B(r+s) \tag{3}$$

for real (or complex) r and s. In its extension to $n \times n$ one-parameter matrices, discussed in Section 2, the matrix B is thus a *binomial matrix* and its identity (3) is the *Matrix Binomial Theorem*.

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An example in which the operation $*$ in (1) is multiplication is given by the one-parameter matrix

$$L(f) = \begin{bmatrix} f & 0 \\ f' & f \end{bmatrix}$$

satisfying

$$L(f)L(g) = L(fg) \tag{4}$$

for differentiable functions f and g, where f' is the derivative of f. It turns out that an obvious generalization to $n \times n$ matrices embodies Leibniz' Rule for the differentiation of a product. We call it, therefore, the *Matrix Leibniz' Rule*. The parameter in (4) is a differentiable function on an interval, and the parameter operation is pointwise multiplication of functions. In its extension to $n \times n$ one-parameter matrices, discussed in Section 3, L is thus a *Leibniz' Matrix* and its identity (4) is the *Matrix Leibniz' Rule* for the differentiation of a product.

The striking analogy between the Binomial Theorem and Leibniz's Rule, celebrated in **[4]**, is well known. The aim of this article is to extend an analogy between the Binomial Theorem and Leibniz' Rule to an analogy between their matrix counterparts, giving rise to a theory which unifies the *distinct* one-parameter matrices $B(r)$ and $L(f)$. One result of special interest is that Leibniz' Rule for the differentiation of a product can be viewed as a particular case of an obvious theorem according to which the pointwise multiplication of two matrix functions commutes with translation of their (matrix) variable by a (matrix) constant.

2. One-parameter matrices with real (or complex) parameter. If M is a one-parameter matrix whose parameter takes values in the additive group of real (or complex) numbers, then M must satisfy

$$M(r)M(s) = M(r+s). \tag{5}$$

In previous work, we have discovered the validity of (5) in the context of specific instances of $M(x)$. For instance, in **[10]**, with $M(x)$ a matrix representation of a geometrical translation by x units, (5) is evidently satisfied. In **[14]**, **[15]**, the matrix $M(x)$ arises as a Wronskian matrix in connection with generalizations of the hyperbolic sine and cosine. In this section we take up the more general problem of characterizing solutions to (5). As (5) is suggestive of an exponential relation, it comes as little surprise that $M(x)$ takes the form Ae^{Bx} for constant matrices A and B. Exponential matrices shall, therefore, play an important role in all that follows. For a square matrix B, we define e^B in the usual way:

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

Note that the exponential law $e^{B+C} = e^B e^C$ holds whenever $BC = CB$, and in particular,

$$e^{B(r+s)} = e^{Br+Bs} = e^{Br}e^{Bs}$$

for any real numbers r and s. This verifies that $M(x) = e^{Bx}$ is a solution to (5). In fact, we may generalize slightly: Expressions of the form $M(x) = Ae^{Bx}$, where $AB = BA$ and $A^2 = A$, are also solutions. (We will show in Theorem 1 below that the common value of AB and BA is B in this case.) For

$$Ae^{Br}Ae^{Bs} = A\sum_{k=0}^{\infty}\frac{B^k r^k}{k!}Ae^{Bs},$$

and we interchange the second factor of A with every B in the sum to obtain

$$Ae^{Br}Ae^{Bs} = A^2e^{Br}e^{Bs} = Ae^{Br+Bs}.$$

Thus, $M(x) = Ae^{Bx}$ satisfies (5). This is actually a complete characterization of solutions to (5). We formalize this result as

THEOREM 1. *$M(x)$ satisfies (5) if and only if $M(x)$ can be written in the form Ae^{Bx} with $A^2 = A$ and $AB = BA = B$.*

Proof. The preceding paragraph shows that taking $M(x) = Ae^{Bx}$ with $A^2 = A$ and $AB = BA = B$ always provides a solution to (5). For the converse, suppose $M(r+s) = M(r)M(s)$. The proof that $M(x)$ is exponential parallels perfectly that used in the scalar case. Since $M(x)$ is differentiable, we have

$$\lim_{h\to 0}\frac{M(h) - M(0)}{h} = M'(0).$$

More generally, for any x,

$$M'(x) = \lim_{h\to 0}\frac{M(h+x) - M(0+x)}{h} = \lim_{h\to 0}\frac{M(h) - M(0)}{h}M(x),$$

and by the continuity of matrix multiplication, $M'(x) = M'(0)M(x)$. Now, if $C(x)$ is any column of $M(x)$, we have

$$C'(x) = M'(0)C(x),$$

and the elementary theory of systems of differential equations (as in **[1]**) assures that

$$C(x) = e^{M'(0)x}C(0)$$

follows. Since this last equation holds for each column of $M(x)$ separately, we have, in fact,

$$M(x) = e^{M'(0)x}M(0).$$

A similar argument (involving the rows of $M(x)$) shows that

$$M'(x) = M(x)M'(0) \quad \text{and} \quad M(x) = M(0)e^{M'(0)x}$$

also hold. For simplicity of notation, let $M(0) = A$ and $M'(0) = B$. It remains only to show that $A^2 = A$ and $AB = BA = B$. The former follows immediately upon taking $r = s = 0$ in (5), while the latter is a consequence of the identities

$$M'(x) = BM(x) = M(x)B$$

shown above, taken with $x = 0$.

We stress the phrase *can be written in the form* of the statement of Theorem 1. Interestingly, given A and B, Theorem 1 does not allow us to decide immediately if Ae^{Bx} satisfies (5). For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. We find that

$$M(x) = Ae^{Bx} = \begin{bmatrix} e^x & 0 \\ 0 & 0 \end{bmatrix}$$

does satisfy (5), yet $AB \neq BA$ and $M'(0) \neq B$. Evidently, a function $M(x)$ which satisfies (5) may be expressible as Ae^{Bx} in more than one way, and not all such expressions need satisfy the conditions of Theorem 1. In general, if $M(x) = Ae^{Bx}$ then $M(0) = A$ but $M'(0) = AB$ need not equal B. From the proof of Theorem 1, for Ae^{Bx} to satisfy (5) it is necessary that $Ae^{Bx} = Ae^{ABx}$, and then $A^2 = A$ and $A \cdot AB = AB \cdot A = AB$ as well. As the next result shows, these conditions are also sufficient.

THEOREM 2. *$M(x) = Ae^{Bx}$ satisfies (5) if and only if $A^2 = A$ and $AB = ABA$.*

Proof. In light of the preceding discussion we need only show that $A^2 = A$ and $AB = ABA$ imply $M(x) = Ae^{Bx}$ satisfies (5). From $AB = ABA$, we can deduce $AB^k = AB^kA$ for $k \geqslant 1$ by induction. Indeed, if $AB^{k-1} = AB^{k-1}A$, right multiplying by B yields

$$AB^k = AB^{k-1}AB. \tag{6}$$

Right multiplying both sides of (6) by A leads to

$$AB^kA = AB^{k-1}ABA \tag{7}$$

while substituting ABA for the rightmost factor AB in (6) gives

$$AB^k = AB^{k-1}ABA \tag{8}$$

Combining (7) and (8) shows $AB^kA = AB^k$, as desired. Next, we observe that $Ae^BA = Ae^B$, and therefore $Ae^{Br}A = Ae^{Br}$ as well. Finally,

$$Ae^{Br}Ae^{Bs} = Ae^{Br}e^{Bs} = Ae^{B(r+s)}$$

which is what we were to prove.

Theorem 2 allows us to restate Theorem 1 in a modified form, as follows.

THEOREM 3. *If $A^2 = A$ and $AB = ABA$, the function $M(x) = Ae^{Bx}$ is a solution to* (5). *Conversely, if $M(x)$ satisfies* (5), *representations of $M(x)$ as Ae^{Bx} exist, and for any such representation,*

(i) $A^2 = A$,
(ii) $AB = ABA$,
(iii) $A = M(0)$,
(iv) $AB = M'(0)$,

and

(v) $M(x) = Ae^{ABx}$ *is another representation of* $M(x)$.

For an example, let $M(x)$ be the $(n+1) \times (n+1)$ matrix

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ x & 1 & 0 & \dots & 0 \\ x^2 & 2x & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & \begin{bmatrix} n \\ 1 \end{bmatrix} x^{n-1} & \begin{bmatrix} n \\ 2 \end{bmatrix} x^{n-2} & \dots & 1 \end{bmatrix},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the binomial coefficients. As discussed in [**10**], $M(x)$ satisfies (5). Observing that $M(0) = I$, we may apply our results to see that $M(x) = e^{Kx}$, where

$$K = M'(0) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & n-1 & 0 \end{bmatrix}$$

with nonzero entries in the first subdiagonal (where $i - j = 1$). We thus express a known solution to (5) in exponential form. Another attempt to express a given solution to (5) in exponential form may be found in [**8**].

For a second example, let us apply Theorem 1 to generate a solution to (5). We will take A to be the 4×4 identity matrix, and in place of B we will use the matrix

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Matrix N is particularly convenient to use because of the simple pattern its powers exhibit. For $0 \leqslant k < 4$, N^k has 1's on the kth subdiagonal (where $i - j = k$) and 0's elsewhere, while the fourth and all higher powers of N are 0. Thus

$$e^{Nx} = \sum_{k=0}^{4} N^k \frac{x^k}{k!}$$

is easily seen to be

$$B(x)=e^{Nx}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ \frac{x^2}{2!} & x & 1 & 0 \\ \frac{x^3}{3!} & \frac{x^2}{2!} & x & 1 \end{bmatrix}.$$

By Theorem 1, e^{Nx} satisfies (5). That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ r+s & 1 & 0 & 0 \\ \frac{(r+s)^2}{2!} & r+s & 1 & 0 \\ \frac{(r+s)^3}{3!} & \frac{(r+s)^2}{2!} & r+s & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ \frac{r^2}{2!} & r & 1 & 0 \\ \frac{r^3}{3!} & \frac{r^2}{2!} & r & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ \frac{s^2}{2!} & s & 1 & 0 \\ \frac{s^3}{3!} & \frac{s^2}{2!} & s & 1 \end{bmatrix}.$$

In particular, focusing on a single entry, say the lower left corner,

$$\frac{(r+s)^3}{6}=\frac{r^3}{6}+\frac{r^2s}{2}+\frac{rs^2}{2}+\frac{s^3}{6}.$$

Here the *Matrix Binomial Theorem* begins to emerge. In the next section this connection will be discussed more fully.

Before proceeding to that topic, we wish to introduce some notation inspired by the preceding example. We will always denote by N an $n \times n$ matrix with 1's on the first subdiagonal and 0's elsewhere. For $1 \leqslant k < n$, N^k has 1's on the kth subdiagonal, all other entries being 0, and for $k \geqslant n$, N^k is the zero matrix. In addition, we refer to e^{Nx} as $B(x)$, to suggest the connection with the *Matrix Binomial Theorem*, which is discussed in the next section. Where no ambiguity results, the dimension n will be suppressed. The binomial matrix $B(x)$ should not be confused with the *constant* matrix B of Theorems 1, 2 and 3.

3. Matrix Identities for the Binomial Theorem and Leibniz' Rule. The connection between (5) and the Binomial Theorem can be made precise as follows. If N is of dimension n, the identity $B(r+s)=B(r)B(s)$ is equivalent to

$$(r+s)^k=\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix} r^j s^{k-j}, \quad 0 \leqslant k < n.$$

We may even say that $B(r+s)=\mathrm{B}(r)B(s)$ is equivalent to the Binomial Theorem, and call it the *Matrix Binomial Theorem*, if it is understood that the dimension of the matrix B is governed by a parameter n, and that (5) is to hold for all n. We will consider other instances of this situation, a scalar identity involving the parameter n

equivalent to a matrix identity where n determines the size of the matrix. When we call the identities equivalent, or say one is the matrix form of the other, it is always to be understood that the matrix identity holds for all n.

As a generalization of the Binomial Theorem, several authors, e.g. [**5**], [**11**], [**13**], [**16**], have considered sequences of binomial type. As defined by Brown [**5**], the sequence of functions $\{f_k\}$ is said to be of binomial type if $f_0(0) = 1$, and for each $k \geqslant 0$,

$$f_k(r+s) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} f_i(r) f_{k-i}(s). \tag{9}$$

In this context the Binomial Theorem simply asserts that $f_k(x) = x^k$ defines one such sequence. Proceeding as in the example of $B(x)$, given any sequence $\{f_k\}$ we may define a lower triangular matrix F with ij entry $f_{i-j}/(i-j)!$ for $i \geqslant j$, and the identity (9) is then equivalent to $F(r)F(s) = F(r+s)$. In this way, we establish a link between sequences of binomial type and the matrix identity (5).

Now Theorem 1 provides a characterization of sequences of binomial type. From the condition $f_0(0) = 1$ it is easy to show that $f_k(0) = 0$ for $k > 0$. Thus, $\{f_k\}$ is of binomial type if and only if F satisfies (5) and $F(0) = I$. Under these circumstances Theorem 1 says $F(x) = e^{Bx}$ where $B = F'(0)$. Also, noting that

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f_k(x) N^k,$$

we have

$$B = \sum_{k=0}^{\infty} \frac{1}{k!} f_k'(0) N^k.$$

Thus $\{f_k\}$ is of binomial type if and only if

$$\sum_{k=0}^{\infty} \frac{1}{k!} f_k(x) N^k = e^{B(N)x}$$

where $B(N)$ is a series in N. This is analogous to Brown's result that $\{f_k\}$ is of binomial type if and only if

$$\sum_{k=0}^{\infty} \frac{1}{k!} f_k(x) t^k = e^{B(t)x}$$

where $B(t)$ is a power series in t. Indeed, since the ring of formal power series in t is isomorphic to the ring of power series in an infinite dimensional version of N, we may view our result as a finite dimensional version of Brown's. However, Brown's result is proved by purely combinatorial methods and appears in a much more general setting.

The sequences of binomial type are a generalization of the combinatorial principle expressed in the Binomial Theorem. Inasmuch as identity (5) encompasses the

sequences of binomial type, there is some justice in considering it in a still more general setting for this principle. As we have seen, the immediate connection between (5) and the Binomial Theorem is expressed in terms of a single solution to (5), namely $B(x)$. This is the obvious starting point in the attempt to extend the analogy between the Binomial Theorem and Leibniz' Rule to the more general context of (5).

In accord with the analogy between the Binomial Theorem and Leibniz' Rule, the natural counterpart of the binomial matrix, $B(x)$, is Leibniz' matrix, $L(f)$, given by the equation

$$L(f)=\begin{bmatrix} f & 0 & 0 & \cdots & 0 \\ f' & f & 0 & \cdots & 0 \\ \dfrac{f''}{2!} & f' & f & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{f^{(n-1)}}{(n-1)!} & \dfrac{f^{(n-2)}}{(n-2)!} & \dfrac{f^{(n-3)}}{(n-3)!} & \cdots & f \end{bmatrix},$$

where f is a suitably differentiable function. Indeed, with this definition of $L(f)$, Leibniz' Rule is equivalent to

$$L(f)L(g)=L(fg). \tag{10}$$

Moreover, if we view $L(f)$ as the result of applying to f an operator

$$L=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ D & 1 & 0 & \cdots & 0 \\ \dfrac{D^2}{2!} & D & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{D^{n-1}}{(n-1)!} & \dfrac{D^{n-2}}{(n-2)!} & \dfrac{D^{n-3}}{(n-3)!} & \cdots & 1 \end{bmatrix},$$

we observe that $L=e^{ND}$, where D is the scalar differential operator. Thus, we may interpret Leibniz' Rule as one instance of a matrix differential operator which satisfies (10). In this sense, the set of solutions to (10) encompasses generalizations of Leibniz' Rule in the same way solutions to (5) encompass generalizations of the Binomial Theorem. Thus, the analogy that links the Binomial Theorem and Leibniz' Rule is seen to extend to identities (5) and (10), and, as we shall see, to their solutions, as well.

Note that (10) is the defining identity for a one-parameter matrix where the parameter takes values in a semigroup of functions, the operation being pointwise multiplication. In this context, it should be clear that the entire motivation for

considering one-parameter matrices of this type is to allow the analogy between the Binomial Theorem and Leibniz' Rule to extend to one-parameter matrices. Further discussion of the most appropriate domain for L and f consistent with this context will appear in the next section.

The matrix forms of the Binomial Theorem and Leibniz' Rule provide a neat format for expressing the corresponding combinatorial relationships. As an illustration, consider the extension of the Binomial Theorem to expressions of the form $(u_1 + u_2 + u_3 + \cdots + u_m)^n$. Specifically, if $u = u_1 + u_2 + u_3 + u_4$, we may obtain $u^3/3!$ as the $(4,1)$ entry of $B(u)$. By (5),

$$B(u) = B(u_1)B(u_2)B(u_3)B(u_4),$$

and the $(4,1)$ entry of this product is given by the usual summation. Since the nonzero entries of $B(r)$ are $r^{i-j}/(i-j)!$ for $i \geqslant j$, we have

$$\begin{aligned} u^3 &= 3! \sum_{i=1}^{4} \sum_{j=1}^{i} \sum_{k=1}^{j} \frac{u_1^{4-i}}{(4-i)!} \frac{u_2^{i-j}}{(i-j)!} \frac{u_3^{j-k}}{(j-k)!} \frac{u_4^{k-1}}{(k-1)!} \\ &= \sum_{i=1}^{4} \sum_{j=1}^{i} \sum_{k=1}^{j} \frac{3!}{(4-i)!(i-1)!} \frac{(i-1)!}{(i-j)!(j-1)!} \frac{(j-1)!}{(j-k)!(k-1)!} u_1^{4-i} u_2^{i-j} u_3^{j-k} u_4^{k-1} \\ &= \sum_{i=1}^{4} \sum_{j=1}^{i} \sum_{k=1}^{j} \binom{3}{i-1} \binom{i-1}{j-1} \binom{j-1}{k-1} u_1^{4-i} u_2^{i-j} u_3^{j-k} u_4^{k-1}. \end{aligned}$$

In exactly the same way, we can derive an expansion for $(u_1u_2u_3u_4)'''$, where the u_i are now functions. Employing the matrix form of Leibniz' Rule and (10), we find that it suffices to compute the identical matrix product entry as above, save that exponents now refer to derivatives instead of powers. This derivation of expansions for $(u_1 + u_2 + u_3 + u_4)^3$ and $(u_1u_2u_3u_4)'''$ may be compared to the approach of Mazkewitsch [12], which does not involve matrices.

Incidentally, when specific functions are chosen for f and g, equation (10) can lead to curious identities. For example, with $f(x) = x^2$, $g(x) = x^3$, and $n = 2$, we observe

$$\begin{pmatrix} x^2 & 0 & 0 \\ 2x & x^2 & 0 \\ 1 & 2x & x^2 \end{pmatrix} \cdot \begin{pmatrix} x^3 & 0 & 0 \\ 3x^2 & x^3 & 0 \\ 3x & 3x^2 & x^3 \end{pmatrix} = \begin{pmatrix} x^5 & 0 & 0 \\ 5x^4 & x^5 & 0 \\ 10x^3 & 5x^4 & x^5 \end{pmatrix}.$$

Or, with $f(x) = \tan x$ and $g(x) = \cot x$, since $L(fg) = L(1) = I$, we observe that $L(f)$ and $L(g)$, i.e.,

$$\begin{pmatrix} \tan x & 0 & 0 \\ \sec^2 x & \tan x & 0 \\ \sec^2 x \tan x & \sec^2 x & \tan x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cot x & 0 & 0 \\ -\csc^2 x & \cot x & 0 \\ \csc^2 x \cot x & -\csc^2 x & \cot x \end{pmatrix}$$

are inverse of one another. More generally, the inverse of $L(f)$ is $L(1/f)$.

As noted earlier, in the matrix form of Leibniz' Rule, we may write $L = e^{ND}$ in just the same way we found $B(x) = e^{Nx}$ earlier. However, while the exponential form of B immediately elucidates (5), the corresponding form of L sheds little light on the significance of (10). There is an interpretation of e^{ND} as a translation operator that remedies this situation, but its development will be postponed until after solutions to (10) have been characterized.

4. Solutions to Identity (10). In the preceding section we have seen that identities (5) and (10) are linked in a natural way by the analogy connecting the Binomial Theorem and Leibniz' Rule. It is less clear that the solutions should be so related. Identity (5) is not the kind of behavior one normally expects from a matrix-valued real function, and the suggestion of an exponential solution is self-evident. Identity (10) is also unexpected behavior, when dealing with differential operators, but there is no particular reason to suspect the involvement of exponential functions. Still, the analogy already developed is sufficiently compelling that little courage is required to guess an exponential form for solutions to (10). It then remains to establish an appropriate context and verify the guess.

When we seek to characterize solutions to (10), over what set shall the operator L be allowed to vary? In the one example at hand,

$$L = \sum_{k=0}^{n} \frac{N^k}{k!} D^k,$$

so polynomials in D with matrix coefficients are a natural choice. However, since we are expecting an exponential characterization, we shall allow L to be a power series in D, that is,

$$L = \sum_{k=0}^{\infty} A_k D^k,$$

where A_k is a square matrix. (Although we will assume A_k is constant, the arguments go through with little modification if A_k is allowed to have variable entries, i.e., A_k has entries that are functions.) With this domain for L, we will be able to establish a characterization of solutions to (10) in the form Ae^{BD}. Here

$$e^{BD} = \sum_{k=0}^{\infty} \frac{B^k}{k!} D^k$$

with D the ordinary differential operator, and in the expression $e^{BD}(f(x))$, x is implicitly restricted to a point of convergence of

$$\sum_{k=0}^{\infty} \frac{B^k}{k!} f^{(k)}(x).$$

THEOREM 4. *Let $L = Ae^{BD}$ where A and B are square matrices, $A^2 = A$, and $AB = BA$. If $L(f)$ and $L(g)$ exist, then* (10) *holds.*

Proof. We begin with the special case $A = I$. The argument will parallel the proof that the scalar exponential series defines a function which satisfies the exponent law (5). By definition,

$$e^{BD}f(x)g(x) = \sum_{k=0}^{\infty} \frac{B^k}{k!} D^k[f(x)g(x)].$$

Applying Leibniz' Rule (and suppressing x) we find

$$\begin{aligned} e^{BD}(fg) &= \sum_{k=0}^{\infty} \frac{B^k}{k!} \sum_{j=0}^{k} \binom{k}{j} D^j(f) D^{k-j}(g) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{B^j D^j(f)}{j!} \frac{B^{k-j} D^{k-j}(g)}{(k-j)!}. \end{aligned}$$

Now, interchange the order of summation and pull out the factors independent of k to arrive at

$$e^{BD}(fg) = \sum_{j=0}^{\infty} \frac{B^j D^j(f)}{j!} \sum_{k=j}^{\infty} \frac{B^{k-j} D^{k-j}(g)}{(k-j)!}.$$

We recognize the second series at right as $e^{BD}(g)$, regardless of the value of j, and thus

$$e^{BD}(fg) = e^{BD}(f) e^{BD}(g).$$

This completes the proof for the case $A = I$.

For the general case, we must show that

$$Ae^{BD}(fg) = (Ae^{BD}f)(Ae^{BD}g).$$

As in the discussion preceding Theorem 1, we may interchange the factors A and $e^{BD}f$ to produce

$$(Ae^{BD}f)(Ae^{BD}g) = A^2(e^{BD}f)(e^{BD}g).$$

Then, since $A^2 = A$ and $(e^{BD}f)(e^{BD}g) = e^{BD}(fg)$, the desired conclusion follows.

For a converse to Theorem 4, suppose

$$L = \sum_{k=0}^{\infty} A_k D^k,$$

where $\{A_k\}$ is a sequence of square matrices. In addition, assume (10) holds whenever all the series involved are defined. Then, in particular, (10) must hold whenever f and g are polynomials. Let $f(x) = 1$ and $g(x) = x$, and observe that $L(1) = A_0$ and $L(x) = A_0 x + A_1$. Thus

$$L(1)L(x) = L(x) \quad \text{implies} \quad A_0^2 x + A_0 A_1 = A_0 x + A_1,$$

and we conclude $A_0^2 = A_0$ and $A_0 A_1 = A_1$. Reversing the roles of f and g leads to $A_1 A_0 = A_1$ as well, so that A_0 and A_1 commute. We now show that for $n \geq 1$, $A_n = A_1^n/n!$. Proceeding by induction, suppose

$$A_{n-1} = \frac{A_1^{n-1}}{(n-1)!},$$

and consider the identity $L(x^{n-1})L(x) = L(x^n)$. Each of the factors $L(x^k)$ is a polynomial, and equating the constant terms on each side produces

$$(n-1)!A_{n-1}A_1 = n!A_n.$$

Therefore, $A_n = A_1^n/n!$ holds for all $n \geq 1$. Since $A_0 A_1 = A_1$, we may choose to express A_n as $A_0 A_1^n/n!$, and write

$$L = A_0 \sum_{k=0}^{\infty} \frac{A_1^k D^k}{k!}.$$

That is, $L = A_0 e^{A_1 D}$ where $A_0^2 = A_0$ and $A_0 A_1 = A_1 A_0 = A_1$. Combining this result with Theorem 4, we have the following complete characterization.

THEOREM 5. *The operator* $L = \sum_{k=0}^{\infty} A_k D^k$ *is multiplicative* (*where defined*) *if and only if* $L = A_0 e^{A_1 D}$, *where* $A_0^2 = A_0$ *and* $A_1 A_0 = A_0 A_1 = A_1$.

In applying Theorem 5, a situation arises similar to that discussed after Theorem 1. A single operator may be expressed in more than one way as Ae^{BD}. Note that if L is expressible as a power series $\sum_{k=0}^{\infty} A_k D^k$, the coefficient matrices A_k are determined by the effects of L on the functions x^k, $k \geq 0$. For example, $A_0 = L(1)$ and $A_1 = L(x) - xL(1)$. This shows that the series representation is *unique* even though the exponential form is not. In general, the power series for Ae^{BD} will have $A_0 = A$ and $A_1 = AB$. If this operator is to be multiplicative, Theorem 5 requires $A_0^2 = A_0$ and $A_0 A_1 = A_1 A_0$. Thus, a necessary condition is $AB = ABA$. In fact, when combined with $A^2 = A$, this condition is sufficient as well, and we have the following theorem.

THEOREM 6. *If* $A^2 = A$ *and* $AB = ABA$, *the operator* $L = Ae^{BD}$ *satisfies identity* (10). *Conversely, if* L *satisfies* (10), *representations of* L *as* Ae^{BD} *exist, and for any such representation,*

(i) $A^2 = A$,
(ii) $AB = ABA$,
(iii) $A = L(1)$,
(iv) $AB = L(x) - xL(1)$,

and

(v) $L = Ae^{ABD}$ *is another representation of* L.

Theorem 6 parallels Theorem 3 and the proofs are nearly identical. For this reason, the proof of Theorem 6 is omitted.

As a final topic, we discuss an interpretation of e^{BD} that sheds some light on the multiplicative identity (10). To begin, consider the scalar operator e^{aD} where a is a fixed real number. For an analytic function f, we have

$$\begin{aligned} e^{aD}f(x) &= \sum_{k=0}^{\infty} \frac{a^k D^k}{k!} f(x) \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} a^k \\ &= f(x+a). \end{aligned}$$

That is, e^{aD} is just a translation by a units. In this context, (10) is perfectly natural. It merely says that the pointwise multiplication of two functions commutes with translation of the independent variable by a: one may translate each and then multiply, or multiply and then translate, with the same result in each case.

More generally, we may extend the algebra of differentiable functions to an algebra of matrices by defining

$$f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} A^k,$$

with convergence assured if the spectrum of A is contained in the radius of convergence of f [**6**, p. 113]. Working with this definition, we can easily show that

$$f(A+B) = \sum_{k=0}^{\infty} \frac{f^{(k)}(A)}{k!} B^k$$

as long as $AB = BA$. At the same time, a matrix differential operator

$$L = \sum_{k=0}^{\infty} A_k D^k$$

may be allowed to operate on a matrix function $M(x)$ in an obvious way:

$$L(M(x)) = \sum_{k=0}^{\infty} A_k \cdot M^{(k)}(x).$$

In this context, we have $L(f(x)) = L(f(x)I) = L(f(xI))$, and in particular,

$$e^{BD}f(xI) = \sum_{k=0}^{\infty} \frac{B^k}{k!} f^{(k)}(xI) = f(xI+B).$$

Thus, we still may interpret e^{BD} as a translation (at least in its effect on the scalar matrices $F(xI)$).

Now let us reconsider Theorem 5. The theorem concerns those matrix operators formally expressible as a power series in D, and the action of such operators on real functions $f(x)$. If we broaden the context so that the operators act on matrix functions $M(x)$ (as above), then the exponential operators e^{BD} are actually translations. Ignoring the matrix A_0 for a moment, Theorem 5 says that the product preserving operators are essentially the translations.

To be more accurate, we must account for the matrix A_0. Since $A_0^2 = A_0$, it is a *projection* (see [**9**, p. 224, exercise 17]), and the condition $A_0 A_1 = A_0$ implies that the range of e^{AD} is already contained in the image of this projection. Thus, Theorem 5 shows that the only product preserving operations are made up of translations and compatible projections.

5. Conclusion. Our results illustrate on an elementary level the application of matrix algebra in combinatorial problems. More specifically, the combinatorial relationships embodied in the Binomial Theorem and Leibniz' Rule find ready expressions in matrix form, and this allows the powerful heuristics of matrix algebra to be brought into play. Although it is true that the exponential flavor of (5) ultimately inspires our main results, the importance of the matrix formulation, both as a unifying force and as a source of intuition and insight, should not be underestimated. The interested reader is encouraged to consult reference [**3**] as an illustration of this point at a more advanced level.

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Binomial Identities and Hypergeometric Series

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1. Introduction. Combinatorial problems arise in many areas of mathematics and theoretical physics and their solutions often involve the evaluation of a sum of products of binomial coefficients. There are various methods for evaluating such sums and some procedures are discussed in Riordan [**13**] and Knuth [**11**]. Knuth has written that there are thousands of binomial identities; many excellent mathematicians have expended considerable amounts of ingenuity in proving these. In fact, contrary to the prevailing view, there is a very small number of essentially different binomial identities, and thus a great deal of mathematical ingenuity has been needlessly wasted. Many mathematicians have been unable to recognize that a given binomial identity is actually equivalent to one already known because the notation for a single binomial coefficient is very misleading when used to express sums of their products. This notation all too often serves to disguise several essentially identical sums and makes them appear very different. One reason for this is that binomial coefficients can be taken apart and then rearranged to take many different forms.

In this paper we shall explain how to write most single series of products of binomial coefficients in a canonical fashion, so that the real character of the series is easily discernible. This method was used by Euler and Gauss. These canonical series are known as hypergeometric series. The classical notation for hypergeometric series is easy to learn and use. This notation expresses explicitly certain important features of the sum; this allows for the kind of classification scheme which makes standardization possible. Here we consider some of the most important hypergeometric identities. These cover most of the single sums of products of binomial coefficients where the series can be summed. Thus, we shall show why the hypergeometric identities should be regarded as the standard forms. A number of examples taken from various sources will be given to illustrate how sums of binomial coefficients can be reduced to standard form.

The following section contains definitions and statements of four main hypergeometric identities. Since the proofs of these identities are unrelated to the applications, which are the prime concern of this paper, the proofs have been relegated to the fourth and last section. The third section contains the examples. The reader

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should also consult Andrews **[2]** where the view presented here is explicitly stated, and which contains examples complementing those presented here. For an interesting history of these identities, one may see Askey **[4]**, **[5]**.

2. Hypergeometric series. In general, a hypergeometric series (or to some, a generalized hypergeometric series) is a series

$$\sum C_n \tag{2.1}$$

such that C_{n+1}/C_n is a rational function of n. Note that this is a situation where the ratio test, the simplest of all convergence tests, applies. Also, if C_n is a product of binomial coefficients, then C_{n+1}/C_n is of this form. We can factor the rational function as

$$\frac{C_{n+1}}{C_n} = \frac{(n+a_1)\cdots(n+a_p)x}{(n+b_1)\cdots(n+b_q)(n+1)} \tag{2.2}$$

and the series can then be written as a constant C_0 times

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, \tag{2.3}$$

where $(\alpha)_n$ are the shifted factorials defined by

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad (\alpha)_0 = 1. \tag{2.4}$$

The series (2.3) is usually denoted by

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right).$$

If $x = 1$, then we omit it. The case $p = q + 1$ arises very often and we shall consider only such cases. The parameters a_i and b_i are complex numbers, but for all our applications they will be real. Moreover, only the finite series is of interest to us here, but if the series is infinite, it converges for $|x| < 1$; if $x = 1$, then we require $(\Sigma b_i - \Sigma a_i) > 0$.

In the applications to binomial identities, q is very small, usually one or two, and the parameters $a_1, \ldots, a_p$, $b_1, \ldots, b_q$ satisfy certain relations. These relations play an important role in the classification of the series and contribute to the power of this method when applied to binomial sums. We shall say that a series is k-balanced if $x = 1$, if one of the a_i's is a negative integer, and if

$$k + \sum a_i = \sum b_i. \tag{2.5}$$

The case $k = 1$ is most important and then the series is called balanced or Saalschützian. (Note that, if a_i is a negative integer for some i, then the series (2.3) must be finite.) The series is called well-poised if

$$1 + a_1 = b_1 + a_2 = \cdots = b_{p-1} + a_p. \tag{2.6}$$

Recall that we have taken $q = p - 1$.

We now state the four hypergeometric identities which occur very often in practice. Numerous binomial identities can be reduced to these. From now on, we assume that n denotes a positive integer.

The Chu-Vandermonde identity:

$$ {}_2F_1\left(\begin{matrix}-n, & -b\\ & c\end{matrix}\right) = \frac{(c+b)_n}{(c)_n}. \tag{2.7}$$

The Pfaff-Saalschütz identity:

$$ {}_3F_2\left(\begin{matrix}-n, & -a, & -b\\ & c, & d\end{matrix}\right) = \frac{(c+a)_n(c+b)_n}{(c)_n(c+a+b)_n}, \tag{2.8}$$

where $d = 1 - a - b - n - c$, that is, the series is balanced.

The Sheppard-Andersen identity:

$$ {}_3F_2\left(\begin{matrix}-n, & -a, & -b\\ & c, & 2-n-a-b-c\end{matrix}\right)$$

$$ = \frac{(c+b-1)_n(c+a)_n}{(c+a+b-1)_n(c)_n}\left[1 - \frac{a}{(c+b-1)(a+c+n-1)}\right]. \tag{2.9}$$

Note that the series (2.9) is 2-balanced. The first two identities are quite old; Chu published the first one in 1303. The second identity was obtained by Pfaff [**12**] in 1797, then forgotten and rediscovered by Saalschütz [**14**] in 1890. Sheppard [**15**] published the identity for a k-balanced ${}_3F_2$ in 1912 and the particular case $k = 2$ was rediscovered by Andersen [**1**] in some work in probability theory.

The final identity is for a well-poised ${}_3F_2$ and is due to Dixon [**7**]. This identity is best expressed in terms of the gamma function $\Gamma(s)$. This is defined for $s > 0$ by the integral

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx, \tag{2.10}$$

and extended to other real values except $s = 0, -1, -2, \dots,$ by

$$\Gamma(s+1) = s\Gamma(s). \tag{2.11}$$

Some important properties of this function are

$$\Gamma(s+1) = s! \quad \text{if } s \text{ is an integer} \geqslant 0, \tag{2.12}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin s\pi}, \tag{2.13}$$

and

$$\lim_{s\to\infty} \frac{\Gamma(s+1)e^s}{\sqrt{2\pi s}\,s^s} = 1 \quad \text{(Stirling's formula)}. \tag{2.14}$$

Dixon's identity is:

$$ {}_3F_2\left(\begin{matrix} a, & -b, & -c \\ & a+b+1, & a+c+1 \end{matrix}\right) $$
$$ = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a+b)\Gamma(1+a+c)\Gamma\left(1+\frac{a}{2}+b+c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}+b\right)\Gamma\left(1+\frac{a}{2}+c\right)\Gamma(1+a+b+c)}. \quad (2.15) $$

Note that this series can be infinite; then we require $a + 2b + 2c + 2 > 0$ for convergence.

We shall need the following elementary identities to reduce sums of products of binomial coefficients to hypergeometric series:

$$ (f)_{n-k} = \frac{(-1)^k (f)_n}{(-f-n+1)_k}, \quad (2.16) $$

and if $f = 1$ we get

$$ (n-k)! = \frac{(-1)^k n!}{(-n)_k}. \quad (2.17) $$

$$ (a-n)_k = (-1)^k(-a+n-k+1)_k, \quad (2.18) $$

and

$$ (a)_{2k} = 2^{2k}\left(\frac{a}{2}\right)_k\left(\frac{a+1}{2}\right)_k. \quad (2.19) $$

These identities are very easily proven. For example, the right side of (2.16) is

$$ \frac{f(f+1)\cdots(f+n-1)}{(f+n-1)\cdots(f+n-k)} = f(f+1)\cdots(f+n-k-1) $$
$$ = (f)_{n-k}. \quad (2.20) $$

The relations (2.18) and (2.19) are proven in a similar way.

3. Examples. We now give a few examples to show how sums of products of binomial coefficients can be reduced to hypergeometric series. Moreover, we illustrate that though such sums come in many disguises, they reduce to very few different kinds of hypergeometric series. The reader can best be convinced of this by taking a number of binomial identities from a textbook or the problem section of the MONTHLY and verifying them by the method explained here.

EXAMPLE 1. The following is problem E 3065 which appeared in this MONTHLY, December 1984. One has to evaluate in closed form the sum

$$S = \sum_{j=0}^{n} (-1)^j \frac{\binom{k}{j}\binom{k-1-j}{n-j}}{j+1}. \tag{3.1}$$

We shall reduce it to a hypergeometric series which will show that it is merely the Chu-Vandermonde series in disguise. The first step is to rewrite (3.1) in factorials to get

$$S = \sum_{j=0}^{n} (-1)^j \frac{k!}{j!(k-j)!} \frac{(k-1-j)!}{(n-j)!(k-1-n)!} \frac{1}{j+1}. \tag{3.2}$$

Next we take terms which do not depend on j outside the summation and the rest of the terms we write as shifted factorials $(\alpha)_j$. To do the latter we use (2.17) and arrive at

$$\begin{aligned} S &= \frac{(k-1)!}{(k-1-n)!n!} \sum_{j=0}^{n} \frac{(-k)_j}{j!} \frac{(-n)_j}{(-k+1)_j} \frac{1}{j+1} \\ &= \binom{k-1}{n} \sum_{j=0}^{n} \frac{(-k)_j(-n)_j}{(1)_{j+1}(-k+1)_j}. \end{aligned} \tag{3.3}$$

The last series would be a standard form of a hypergeometric series except for the term $(1)_{j+1}$, so we effect the following changes:

$$S = \binom{k-1}{n} \frac{(-k)}{(n+1)(k+1)} \sum_{j=0}^{n} \frac{(-k-1)_{j+1}(-n-1)_{j+1}}{(1)_{j+1}(-k)_{j+1}}. \tag{3.4}$$

(Note that $(-k)_j = (-k-1)_{j+1}/-(k+1)$ and a similar relation is true for the other terms.)

If we set $j+1 = l$ the sum may be written as

$$\sum_{l=1}^{n+1} \frac{(-k-1)_l(-n-1)_l}{(1)_l(-k)_l} \tag{3.5}$$

and if we add one to this sum we get the hypergeometric series ${}_2F_1\left(\begin{matrix} -n-1, & -k-1 \\ & -k \end{matrix}\right)$.

(Recall that a hypergeometric series begins with one.) Thus,

$$S = \binom{k-1}{n}\frac{k}{(n+1)(k+1)}\left[1 - {}_2F_1\left(\begin{matrix}-n-1, & -k-1\\ & -k\end{matrix}\right)\right]$$

$$= \binom{k-1}{n}\frac{k}{(n+1)(k+1)}\left[1 - \frac{(1)_{n+1}}{(-k)_{n+1}}\right], \quad \text{by (2.7).} \tag{3.6}$$

The last expression is easily simplified to

$$\frac{1}{k+1}\left[\binom{k}{n+1} + (-1)^n\right].$$

The following are three more cases of the Chu-Vandermonde identity as the reader can easily verify:

$$\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}, \quad \text{where } r \text{ is real}, \quad n = \text{integer} \geqslant 0; \tag{3.7}$$

$$\sum_{k}\binom{r}{k}\binom{s}{n+k} = \binom{r+s}{r+n}, \quad n = \text{integer}, \quad r = \text{integer} \geqslant 0; \tag{3.8}$$

$$\sum_{k}(-1)^k\binom{r}{k}\binom{s+k}{n} = (-1)^r\binom{s}{n-r}, \quad n, r \text{ as in (3.8).} \tag{3.9}$$

We now see that the verification of many binomial identities can be reduced to a routine calculation by using hypergeometric series.

EXAMPLE 2. The next example is the most difficult of the worked problems on binomial identities in Knuth [**11**]. The problem is to evaluate

$$S = \sum_{k\geqslant 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}. \tag{3.10}$$

This is only the Pfaff-Saalschütz identity in another form. In showing this we also illustrate the use of the duplication formula (2.19). We write S as

$$\sum_{k\geqslant 0}\frac{(n+k)!(2k)!(-1)^k}{(m+2k)!(n-m-k)!k!k!(k+1)}$$

$$= \frac{n!}{m!(n-m)!}\sum_{k\geqslant 0}\frac{(n+1)_k(-n+m)_k(1)_{2k}}{(m+1)_{2k}(1)_k(1)_{k+1}}, \quad \text{by (2.17).} \tag{3.11}$$

We now rewrite $(1)_{2k}, (m+1)_{2k}$ using (2.19) to get

$$S = \binom{n}{m}\sum_{k\geqslant 0}\frac{(n+1)_k(-n+m)_k(1)_k\left(\frac{1}{2}\right)_k}{\left(\frac{m+1}{2}\right)_k\left(\frac{m}{2}+1\right)_k(1)_k(1)_{k+1}}$$

$$= \binom{n}{m}\frac{\frac{m}{2}\left(\frac{m-1}{2}\right)}{(-1-n+m)n\left(-\frac{1}{2}\right)}\sum_{k\geqslant 0}\frac{(-1-n+m)_{k+1}(n)_{k+1}\left(-\frac{1}{2}\right)_{k+1}}{\left(\frac{m-1}{2}\right)_{k+1}\left(\frac{m}{2}\right)_{k+1}(1)_{k+1}}$$

$$= \binom{n}{m}\frac{m(m-1)}{2n(n+1-m)}\left[{}_3F_2\left(\begin{matrix}-1-n+m,\ n,\ -\frac{1}{2}\\ \frac{m-1}{2},\ \frac{m}{2}\end{matrix}\right)-1\right]$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{\left(\frac{m-1}{2}-n\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}}{\left(\frac{m-1}{2}\right)_{n+1-m}\left(\frac{m}{2}-n\right)_{n+1-m}}-1\right],\quad \text{by (2.8)}$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{\left(\frac{m+1}{2}\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}}{\left(\frac{m-1}{2}\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}}-1\right],\quad \text{by (2.18)}.$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{n+\frac{1}{2}-\frac{m}{2}}{\frac{m-1}{2}}-1\right]$$

$$= \binom{n-1}{m-1}. \tag{3.12}$$

The reader may like to verify that $\sum_{k\geqslant 0}\binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n}$, though somewhat different in appearance from (3.10), also reduces to the Pfaff-Saalschütz series. This sum is also from Knuth [**11**].

I. J. Good [**9**] encountered the following sum in some work in probability theory:

$$\sum_{\nu=0}^{s}(-1)^{\nu}\binom{\beta}{\nu}\binom{\beta+s-\nu}{\beta}\frac{\alpha}{\alpha+s-\nu}. \tag{3.13}$$

He remarks that it is difficult to see directly that this sum is positive when $\alpha > \beta$. However, (3.13) is equal to

$$\frac{(\beta+s)!}{s!\beta!}\frac{\alpha}{\alpha+s}\,{}_3F_2\left(\begin{matrix}-s,\ -\beta,\ -\alpha-s\\ -\beta-s,\ -\alpha-s+1\end{matrix}\right), \tag{3.14}$$

which is again the Pfaff-Saalschütz series (2.8) and the value of the sum (3.13) is

$$\frac{(\alpha-\beta)_s}{(\alpha+1)_s}, \quad \text{which is positive when } \alpha > \beta. \tag{3.15}$$

Now consider the identity,

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k} = \binom{n+k}{k}^2, \tag{3.16}$$

where k and n are nonnegative integers. Takács [**16**] has written a brief history of the various proofs of this identity which have been given in the past fifty years; some of these are quite complicated. However, L. Carlitz pointed out that this identity too is a particular case of the Pfaff-Saalschütz summation (2.8). The reader will not find it difficult to verify this.

The sums above reduced to a balanced ${}_3F_2$. The next example from Riordan [**13**, p. 87] reduces to a 2-balanced ${}_3F_2$. The problem is to evaluate

$$\sum_{k=0}^{n}\frac{2n}{n+k}\binom{n+k}{2k}\binom{2k}{k}(k+p)^{-1}(-1)^k. \tag{3.17}$$

It is easily seen that (3.17) is

$$\frac{2}{p}\,{}_3F_2\left(\begin{matrix}-n,\ n,\ p\\ 1,\ p+1\end{matrix}\right), \quad \text{which can be evaluated by (2.9).} \tag{3.18}$$

EXAMPLE 3. We now consider examples which reduce to Dixon's well-poised ${}_3F_2$ given by (2.15). Problem 62 on page 73 of Knuth [**11**] is to show that

$$\sum_{k=-l}^{l}(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k} = \frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(m+n)!(n+l)!l!m!n!}, \tag{3.19}$$

where we are assuming that $l = \min(l, m, n)$. To write (3.19) as a hypergeometric series, set $j = k + l$ in (3.19) to get

$$(-1)^l\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\binom{2m}{m-l+j}\binom{2n}{n-l+j}. \tag{3.20}$$

By the methods already explained, this reduces to the series

$$\frac{(-1)^l(2m)!(2n)!}{(m-l)!(m+l)!(n-l)!(n+l)!}\,{}_3F_2\left(\begin{matrix}-2l, & -m-l, & -n-l\\ & m-l+1, & n-l+1\end{matrix}\right), \tag{3.21}$$

which is Dixon's well-poised ${}_3F_2$. However, Dixon's result cannot be applied directly since we get the term $\Gamma(1-l)/\Gamma(1-2l)$ on the right, which is undefined. To take care of this difficulty we consider the following case of Dixon's formula:

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-2l-2\varepsilon, & -m-l-\varepsilon, & -n-l-\varepsilon\\ & m-l-\varepsilon+1, & n-l-\varepsilon+1\end{matrix}\right)\\ &\quad=\frac{\Gamma(1-l-\varepsilon)\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon)\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}.\end{aligned} \tag{3.22}$$

We now apply the formula (2.13) to the right side of (3.22) to get

$$\frac{\sin\pi(2l+2\varepsilon)}{\sin\pi(l+\varepsilon)}\cdot\frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)}\cdot\frac{\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}.$$

We let $\varepsilon \to 0$ and obtain (3.19). Riordan [**13**, p. 89] has the following series:

$$\sum_{k=1}^{m} 2k\binom{2p}{k+p}\binom{2n}{k+n}, \tag{3.23}$$

and this too reduces to Dixon's sum.

4. Proofs of the hypergeometric identities. In this section we give inductive proofs of the identities stated in Section 2. There are many other proofs and the reader can consult Askey [**3**] or Bailey [**6**] for these. Askey's analytic proofs best indicate why these identities hold.

We begin by proving the Pfaff-Saalschütz theorem. This proof is due to Dougall [**8**]. We first observe the symmetry in a, b and n by rewriting (2.8) as

$${}_3F_2\left(\begin{matrix}-a, & -b, & -n\\ & c, & d\end{matrix}\right)=\frac{\Gamma(a+c+n)\Gamma(b+c+n)\Gamma(a+b+c)\Gamma(c)}{\Gamma(c+a)\Gamma(b+c)\Gamma(c+n)\Gamma(a+b+c+n)}, \tag{4.1}$$

where we have used property (2.11) of the gamma function. Now since $d = 1 - a - b - n - c$, to prove (2.8) we must show that

$$(c)_n(c+a+b)_n\sum_{j=0}^{n}\frac{(-n)_j(-a)_j(-b)_j}{j!(c)_j(1-a-b-n-c)_j}=(c+a)_n(c+b)_n. \tag{4.2}$$

Note that it follows from (2.16) that both sides of (4.2) are polynomials in b of degree n. Therefore, it is sufficient to prove that they are equal for $n+1$ distinct values of b. Clearly the result is true for $n = 0$. Assume the result true for $n = 0, 1, \dots, k-1$. Now set $n = k$. By symmetry in b and n, it follows from the

inductive hypothesis that (4.2) is true for $b = 0, 1, \ldots, k-1$. If we can find one more value of b for which the relation is true, we would be done. Observe that (2.16) implies

$$\frac{(c+a+b)_n}{(1-a-b-n-c)_j} = (-1)^j (c+a+b)_{n-j}. \tag{4.3}$$

Thus, when $b = -a-c$, both sides of (4.2) are equal to $(c+a)_k(-a)_k$ and the result is proved.

We can derive Chu-Vandermonde from (4.1). Let $a = m$, an integer, let $n \to \infty$ and use Stirling's formula (2.14) together with (2.11) to get

$${}_2F_1\left(\begin{matrix}-m, & -b \\ & c\end{matrix}\right) = \frac{(c+b)_m}{(c)_m}. \tag{4.4}$$

An inductive argument can be used to prove the Sheppard-Andersen identity as well. The identity of Dixon lies a little deeper. Dougall showed that a much more general identity could be proved by the method used to prove (4.1). The reader may reconstruct the proof himself or consult Dougall's paper. The argument is also reproduced in Bailey **[6]** and Hardy **[10]**. The identity gives the sum of a very well-poised 2-balanced ${}_7F_6$.

$${}_7F_6\left(\begin{matrix}a, \ 1+\frac{1}{2}a, & -b, \ -c, \ -d, \ -e, \ -n \\ \frac{1}{2}a; & 1+a+b, \ 1+a+c, \ 1+a+d, \ 1+a+e, \ 1+a+n\end{matrix}\right)$$

$$= \frac{(1+a)_n(1+a+b+c)_n(1+a+b+d)_n(1+a+c+d)_n}{(1+a+b)_n(1+a+c)_n(1+a+d)_n(1+a+b+c+d)_n}, \tag{4.5}$$

where $1 + 2a + b + c + d + e + n = 0$ and n is a positive integer. The last relation simply means that the series is 2-balanced. The adjective "very" in very well-poised refers to the factor

$$\frac{\left(\frac{a}{2}+1\right)_k}{\left(\frac{a}{2}\right)_k} = \frac{a+2k}{a}$$

in the series. To derive Dixon's identity we merely set $d = -\frac{1}{2}a$ in (4.5), let $n \to \infty$, apply Stirling's formula (2.14), and use (2.11).

Acknowledgement. I would like to thank Paul Campbell for suggesting that I write this paper and Richard Askey for reading various versions of the paper and for his helpful comments and suggestions.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics.*

A Conjecture Related to Chi-Bar-Squared Distributions

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Consider a polyhedral, convex, closed cone C in $\mathbb{R}^n$. We write $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ and $\|x\| = \langle x, x\rangle^{1/2}$. Let $P:\mathbb{R}^n \to C$ be the projection mapping onto C, which is defined as assigning to each $x \in \mathbb{R}^n$ the closest point in C, that is

$$\|x - P(x)\| = \min_{y\in C}\|x - y\|.$$

Let $X = (X_1, \dots, X_n)$ be a multivariate random variable having the standard

normal distribution, i.e., $X \sim \mathcal{N}(0, \mathrm{I})$. Define numbers $w_i(C)$, $i = 0, 1, \ldots, n$, to be the probabilities that $P(X)$ belongs to a face of C of dimension i. If C has no i-dimensional faces, then by definition, $w_i(C)$ is zero.

CONJECTURE. *For every polyhedral, convex, closed cone C, which is not a linear subspace of $\mathbb{R}^n$, one has*

$$\sum_{i=0}^{n} (-1)^i w_i(C) = 0. \tag{1}$$

The numbers $w_i(C)$ are related to the random variable $\bar{\chi}^2$ representing the squared distance to C,

$$\bar{\chi}^2 = \|X - P(X)\|^2.$$

It can be shown that $\bar{\chi}^2$ has the distribution which is a mixture of chi-squared distributions with weights $w_i(C)$ (see [3], [4], [5], [7]). Consequently the problem of finding this distribution (called the chi-bar-squared distribution) is reduced to evaluation of weights $w_i(C)$. Unfortunately it is not easy to calculate numbers $w_i(C)$ for an arbitrary cone C, although in some simple cases the solution is known in a closed form (e.g. [**1**, pp. 134–148], [3], [7]). Of course, established identities between $w_i(C)$ may facilitate the problem.

There is extensive theoretical and numerical evidence supporting the Conjecture (cf. [**1**, p. 174], [**2**], [**6**], [7]). For instance, if C is the nonnegative orthant $\mathbb{R}^n_+ = \{x : x_i \geq 0\}$, then $w_i(C) = 2^{-n}\binom{n}{i}$ and (1) follows. For $n = 2$ it can be easily verified that $w_1(C) = 1/2$, which implies (1). Already for $n = 3$ the identity (1) in all its generality is not trivial. In $\mathbb{R}^3$ the weights have a simple geometric interpretation ([7], [**8**]) and this leads to a proof in this case. Namely, the weights $w_0(C)$, $w_1(C)$, $w_2(C)$ $w_3(C)$ are given by $\beta(C^0)/4\pi$, $\alpha(C^0)/4\pi$, $\alpha(C)/4\pi$, $\beta(C)/4\pi$, where $\beta(C)$ denotes the solid angle of C, $\alpha(C)$ denotes the plane angle corresponding to the surface of C, and C^0 denotes the dual (polar) cone of C,

$$C^0 = \{y \in \mathbb{R}^n : \langle y, x \rangle \leq 0, \forall x \in C\}.$$

Consequently (1) becomes

$$\alpha(C^0) + \beta(C) = 2\pi. \tag{2}$$

By passing to a limit, the weights $w_i(C)$ can be associated with any convex (not necessarily polyhedral) cone C [**7**]. It will be of certain interest if one can extend the geometric interpretation above ($n = 3$) to higher dimensions. Hopefully this will lead to a generalization of the geometrical identity (2).

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NOTES

EDITED BY CAROL G. CRAWFORD, RICHARD LIBERA, AND ANITA E. SOLOW

Curvature, Circles, and Conformal Maps

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We know that Möbius transformations, namely maps of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad ad-bc \neq 0,$$

map circles to circles (where straight lines are considered to be circles through ∞) and, conversely, that any circle preserving meromorphic map of the extended complex plane onto itself is Möbius ([**2**], p. 90). In this note we recall the effect of a conformal map on the curvature of a plane curve and, with very little effort, obtain two particularly strong *local* versions of the converse. Although these results have not, as far as I know, been noted before, the real purpose of this note is to provide an attractive illustration of the use of the Schwarzian derivative and to suggest further problems.

To state our results, consider a function f meromorphic in a domain D in the extended complex plane $\mathbb{C}_\infty$ and for each $n = 1, 2, \ldots, +\infty$, let A_n be the set of z in D with the property that there are at least n circular arcs through z (in different directions) that are mapped to circular arcs by f. With this notation, we have the following theorem:

THEOREM. *If either*
(i) *A_3 has a point of accumulation in D, or*
(ii) *A_∞ has at least three points, then f is Möbius and (of course) $A_3 = A_\infty = \mathbb{C}_\infty$. This is sharp in the sense that*
(iii) *there exists a non-Möbius f with $A_2 = D$, and*
(iv) *there exists a non-Möbius f with A_∞ containing exactly two points.*

Despite (iii) and (iv), it will be apparent from the proof that we obtain slightly stronger results than both (i) and (ii).

Proof. The two examples required to establish (iii) and (iv) are elementary. The function log defined on the upper half-plane D has $A_2 = D$ as all of the curves $|z| = a$ and $\arg(z) = b$ are mapped to straight lines. The function $z \mapsto z^n$ maps every straight line through the origin to another such, so, in this case, $A_\infty = \{0, \infty\}$.

We now establish (i) and (ii). The curvature of the curve $t \mapsto r(t)$ in $\mathbb{R}^3$ at the point $r(t)$ is given by

$$k(t) = \frac{|\dot{r}(t) \times \ddot{r}(t)|}{|\dot{r}(t)|^3}$$

where $\dot{r}(t)$ is the derivative of r with respect to t. A plane curve

$$t \mapsto z(t) = x(t) + iy(t)$$

can be identified with the curve $r(t) = (x(t), y(t), 0)$ in $\mathbb{R}^3$ and as

$$\operatorname{Im}\left[\frac{\ddot{z}}{\dot{z}}\right] = |\dot{z}|^{-2} \operatorname{Im}[\ddot{z}\bar{\dot{z}}],$$

a direct computation shows that we can obtain the complex form for $k(t)$, namely

$$k(t) = \frac{1}{|\dot{z}|} \operatorname{Im}\left[\frac{\ddot{z}}{\dot{z}}\right].$$

We observe, in passing, that if $z(t)$ is the restriction of a Möbius transformation z to the real line (containing t), then an easy computation (using the fact that t is real) gives $k(t)$ constant. This shows that $t \mapsto z(t)$ maps the real line $\mathbb{R}$ to a circle and, moreover, it gives the radius of the circle explicitly. From this it is easy to see that any circle maps to a circle; thus it becomes clear from curvature considerations that Möbius maps take circles to circles.

As circles are curves of constant curvature, it is advantageous to compute the derivative $\dot{k}(t)$ of the curvature $k(t)$ and, as we shall see, the resulting formula is best expressed in terms of the Schwarzian derivative. For any function F, the Schwarzian derivative of F at a point z is

$$S_F(z) = \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2$$

(where primes denote differentiation) and it is known (and very easy to prove) that S_F is zero in a domain precisely when F is Möbius. It is this fact that will ultimately enable us to identify f from purely local considerations.

To obtain a formula for $k(t)$, observe that

$$\dot{k}(t) = \frac{1}{|\dot{z}|}\mathrm{Im}\left[\frac{d}{dt}\left(\frac{\ddot{z}}{\dot{z}}\right)\right] + \mathrm{Im}\left[\frac{\ddot{z}}{\dot{z}}\right]\frac{d}{dt}\left(\frac{1}{|\dot{z}|}\right)$$

$$= \frac{1}{|\dot{z}|}\mathrm{Im}[S_z(t)] + \mathrm{Im}\left[\frac{\ddot{z}}{\dot{z}}\right]\left\{\frac{1}{|\dot{z}|}\mathrm{Re}\left[\frac{\ddot{z}}{\dot{z}}\right] + \frac{d}{dt}\left(\frac{1}{|\dot{z}|}\right)\right\},$$

since for any complex number w (in our case, $w = \ddot{z}/\dot{z}$) we have

$$\mathrm{Im}[w^2] = 2\,\mathrm{Im}[w]\,\mathrm{Re}[w].$$

The last term in the expression for $k(t)$ is zero because (writing $z = x + iy$) we have

$$2|\dot{z}|^2\mathrm{Re}\left[\frac{\ddot{z}}{\dot{z}}\right] = 2\,\mathrm{Re}[\bar{\dot{z}}\ddot{z}] = \frac{d}{dz}(|\dot{z}|^2)$$

and

$$\frac{d}{dt}\left(\frac{1}{|\dot{z}|}\right) = \frac{d}{dt}(|\dot{z}|^2)^{-1/2} = \frac{-1}{2|\dot{z}|^3}\frac{d}{dt}(|\dot{z}|^2).$$

Thus we obtain the formula for $\dot{k}(t)$, namely

$$\dot{k}(t) = \mathrm{Im}[S_z(t)]/|\dot{z}|, \tag{1}$$

a formula which apparently was known to G. Pick (see [**1**]).

The Chain Rule for the Schwarzian derivative is

$$S_{f\circ z}(t) = S_f(z(t))[\dot{z}(t)]^2 + S_z(t);$$

so writing K for the curvature of the image curve $f(z(t))$, we have

$$|f'(z)|\dot{K}(t) = \dot{k}(t) + |\dot{z}(t)|\,\mathrm{Im}\left[\left(\frac{\dot{z}}{|\dot{z}|}\right)^2 S_f(z(t))\right].$$

If an arc σ of a circle is mapped by f to another such arc, then both curvature derivatives vanish on σ, and we obtain

$$\mathrm{Im}\left[e^{2i\phi}S_f(z)\right] = 0 \tag{2}$$

for each z on σ, where ϕ is the argument of the tangent vector to σ at z. It is now a matter of translating this information about the Schwarzian derivative into the required form.

First, assume that (i) holds. As S_f is meromorphic and not always ∞, the points in A_3 at which S_f is finite must also accumulate in D. If z is in A_3, then (2) holds for at least three different choices of ϕ, and hence for at least two different choices of $e^{2i\phi}$, neither a real multiple of the other. Assume that $S_f(z)$ is finite; this then implies that $S_f(z) = 0$. Thus the Schwarzian is a meromorphic function with a

non-isolated zero and we deduce that it is identically zero. Thus if (i) holds, then f is Möbius.

Observe that we have derived a stronger result than (i): we can replace points in A_3 in our proof of (i) by points in A_2 provided that the two arcs in question are not orthogonal. This observation is due to D. Barden. Of course, the example log used to verify (iii) has two such orthogonal arcs at every point.

Before proceeding to the proof of (ii), we pause to prove a preliminary result which obviously is related to the ideas discussed here.

LEMMA 1. *Let $g(z) = \Sigma a_n z^n$ be holomorphic and not constant near $z = 0$, and let L_1 and L_2 be two segments crossing at a positive angle θ at the origin. If g maps L_1 and L_2 into the real axis, then $a_n = 0$ for $1 \leqslant n < \pi/\theta$.*

Proof. We may write

$$g(z) = a_0 + a_q z^q + O(z^{q+1}) \quad \text{as } z \to 0,$$

where $a_q \neq 0$. This implies that g maps two curves meeting at an angle θ at $z = 0$ into two curves meeting at an angle $q\theta$ at $g(0)$. With the given hypotheses, we may deduce that $q\theta = n\pi$ for some integer n, so $q \geqslant \pi/\theta$.

We now return to the proof of (ii). First, we may replace f by some composition $h \circ f \circ g$ where g and h are Möbius, for if this composition is Möbius, then so is f. Thus we may assume that the origin is in A_∞, and that near $z = 0$ we have (for some p)

$$f(z) = z^p + O(z^{p+1}) \quad \text{as } z \to 0. \tag{3}$$

Now consider any two circular arcs C_1, C_2 (whose f-images are also circular arcs) meeting at a positive angle θ at the origin and (when extended) at, say z_0. Define

$$g(z) = \frac{zz_0}{z + z_0},$$

so g is Möbius, $g(0) = 0$, $g(\infty) = z_0$, and

$$g'(z) = [g(z)/z]^2. \tag{4}$$

The circular arcs

$$C_1^* = g^{-1}(C_1), \quad C_2^* = g^{-1}(C_2),$$

meet at the origin at an angle θ and, moreover, these are straight line segments as both (when extended) contain $g^{-1}(z_0)$. In addition, the composite map

$$F = f \circ g$$

maps these segments into circular arcs. Now as C_1^* contains $z = te^{i\phi}$ for some ϕ and for all sufficiently small real t, we may apply (2) to F and obtain

$$\operatorname{Im}\left[e^{2i\phi} S_F(z)\right] = 0.$$

As t is real, this is equivalent to

$$\operatorname{Im}\left[z^2 S_F(z)\right] = 0 \qquad (z = te^{i\phi})$$

when z is on C_1^*. The same holds on C_2^*, so the map

$$z \mapsto z^2 S_F(z)$$

maps both segments C_1^*, C_2^* into the real axis. A simple calculation shows that as F has a zero of order p at $z = 0$ (see (3)), $z^2 S_F(z)$ is holomorphic with value $(1 - p^2)/2$ there. Applying Lemma 1, we deduce that

$$z^2 S_F(z) = (1 - p^2)/2 + O(z^q) \tag{5}$$

near $z = 0$ where $q \geqslant \pi/\theta$. As q is an integer and $\theta < \pi$, we see that $q \geqslant 2$. We wish to take θ to be small (and hence q large); however, as F (but not f) depends on z_0 (which depends implicitly on θ), we must first recast this in terms of f. As g is Möbius, the Chain Rule for Schwarzian derivatives yields

$$S_F(z) = S_f(gz)[g'(z)]^2,$$

hence

$$S_f(gz)[zg'(z)]^2 = (1 - p^2)/2 + O(z^q).$$

Using (4) and writing $\zeta = g(z)$, we obtain

$$\zeta^4 S_f(\zeta) = (1 - p^2)z^2/2 + O(z^{q+2}),$$

so

$$\zeta^2 S_f(\zeta) = \frac{(1 - p^2)}{2}\left(\frac{z_0}{z_0 - \zeta}\right)^2 + O(\zeta^q). \tag{6}$$

Of course, the function

$$w \mapsto w^2 S_f(w)$$

is completely determined by f and is independent of the choice of C_1 and C_2. However, our assumptions on C_1 and C_2 have led to the conclusion that this function satisfies (6), namely

$$\zeta^2 S_f(\zeta) = (1 - p^2)\left[\frac{1}{2} + \frac{\zeta}{z_0} + \cdots\right] + O(\zeta^q),$$

where $q \geqslant 2$.

One possibility is that $p = 1$. In this case,

$$\zeta^2 S_f(\zeta) = O(\zeta^q), \quad q \geqslant \pi/\theta,$$

and, since θ can be chosen arbitrarily small (because the origin is in A_∞), we find that S_f is identically zero and so f is Möbius.

The remaining possibility is that $p \geqslant 2$. In this case the coefficient of ζ, namely $(1-p^2)/z_0$, must be independent of the choice of C_1 and C_2. It follows that all circular arcs arising out of the fact that the origin is in A_∞ must (when extended) pass through the same two points, namely 0 and z_0. This in turn means that g, and hence F, is independent of the choice of C_1 and C_2. Returning to (5), we may now choose θ to be arbitrarily small and conclude that

$$S_F(z) = (1-p^2)/2z^2. \tag{7}$$

It is well known that S_F determines F up to a Möbius transformation: explicitly ([**3**], p. 376), the general solution of (7) is $F = u_1/u_2$, where u_1 and u_2 are two linearly independent solutions of the linear differential equation

$$u^{(2)} + \left[(1-p^2)/4z^2\right]u = 0.$$

Two linearly independent solutions are

$$z^{(1+p)/2}, \quad z^{(1-p)/2},$$

so

$$\begin{aligned} F(z) &= \frac{az^{(1+p)/2} + bz^{(1-p)/2}}{cz^{(1+p)/2} + dz^{(1-p)/2}} \\ &= \frac{az^p + b}{cz^p + d}, \qquad ad - bc \neq 0. \end{aligned}$$

In fact, as F has a zero of order p at the origin we see that $b = 0$.

Let us say that two functions f_1 and f_2 are *Möbius equivalent* if $f_2 = h \circ f_1 \circ g$ for some Möbius transformations g and h. Then F and f are Möbius equivalent and we have, in fact, proved the following result.

THEOREM. *If $A_\infty \neq \varnothing$, then either f is Möbius (and $A_\infty = \mathbb{C}_\infty$) or f is Möbius equivalent to some map $z \mapsto z^p (p \geqslant 2)$ and A_∞ has precisely two points.*

The results do not explicitly involve the Schwarzian derivative and one may ask to what extent some similar statements are true for smooth transformations acting on $\mathbb{R}^n$. Ahlfors [**1**] has initiated a discussion of the Schwarzian derivative for maps of $\mathbb{R}^n$ to itself and this may be helpful here.

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Counting the Subgroups of Some Finite Groups

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How many subsets does a finite set have? The answer, of course, is 2^n, where n is the size of the set. We cannot expect such an elementary answer for the number of subgroups of a finite group. In the first place, the order of a group does not determine the number of its subgroups. Still, within many familiar classes of groups, we can specify a group G in terms of certain parameters. In this case we might ask whether there is a simple way to express the number of subgroups of G as a function of the parameters.

For example, a finite cyclic group, Z_m, is determined (up to isomorphism) by its order. The number of subgroups is given by $d(m)$, the number of divisors of m. A nonabelian example is provided by the dihedral groups. The dihedral group D_m $(m > 3)$ is also determined by its order $(2m)$. The number of subgroups is $d(m) + \sigma(m)$, where $\sigma(m)$ is the sum of the divisors of m [1]. In this note we generalize these results to the class of groups which can be formed as cyclic extensions of cyclic groups.

Such a group, G, is of order mn and is generated by a and b where

$$a^m = b^n = e, \quad bab^{-1} = a^r, \quad \text{and} \quad r^n \equiv 1 \pmod{m}.$$

(See [2, pp. 9–111.) This class of groups includes the dihedral groups, the Z-metacyclic groups (i.e., metacyclic in the sense of Zassenhaus [3, pp. 144–145]), and the abelian groups of the form $Z_m \times Z_n$.

In the following theorem $G(m, n, r)$ denotes the cyclic extension of a cyclic group defined by the relations given above, and (x, y) denotes the greatest common divisor of integers x and y.

THEOREM. *For any integers m, n, and r such that $m \geq 2$, $n \geq 2$, $|r| \neq 1$, and $r^n \equiv 1 \pmod{m}$, the number of subgroups of $G(m, n, r)$ is*

$$\sum_{q|m} \sum_{t|n} \left(q, \frac{r^n - 1}{r^t - 1}\right).$$

Note that $(r^n - 1)/(r^t - 1)$ is an integer. Also, although the theorem excludes the cases $r = \pm 1$, $G(m, n, \pm 1)$ is equivalent to $G(m, n, m \pm 1)$, so that the formula may be applied in these cases as well. Furthermore, the formula may be simplified in certain special cases.

EXAMPLE 1. The dihedral group D_m may be defined by

$$a^m = b^2 = e, \quad bab^{-1} = a^{-1} = a^{m-1},$$

so the number of subgroups of D_m is

$$\sum_{q|m} \left(\left(q, \frac{(m-1)^2 - 1}{(m-1) - 1}\right) + 1 \right) = \sum_{q|m} (q + 1) = d(m) + \sigma(m).$$

EXAMPLE 2. If $r = 1$, or equivalently $r = m + 1$, then $G(m, n, m + 1) = Z_m \times Z_n$. In particular, if $(m, n) = 1$, $G(m, n, m + 1) = Z_{mn}$. In the latter case the formula becomes

$$\sum_{q|m}\sum_{t|n}\left(q, \frac{(m+1)^n - 1}{(m+1)^t - 1}\right) = \sum_{q|m}\sum_{t|n} 1 = d(mn)$$

as is well known.

EXAMPLE 3. For any prime p, $G(p, p, 1) = Z_p \times Z_p$ is a 2-dimensional vector space over the field Z_p. Additive subgroups of this vector space are automatically closed under scalar multiplication, so the number of subgroups of $G(p, p, 1)$ is the number of subspaces of the vector space $Z_p \times Z_p$, which is well known to be $p + 3$. It is easy to check that this is the number given by the formula for $G(p, p, p + 1)$.

EXAMPLE 4. Let p be a prime and $N \geqslant 3$. The relations

$$a^{p^{N-1}} = b^p = e, \quad bab^{-1} = a^{1+p^{N-2}}$$

define a nonabelian group of order p^N with a large normal subgroup, $\langle a \rangle$. We leave it to the reader to check that the number of subgroups of $G(p^{N-1}, p, 1 + p^{N-2})$ is 10 if $p = 2$ and $N = 3$ and $(p + 1)N - (p - 1)$ otherwise. It is interesting to note that the abelian groups $G(p^{N-1}, p, 1 + p^{N-1})$, which are also of order p^N, have $(p + 1)N - (p - 1)$ subgroups for any choice of p and N, so that in all but the smallest case where $p = 2$ and $N = 3$ the number of subgroups of the nonabelian group is the same as that of the abelian group of the same order.

We now turn to the proof of the theorem. We simplify the notation by writing G for $G(m, n, r)$. Elements of G are denoted by small letters a, b, g and h. For convenience we define

$$g(x, y) = a^x b^y \in G,$$

for integers x, y and, if $y|n$,

$$R(y) = (r^n - 1)/(r^y - 1) \in Z.$$

For $g \in G$, $\langle g \rangle$ is the subgroup generated by g. For $k \in Z$, A_k will mean $\langle a^k \rangle$. When $k = 1$ we will usually drop the subscript so that $A = A_1 = \langle a \rangle$. Also, $A_k g$ represents a coset of A_k and $\langle A_k g \rangle$ is the subgroup of G/A_k generated by $A_k g$.

We will use three facts whose elementary proofs are left to the reader.

FACT 1. *For each $g \in G$ there is a unique pair $x, y \in Z$, such that* $0 \leqslant x \leqslant m - 1$, $0 \leqslant y \leqslant n - 1$, *and* $g = g(x, y)$.

FACT 2. *For any integers* x_1, x_2, y_1, y_2, $y_1 \geqslant 0$,

$$g(x_1, y_1)g(x_2, y_2) = g(x_1 + x_2 r^{y_1}, y_1 + y_2).$$

FACT 3. *A_k is normal in G for all $k \in Z$.*

We will also use two lemmas.

LEMMA 1. *For any integers x, y, $y \geqslant 0$,*

$$g(x, y)^k \in Ab^{ky}.$$

Proof. By Fact 2

$$g(x, y)^k = g\big(x(1 + r^y + r^{2y} + \cdots + r^{(k-1)y}), ky\big).$$

LEMMA 2. *If $y|n$ then*

$$g(x, y)^{n/y} = g(xR(y), 0).$$

Proof. From Fact 2 we see that

$$\begin{aligned} g(x, y)^{n/y} &= g\big(x(1 + r^y + r^{2y} + \cdots + r^{((n/y)-1)y}), n\big) \\ &= g\left(x\left(\frac{(r^y)^{n/y} - 1}{r^y - 1}\right), 0\right) = g(xR(y), 0). \end{aligned}$$

The proof of the theorem proceeds by establishing a bijection between a certain set T of triples of integers and the set S of subgroups of G. Then counting the elements of T yields the number of subgroups of G. More specifically we will uniquely identify each subgroup H of G by its intersection with A, A_q, and by a coset of A_q, $A_q g(s, t)$, which generates H under the operation of G/A_q. Equivalently we identify H with the triple of integers (q, t, s).

Proof of the Theorem. Let T be the set of triples (q, t, s) of nonnegative integers where $q|m$, $t|n$, and s is such that $0 \leqslant s < q$ and $q|sR(t)$. Let S be the set of subgroups of G.

CLAIM. The mapping $\phi: T \to S$ is a bijection where

$$\phi(q, t, s) = \bigcup_{k=1}^{n/t} \big(A_q g(s, t)\big)^k,$$

that is, $\phi(q, t, s)$ is the union of the cosets of A_q obtained by taking the first n/t powers of $A_q g(s, t)$ in the group G/A_q.

We will prove the claim by proving

(1) ϕ maps T into S, i.e., that $\phi(q, t, s)$ is a subgroup of G.

(2) ϕ is 1-1.

(3) ϕ is onto.

We can then use ϕ to establish the formula. Since ϕ is a bijection, $|T| = |S|$, so we may count the subgroups of G by counting the triples in T. For any given positive integers q and t where $q|m$ and $t|n$, $(q, t, s) \in T$ if and only if s is a multiple of $q/(q, R(t))$ and $0 \leqslant s < q$. There are $(q, R(t))$ such multiples. Therefore,

$$|S| = |T| = \sum_{q|m} \sum_{t|n} (q, R(t)),$$

and so we only need to prove (1), (2), and (3) to complete the proof.

Proof of (1). Observe that

$$\left(A_q g(s,t)\right)^{n/t} = A_q g(s,t)^{n/t} = A_q g(sR(t),0) \qquad \text{(Lemma 2)}.$$

Thus, since $q|sR(t)$, we have

$$\left(A_q g(s,t)\right)^{n/t} = A_q.$$

Hence $\{(A_q g(s,t))^k\colon 1 < k < n/t\}$ is the subgroup $\langle A_q g(s,t)\rangle$ of G/A_q and is therefore closed under the operation of G/A_q. Thus

$$\phi(q,t,s) = \bigcup_{k=1}^{n/t} \left(A_q g(s,t)\right)^k$$

is a subgroup of G.

Proof of (2). To prove that ϕ is 1-1, suppose $\phi(q,t,s) = \phi(q',t',s') = H$, where both triples are in T. This means

$$H = \bigcup_{k=1}^{n/t} \left(A_q g(s,t)\right)^k = \bigcup_{j=1}^{n/t'} \left(A_{q'} g(s',t')\right)^j. \qquad (\#)$$

We will show that $(q,t,s) = (q',t',s')$.

We show first that $q = q'$. From Lemma 1 and Fact 1, if $0 < k < n/t$ and $0 < j < n/t'$, then

$$\left(A_q g(s,t)\right)^k \cap A = \varnothing = \left(A_{q'} g(s',t')\right)^j \cap A.$$

Furthermore, as we saw above

$$\left(A_q g(s,t)\right)^{n/t} = A_q, \quad \text{and} \quad \left(A_{q'} g(s',t')\right)^{n/t'} = A_{q'}.$$

From this we may conclude $A_q = A \cap H = A_{q'}$, which implies $q = q'$.

Now we wish to show that $t = t'$. Suppose $t < t'$. Then since $t'|n$, there is no $j \in Z$ such that $jt' \equiv t \pmod n$. Therefore, by Lemma 1,

$$g(s,t) \notin \bigcup_{j=1}^{n/t'} \left(A_{q'} g(s',t')\right)^j.$$

This contradicts (#), so $t \geqslant t'$. By symmetry, $t' \geqslant t$, so $t = t'$.

Finally,

$$g(s,t) \in H \cap Ab^t = \left(\bigcup_{j=1}^{n/t} \left(A_q g(s',t)\right)^j\right) \cap Ab^t = A_q g(s',t),$$

by Lemma 1. It follows that $a^s \in A_q a^{s'}$ which implies $s \equiv s' \pmod q$. But $0 \leqslant s < q$ and $0 \leqslant s' < q$, so $s = s'$. Therefore, $(q,t,s) = (q',t',s')$.

Proof of (3). Given $H \in S$ we produce a $(q,t,s) \in T$ such that $\phi(q,t,s) = H$.

(i) Let $q = m/|H \cap A|$.

(ii) Let t be the least positive integer such that $H \cap Ab^t \neq \varnothing$.

(iii) Let s be the least nonnegative integer such that $H \cap A_q g(s,t) \neq \varnothing$.

We will show that $(q, t, s) \in T$ and $\phi(q, t, s) = H$. We begin by considering q. Since $H \cap A$ is a subgroup of A it follows that $m/|H \cap A|$ is a nonnegative integer and $q|m$. Also $|H \cap A| = |A_q|$ which implies $H \cap A = A_q$.

Now t exists since $e \in H \cap A = H \cap Ab^n$. We must show $t|n$. Suppose $t \nmid n$. Let kt be the first multiple of t greater than n. Then $kt \equiv t' \pmod n$ for some t', $0 < t' < t$. Since $H \cap Ab^t \neq \varnothing$, there is an integer x such that $g(x, t) \in H$. But $g(x, t)^k \in Ab^{kt} = Ab^{t'}$ violates (ii). Hence $t|n$.

Finally, consider s. Note that $\bigcup_{k=0}^{q-1} A_q a^k = A$, since the expression on the left is the union of all cosets of A_q in A. Also $H \cap Ab^t \neq \varnothing$. Therefore,

$$H \cap A_q g(k, t) = H \cap \left(A_q a^k\right) b^t \neq \varnothing$$

for at least one k, $0 \leqslant k < q$. Hence s exists and $0 \leqslant s < q$. We must show $q|sR(t)$. Since $A_q \subseteq H$ and $H \cap A_q g(s, t) \neq \varnothing$, it follows that $A_q g(s, t) \subseteq H$. Hence $g(s, t) \in H$, so $g(s, t)^{n/t} \in H$. By Lemma 2, $g(s, t)^{n/t} = a^{sR(t)} \in A$. So $a^{sR(t)} \in H \cap A = A_q$. The fact that $q|sR(t)$ follows from the definition of A_q. Thus $(q, t, s) \in T$.

Now to show $\phi(q, t, s) = H$. Recall that

$$\phi(q, t, s) = \bigcup_{k=1}^{n/t} \left(A_q g(s, t)\right)^k.$$

We saw above that $A_q g(s, t) \subseteq H$. Therefore,

$$\phi(q, t, s) \subseteq H. \qquad (*)$$

Let $h \in H$. Then $h = g(x, y)$, for some x and y with $0 \leqslant x < m$, $0 \leqslant y < n$.

CASE 1. $y = 0$. Then $h \in A$ so $h \in H \cap A = A_q$. Since

$$\left(A_q g(s, t)\right)^{n/t} = A_q a^{sR(t)} = A_q,$$

we have $A_q \subseteq \phi(q, t, s)$. Hence $h \in \phi(q, t, s)$.

CASE 2. $y \neq 0$. We may further assume that $y \geqslant t$ since $0 < y < t$ contradicts (ii).

We first show that $t|y$. Suppose $t \nmid y$. Then $y = kt + y'$ for some positive integers k, y', with $1 \leqslant y' < t$. Consider

$$g = g(s, t)^{(n/t-k)} g(x, y) \in H.$$

By Lemma 1, g is in $Ab^{n+y'} = Ab^{y'}$, but this contradicts (ii). Hence $t|y$.

Now consider $K = (A_q(a^s b^t))^{y/t}(A_q(a^x b^y))^{-1}$ in G/A_q. Clearly,

$$K = A_q\left(a^s b^t\right)^{y/t} b^{-y} a^{-x}.$$

By Lemma 1, $g(s, t)^{y/t} \in Ab^y$, so $(a^s b^t)^{y/t} b^{-y} a^{-x} \in A$. Hence $K \subseteq A$, and since $K \subseteq H$, we have $K \subseteq H \cap A = A_q$. But K is a coset of A_q, so $K = A_q$. This means

K is the identity of G/A_q, which implies

$$\left(A_q(a^s b^t)\right)^{y/t} = A_q a^x b^y.$$

Hence $a^x b^y \in (A_q(a^s b^t))^{y/t}$, so $g(x, y) \in \phi(q, t, s)$. This completes Case 2.

Case 1 and Case 2 show that $H \subseteq \phi(q, t, s)$, and together with $(*)$ this means $H = \phi(q, t, s)$. Therefore, ϕ is onto.

Acknowledgement. The author would like to thank Professor Steven P. Galovich of Carleton College for suggesting the problem, and for discussions which were helpful in obtaining and presenting the result.

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The Diophantine Equation $X^2 + 7 = 2^n$

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Ramanujan conjectured in 1913 that the Diophantine equation $X^2 + 7 = 2^n$ has solutions for exactly five values of n, namely $n = 3, 4, 5, 7$, and 15. This was proved by Nagell [7] in 1948 and by others, using several different proofs, in the late 1950's and early 1960's. A detailed history of this problem and some of its generalizations can be found in Cohen [**2**], [**3**]. There are interesting applications of this result to coding theory by Shapiro and Slotnick [**8**] and to differential algebra by Mead [**5**].

The proof of the Ramanujan conjecture provides a good application of unique factorization in the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$. In the early papers, unique factorization was used to reduce the problem to a question about a Fibonacci-type integer sequence. This question was then settled using p-adic methods. A somewhat different approach, which still depends upon unique factorization but then uses congruences with algebraic numbers, appears in the textbooks by Mordell [**6**] and Stewart and Tall [**9**]. We present here an elementary proof of the Ramanujan conjecture based upon some ideas of Beukers [**1**]. Unique factorization in $\mathbb{Q}(\sqrt{-7})$ plays its usual role, but the critical result on the related integer sequence is obtained by purely arithmetic means.

If $X^2 + 7 = 2^n$, then clearly $n \geqslant 3$ and X is odd. In $\mathbb{Q}(\sqrt{-7})$, the left-hand side factors as $(X + \sqrt{-7})(X - \sqrt{-7})$ and, if $X \in \mathbb{Z}$, both factors are algebraic integers. If R denotes the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{-7})$, then $R = \{a + b\omega | a, b \in \mathbb{Z}\}$, where $\omega = (1 + \sqrt{-7})/2$, and it is known that R has unique factorization. Note that $\omega + \bar{\omega} = 1$ and $\omega\bar{\omega} = 2$, from which it follows that

$\omega^2 = \omega - 2$. The equation $X^2 + 7 = 2^n$ now factors in R more precisely as

$$(Y + \omega)(Y + \bar{\omega}) = \omega^{n-2}\bar{\omega}^{n-2}, \tag{1}$$

where $Y = (X - 1)/2$, $Y \in \mathbb{Z}$. Since $\omega\bar{\omega} = 2$, and neither ω nor $\bar{\omega}$ is a unit of R, both are irreducible in R. Also, $2 = \omega\bar{\omega}$ does not divide either $Y + \omega$ or $Y + \bar{\omega} = Y + 1 - \omega$ in R. Thus the decomposition of $Y + \omega$ into irreducibles in R must be, up to a unit, either a power of ω or a power of $\bar{\omega}$. Whichever it is, the other factor, $Y + \bar{\omega}$, is the same power of the conjugate. Thus the power in question must be exactly $n - 2$. Since the only units of R are ± 1, it follows that

$$\omega^{n-2} = \pm(Y + \omega) \quad \text{or} \quad \omega^{n-2} = \pm(Y + \bar{\omega}) = \pm(Y + 1 - \omega). \tag{2}$$

Thus the integer ω^{n-2}, when expressed in the form $a + b\omega$ for $a, b \in \mathbb{Z}$, must have $b = \pm 1$. Conversely, if $\omega^{n-2} = a \pm \omega$ for $a \in \mathbb{Z}$, then multiplying this equation by its conjugate, one obtains $(2a \pm 1)^2 + 7 = 2^n$ and the equation of the title has a solution.

The problem, then, is to determine exactly those powers of ω that can be expressed in the form $a \pm \omega$, $a \in \mathbb{Z}$. If we write $\omega^n = a_n + b_n\omega$ for $a_n, b_n \in \mathbb{Z}$, then from $\omega^2 = \omega - 2$, we have $a_{n+1} = -2b_n$ and $b_{n+1} = a_n + b_n$, which together imply that $b_{n+2} = b_{n+1} - 2b_n$. Thus the sequence of rational integers b_n is completely determined by this binary linear recurrence and the initial values $b_1 = b_2 = 1$. The sequence begins:

$$1, 1, -1, -3, -1, 5, 7, -3, -17, -11, 23, 45, -1, -91, -89, \ldots. \tag{3}$$

Since $b_1 = b_2 = 1$ and $b_3 = b_5 = b_{13} = -1$, we are provided with the five solutions to the equation $X^2 + 7 = 2^n$ noted by Ramanujan. The problem is to prove that there are no further occurrences of ± 1 in the sequence (3).

Most proofs of the Ramanujan conjecture are identical to this point, but there have been several proofs given for the following theorem. The proof below is relatively simple and straightforward.

THEOREM. *Let the sequence of rational integers b_n be defined by the equations: $b_1 = b_2 = 1$, $b_{n+2} = b_{n+1} - 2b_n$. Then $b_n = \pm 1$ only for $n = 1, 2, 3, 5$, and 13.*

Proof. We first establish an important identity for the sequence b_n. Note that $\omega = 1 - \bar{\omega} = b_2 - b_1\bar{\omega}$ and, by induction, we have

$$\omega^n = b_{n+1} - b_n\bar{\omega}. \tag{4}$$

Taking conjugates and subtracting, we obtain

$$b_n = \frac{\omega^n - \bar{\omega}^n}{\omega - \bar{\omega}}. \tag{5}$$

We also have the binomial expansion

$$\omega^{nk} = b_{n+1}^k + \sum_{j=1}^{k} (-1)^j \binom{k}{j} b_{n+1}^{k-j} b_n^j \bar{\omega}^j, \qquad k \geqslant 1, \tag{6}$$

from which we obtain

$$\omega^{nk+1} = b_{n+1}^k \omega - 2kb_{n+1}^{k-1} b_n + 2b_n^2 \sum_{j=2}^{k} (-1)^j \binom{k}{j} b_{n+1}^{k-j} b_n^{j-2} \bar{\omega}^{j-1}, \qquad k \geqslant 2. \quad (7)$$

Taking the difference between equation (7) and its conjugate, we obtain from (5) the integer identity

$$b_{nk+1} = b_{n+1}^k - 2b_n^2 \sum_{j=2}^{k} (-1)^j \binom{k}{j} b_{n+1}^{k-j} b_n^{j-2} b_{j-1}, \qquad k \geqslant 2. \quad (8)$$

We next consider the sequence (3) (mod 16):

$$1, 1, 15, \underbrace{13, 15, 5, 7,}\ \underbrace{13, 15, \dots}. \quad (9)$$

Clearly, the sequence (9) repeats with a period of 4. Thus $b_n = 1$ only for $n = 1, 2$ and, if $b_n = -1$, then $n = 3$ or $n = 4k + 1$ for some $k \geqslant 1$. We want to show that the only additional possibilities are $n = 5$ or 13.

Suppose $b_{4k+1} = -1$ for some $k \geqslant 1$. For $k = 1$, we have $b_5 = -1$. If $k \geqslant 2$, then by (8),

$$-1 = b_{4k+1} = b_5^k - 2b_4^2 \sum_{j=2}^{k} (-1)^j \binom{k}{j} b_5^{k-j} b_4^{j-2} b_{j-1}. \quad (10)$$

Since $b_4 = -3$, we have $-1 \equiv (-1)^k (\operatorname{mod} 3)$, so that k is odd, $b_5^k = -1$, and the sum in equation (10) vanishes:

$$\sum_{j=2}^{k} (-1)^j \binom{k}{j} 3^{j-2} b_{j-1} = 0. \quad (11)$$

Note that equation (11) is satisfied for $k = 3$, consistent with the fact that $b_{13} = -1$. For $k \geqslant 5$, however, we divide (11) by $k(k-1)$ to obtain

$$\frac{1}{2} - \frac{k-2}{2} + \sum_{j=4}^{k} (-1)^j \binom{k-2}{j-2} \frac{3^{j-2}}{j(j-1)} b_{j-1} = 0. \quad (12)$$

Now for $j \geqslant 4$, each term in the sum above is an integer times $3^{j-2}/j(j-1)$. These rational numbers: $3^2/4 \cdot 3, 3^3/5 \cdot 4, 3^4/6 \cdot 5, \dots$, when reduced, all still have numerators divisible by 3. Thus

$$\frac{1}{2} - \frac{k-2}{2} = \frac{3-k}{2}$$

must have the same property, implying that 3 divides k.

Replacing k by $3k$, we look for those odd values of k such that $b_{12k+1} = -1$. For $k = 1$ we have $b_{13} = -1$. If $k \geqslant 3$, then again by (8),

$$-1 = b_{12k+1} = b_{13}^k - 2b_{12}^2 \sum_{j=2}^{k} (-1)^j \binom{k}{j} b_{13}^{k-j} b_{12}^{j-2} b_{j-1}. \quad (13)$$

Since k is odd and $b_{13} = -1$, the sum above also vanishes. Dividing the sum by

$k(k-1)$ and using the fact that $b_{12} = 45$, we obtain

$$\frac{1}{2} + \sum_{j=3}^{k} \binom{k-2}{j-2} \frac{45^{j-2}}{j(j-1)} b_{j-1} = 0. \tag{14}$$

In this case, each fraction in the sum, when reduced, has numerator divisible by 5, so that $\frac{1}{2}$ should also have this property. This provides a contradiction to the assumption that $k \geqslant 3$.

It turns out that the role of unique factorization in $\mathbb{Q}(\sqrt{-7})$ is not as critical as the exposition here might indicate. Using the unique decomposition of ideals into products of prime ideals in quadratic fields, one can apply the techniques above generally to solve, for example, all Diophantine equations of the form $X^2 + (4q - 1) = 4q^n$, where q is an arbitrary prime. For more on this, see the more-detailed paper of the author [4].

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THE TEACHING OF MATHEMATICS

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The Tumbling Box

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Introduction. Toss a rigid body, such as a book or an empty cereal box, in the air three times, each time giving it a spin about one of its axes. It is perhaps surprising to learn that the box will always rotate stably about two of its three axes, but will

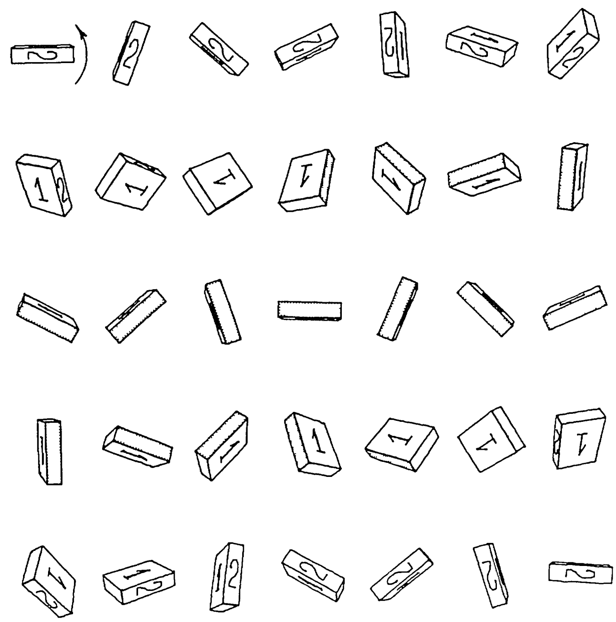

FIG. 1. Numerical plot of a box tumbling in air. The box is viewed from a distance of 5 box lengths and along the direction of the fixed angular momentum vector $\vec{L}$. In these "snapshots," the unstable axis #2 is misaligned from $\vec{L}$ by 5°. The principal moments of inertia of the box are in the ratio $I_1 : I_2 : I_3 :: 6 : 4 : 3$.

wobble and somersault unstably about the third (see Fig. 1). This fact is a well-known result from classical mechanics (see [**4**] pp. 116ff. or [**1**] pp. 142–145), but, unfortunately, one with which too few mathematicians seem to be familiar. The tumbling box problem presents a very beautiful and natural example of a system of nonlinear differential equations, one which can be made appropriate for an undergraduate course in the subject, but one which fails to appear in any of the elementary texts. In addition, it is one of those rare problems whose purely mathematical solution is also easily verified empirically.

I would like to express my gratitude to Professor Alar Toomre of M.I.T. for introducing me and countless others to this particular version of the problem (see, for example, [**2**] problem 4.51, pp. 202ff.), for encouraging me to present it here, and for allowing me to reproduce his splendid diagrams in Figs. 1 and 2.

Some Physics. The basic law of motion for any vector function of a rigid body in space is Euler's equation. If $\vec{A}(t)$ is any vector function pertaining to the body, the formula states

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{A}, \tag{1}$$

where $(d\vec{A}/dt)_{\text{space}}$, $(d\vec{A}/dt)_{\text{body}}$ are the rates of change respectively measured in fixed spatial coordinates and in coordinates relative to the principal axes of the object, and where $\vec{\omega}$ is angular velocity (see, for instance, [3] p. 133). This formula says essentially that $d\vec{A}/dt$ consists of both a translational and a rotational component, the latter being given by $\vec{\omega} \times \vec{A}$.

For our tumbling box, we are concerned with the case $\vec{A} = \vec{L}$, the angular momentum. If we toss the box by giving it a spin as described in the introduction, we are introducing a constant angular momentum vector. Hence $(d\vec{L}/dt)_{\text{space}} = \vec{0}$ and so (1) becomes

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{body}} = \vec{L} \times \vec{\omega}. \tag{2}$$

The Differential System. In the sequel, we assume that all coordinates are measured relative to the axes of the box and henceforth we will drop the subscript "body." Then $\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, where ω_j is the jth component of angular velocity and I_j the moment of inertia about the jth principal axis of the box. If we assume that the box is uniform and has distinct dimensions (so that the box may indeed be considered to look like a book), then we may take $I_1 > I_2 > I_3 > 0$. Then, in coordinates, (2) becomes the system

$$\begin{cases} \dot{L}_1 = (I_2 - I_3)\omega_2\omega_3, \\ \dot{L}_2 = (I_3 - I_1)\omega_1\omega_3, \\ \dot{L}_3 = (I_1 - I_2)\omega_1\omega_2. \end{cases}$$

Equivalently, since $\omega_j = L_j/I_j$, $j = 1,2,3$, we have

$$\begin{cases} \dot{L}_1 = \left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right)L_2L_3, \\[2ex] \dot{L}_2 = \left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right)L_1L_3, \\[2ex] \dot{L}_3 = \left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right)L_1L_2. \end{cases} \tag{3}$$

It is easy to check that from (3) it follows that

$$L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3 = 0. \tag{4}$$

Integrating, we find

$$L_1^2 + L_2^2 + L_3^2 = C.$$

Hence we see that the trajectories of (3) must all lie on spheres centered at the

origin. For simplicity, let us consider only the case $C = 1$. We have reduced our problem to that of finding solutions on a single *phase sphere*.

Linearizations. Now we make a standard local analysis. It is not difficult to see that (3) has six isolated critical points at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. By using Taylor's formula for several variables near each critical point, we may approximate (3) locally by the following six linear systems:

$$\text{near } (\pm 1, 0, 0) \qquad \begin{cases} \dot{L}_1 = 0, \\ \dot{L}_2 = \pm\left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) L_3, \\ \dot{L}_3 = \pm\left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) L_2, \end{cases} \tag{5a}$$

$$\text{near } (0, \pm 1, 0) \qquad \begin{cases} \dot{L}_1 = \pm\left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) L_3, \\ \dot{L}_2 = 0, \\ \dot{L}_3 = \pm\left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) L_1, \end{cases} \tag{5b}$$

$$\text{near } (0, 0, \pm 1) \qquad \begin{cases} \dot{L}_1 = \pm\left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) L_2, \\ \dot{L}_2 = \pm\left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) L_1, \\ \dot{L}_3 = 0. \end{cases} \tag{5c}$$

Since in each of (5a), (5b), and (5c) one of L_1, L_2, or L_3 is constant to first order, we may regard each of these linear systems as being two-dimensional by "ignoring" the constant variable. With such a simplification, the characteristic equation of (5a) is $x^2 - \alpha = 0$, where

$$\alpha = (1/I_1 - 1/I_3)(1/I_2 - 1/I_1) < 0 \quad (\text{for } I_1 > I_2 > I_3).$$

Hence the corresponding eigenvalues are pure imaginary and thus the linear critical points at $(\pm 1, 0, 0)$ are *centers*. Similarly, the characteristic equation of (5c) is $x^2 - \beta = 0$, where

$$\beta = (1/I_1 - 1/I_3)(1/I_3 - 1/I_2) < 0,$$

so at $(0, 0, \pm 1)$ we also have centers. However, (5b) has characteristic equation $x^2 - \gamma = 0$, where

$$\gamma = (1/I_2 - 1/I_1)(1/I_3 - 1/I_2) > 0.$$

Thus the eigenvalues of (5b) are real and of opposite sign and hence the critical points at $(0, \pm 1, 0)$ are *saddle points* (and are unstable).

Unfortunately, the local analysis above does not yet afford a complete solution because of the centers resulting from (5a) and (5c). A center singularity is a "borderline case" in that the original nonlinear system possibly could have a singularity of a different type (see, for example, [2, p. 183]). However, analogously to (4), we may check that from (3),

$$\frac{I_2 I_3}{I_2 - I_3} L_1 \dot{L}_1 + \frac{I_1 I_3}{I_1 - I_3} L_2 \dot{L}_2 = 0, \tag{6a}$$

$$\frac{I_1 I_3}{I_1 - I_3} L_2 \dot{L}_2 + \frac{I_1 I_2}{I_1 - I_2} L_3 \dot{L}_3 = 0. \tag{6b}$$

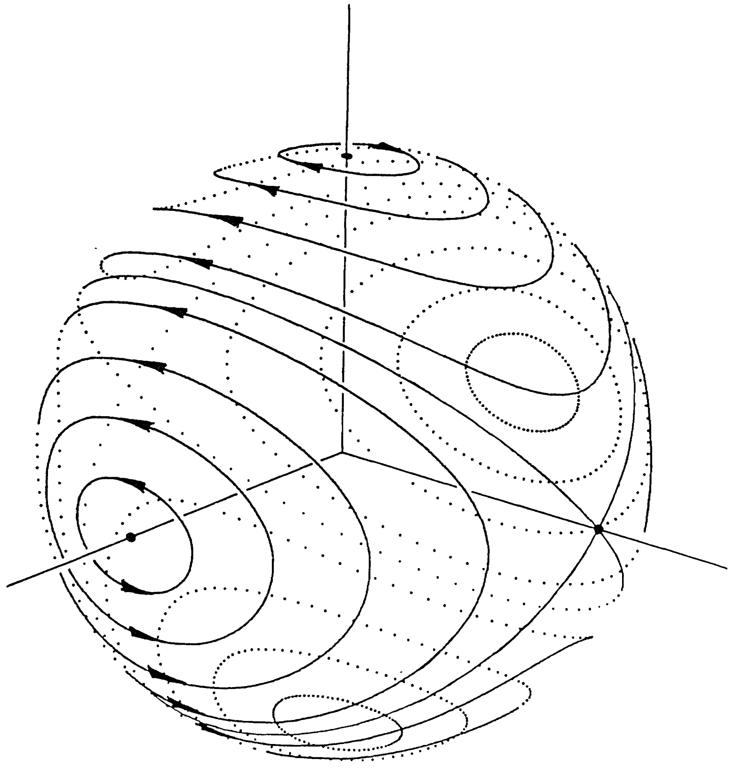

FIG. 2. Trajectories of $(\frac{d\vec{L}}{dt})_{\text{body}} = \vec{L} \times \vec{\omega}$ on the phase sphere $L_1^2 + L_2^2 + L_3^2 = 1$.

These equations integrate respectively to

$$\frac{I_2}{I_2 - I_3}L_1^2 + \frac{I_1}{I_1 - I_3}L_2^2 = C_1, \tag{7a}$$

$$\frac{I_3}{I_1 - I_3}L_2^2 + \frac{I_2}{I_1 - I_2}L_3^2 = C_2. \tag{7b}$$

Equations (7a) and (7b) describe elliptical cylinders with axes the z and x axes, respectively. For sufficiently small C_1 and C_2, the intersections of these cylinders with the phase sphere are closed curves about the z and x axes. Hence the centers remain stable centers when we pass from the linearizations to the nonlinear system (3). See Fig. 2 for a sketch of the trajectories on the phase sphere, along with arrows indicating direction with increasing time. From this diagram, it is apparent that rotations about either the longest or the shortest axis are non-asymptotically stable, that rotations about the middle axis are unstable, and, furthermore, that this unstable motion will tend to wobble around one of the two types of stable rotations.

Finally, we remark that if any two of the principal moments of inertia are equal, then (3) immediately reduces to a simpler linear system which is easily seen to yield only stable rotations.

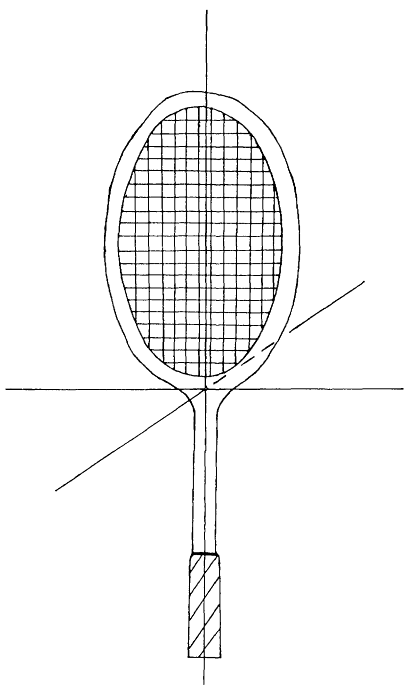

FIG. 3. The tumbling tennis racquet. Rotations about axis 2 will exhibit Eulerian wobble.

Note for Tennis Players. Essentially the same analysis as that employed above can also be applied to the tossing of tennis racquets to verify the existence of stable and unstable rotations (see Fig. 3). One finds that spinning the racquet about the axis which is perpendicular to the neck and lies in the "plane" of the racquet results in the Eulerian wobble described above. Tennis racquets provide excellent examples of rigid bodies with distinct principal moments of inertia, since they have handles which make them convenient to throw.

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The Multiplication Theorem for Fredholm Operators

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A bounded linear operator T from one Banach space to another is called a Fredholm operator if its kernel is of finite dimension and its range is of finite codimension; one then defines $\operatorname{ind}(T)$, the index of T, to be the difference between the dimension of the kernel of T and the codimension of the range of T. A Fredholm operator automatically has a closed range, a property that in many treatments is incorporated as part of the definition. The resulting redundancy is not a practical disadvantage, for, invariably, in checking in a concrete case that an operator satisfies the requirements of the definition, when the range of the operator is not obviously closed, one shows it is closed as part of the argument that shows it has a finite codimension. Moreover, the instructor who uses the longer definition can, without interrupting the logical flow of the lectures, assign to the students the task of proving that the two definitions are equivalent. The proof is a satisfying application of the open mapping theorem [**8**].

The theorem referred to in the title is one of the central results about Fredholm operators. It states: *If T is a Fredholm operator from X to Y and S is a Fredholm operator from Y to Z, then ST is a Fredholm operator whose index is the sum of the indices of S and T.* The main point I wish to make here is that this theorem is a purely algebraic result whose general case is easily reduced to the finite-dimensional one, that case being an immediate consequence of the fundamental theorem of linear algebra (which states: *For a linear operator acting on a finite-dimensional vector space, the dimensions of the kernel and the range add up to the dimension of the space*). I was prompted to write this note because the proofs of the multiplication

theorem I have seen in the literature [3]–[8] do not clearly reveal its underlying simplicity.

To prove the multiplication theorem, let X, Y, Z, S and T be as in its statement, except that we can ignore the topologies on X, Y and Z, regarding them purely as vector spaces. If they are finite-dimensional, the fundamental theorem of linear algebra implies that $\operatorname{ind}(T) = \dim X - \dim Y$, together with similar expressions for $\operatorname{ind}(S)$ and $\operatorname{ind}(ST)$, from which the equality $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$ follows.

The reduction of the general case to the finite-dimensional case involves complementary subspaces. Two subspaces of a vector space are called complementary subspaces of each other if their intersection is trivial and their sum is the whole space (in other words, if the whole space is their direct sum). The following result is needed: *If E and F are subspaces of a vector space and their intersection is trivial, then E has a complementary subspace that contains F.* The proof is a standard application of Zorn's lemma. In one of the two cases of interest below, E is finite-dimensional and F is trivial, while in the other E has a finite codimension. (Zorn's lemma is not needed in the latter case.)

We shall produce complementary subspaces X_0 and X_1 of X, Y_0 and Y_1 of Y, and Z_0 and Z_1 of Z with the following properties:

(i) X_0, Y_0 and Z_0 are finite-dimensional;
(ii) $TX_0 \subset Y_0$ and $SY_0 \subset Z_0$;
(iii) $TX_1 = Y_1$ and $SY_1 = Z_1$;
(iv) $\ker T \cap X_1$ and $\ker S \cap Y_1$ are trivial.

That will effect the desired reduction.

We define X_0 to be $T^{-1}(\ker S)$; it is clearly finite-dimensional. We let X_1 be any complementary subspace of X_0, and we let Y_1 equal TX_1. As the subspace Y_1 then has a finite codimension in TX, which has a finite codimension in Y, it also has a finite codimension in Y. The intersection of Y_1 and $\ker S$ obviously being trivial, there is a complementary subspace Y_0 of Y_1 that contains $\ker S$. Let Z_1 be SY_1. Reasoning as above, we see that Z_1 has a finite codimension in Z. From the inclusion $\ker S \subset Y_0$ one easily deduces that Z_1 and SY_0 have a trivial intersection, so Z_1 has a complementary subspace Z_0 that contains SY_0. Properties (i)–(iv) obviously being satisfied, the proof of the theorem is complete.

An instructor who uses the longer of the two definitions of Fredholm operator can topologize the proof above so as to prove without much added effort that ST has a closed range. For that one must take X_1 to be closed, which is possible by the elementary result that every finite-dimensional subspace of a Banach space has a closed complementary subspace; the proof involves the Hahn-Banach theorem (so Zorn's lemma does not go away). One must then show that $Y_1(= TX_1)$ and $Z_1(= SY_1)$ are closed. It is no more difficult to show that a Fredholm operator in general maps closed subspaces onto closed subspaces, the proof of which uses the open mapping theorem and is nearly identical to the proof of the equivalence of the

two definitions of Fredholm operator. The last circumstance has persuaded me that the purely algebraic definition is the more natural starting point for the theory.

The reasoning above can easily be adapted to yield the multiplication theorem for unbounded Fredholm operators, the case treated in [**6**] and [7].

The multiplication theorem for Fredholm operators originated with F. V. Atkinson [**1**]. An interesting article by J. Dieudonné describing the early development of the notion of index has recently appeared [**2**].

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PROBLEMS AND SOLUTIONS

*For instructions about submitting **proposed** problems for publication in this department see the inside front cover. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the address given on the inside front cover.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a

more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

For instructions about submitting solutions of these Elementary Problems, which should be mailed by May 31, 1987, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3183*. *Proposed by Murray S. Klamkin, University of Alberta, Canada.*

Let P' denote the convex n-gon whose vertices are the midpoints of the sides of a given convex n-gon P. Determine the extreme values of

(i) Area P'/Area P,

(ii) Perimeter P'/Perimeter P.

E 3184. *Proposed by Calin P. Popescu, Bucharest, Romania.*

Suppose the roots of the polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$, $n \geqslant 2$, are real numbers lying in $(0,1)$ and that $a_n > 0$. Show that $\sum_{i=k}^{n-2} \binom{i}{k} a_i > 0$, where $0 \leqslant k \leqslant n-2$.

E 3185. *Proposed by Raymond E. Spaulding, Radford University, VA.*

Let P be a point in the interior of an equilateral triangle, and let S be the sum of the perpendicular distances to the three sides of the triangle from P. In Euclidean geometry, the sum S always equals the altitude of the triangle. In Lobachevskian geometry, prove that S is less than any altitude. In addition, find the position of P which would give a minimum value for S.

E 3186. *Proposed by I. A. Sakmar, University of Western Ontario.*

On a current TV game show, three contestants for the final showcase prizes are selected with the spin of a wheel, which is divided into twenty equal sections numbered in steps of 5 from 5 to 100. The aim is to get as close to 100 as possible without exceeding a sum of 100 in two spins. Each contestant completes one or both spins before the next contestant's turn, and if either the 1st or 2nd contestant exceeds 100, that contestant is out of contention. In case of a tie, the first contestant to attain that score is chosen.

In the running of the program, it is implied that it is a good policy for the first contestant to stop if the number from the first spin is greater than 50. What is the optimal number for the first contestant to stay on his first spin?

E 3187. *Proposed by Stephen L. Lipscomb, Mary Washington College, and Allen J. Schwenk, Western Michigan University.*

A *vertex deleted* subgraph $G - v$ of a graph G is formed by removing one vertex v and every edge incident with it. A graph is called *asymmetric* if it has no nontrivial automorphisms (symmetries).

(a) Find a smallest possible asymmetric graph all of whose vertex deleted subgraphs are also asymmetric.
(b) Same as (a) but also require that no pair of vertex deleted subgraphs be isomorphic.

E 3188. *Proposed by Ioan Tomescu, University of Bucharest, Romania.*

Let X be a nonempty set having n elements and C be a color set with $p \geqslant 1$ elements. Find the greatest number p satisfying the following property:

If we color in an arbitrary way each subset of X with colors from C such that each subset receives only one color, then there exist two distinct subsets A, B of X such that the sets

$$A, B, A \cup B, A \cap B$$

have the same color.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Old (Old) Result

E 2949 [1982, 334]. *Proposed by Man-Duen Choi, University of Toronto.*

Let $n_0 < n_1 < \cdots < n_k$ by any given $k + 1$ integers. Show by elementary means that the integer $\prod_{k \geqslant j > i \geqslant 0}(n_j - n_i)$ is exactly divisible by $\prod_{k \geqslant j > i \geqslant 0}(j - i) = 1!2! \cdots k!$. [An abstract indirect proof of this result appeared in H. Weyl's book, *The Classical Groups*, Chapter VII, Section 5, p. 201, Princeton University Press, 1939.]

Editorial comment. This problem is E 2637 [1977, 134; 1978, 386], as was pointed out by many readers. In addition to the numerous comments in that location, Leonard Carlitz provided the following reference: H. W. Segar, Messenger of Math., 22 (1892–93), 59 and 23 (1893–94), 31–36.

A Prime Condition

E 3005 [1983, 400]. *Proposed by O. Krafft, Aachen, West Germany.*

Let $n \geqslant 5$ be an integer. Show that n is a prime if and only if for every partition of n into four positive integers, $n = n_1 + n_2 + n_3 + n_4$, $n_{i_1}n_{i_2} \neq n_{i_3}n_{i_4}$, for each permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$.

Composite Solution. If n is composite we may write n as $(a+1)(b+1)$ where a and b are positive integers. Thus, $n = ab + a + b + 1$ is a partition of the desired form.

Conversely suppose, without loss of generality, that $n = n_1 + n_2 + n_3 + n_4$ and $n_1 \cdot n_2 = n_3 \cdot n_4$. Let the fraction $n_1/n_4 = n_3/n_2$ be written in lowest terms r/s. Therefore, there exist positive integers h and k such that $n_1 = kr$, $n_4 = ks$, $n_3 = hr$, and $n_2 = hs$. Hence,

$$n = n_1 + n_2 + n_3 + n_4 = (h+k)(r+s)$$

and n is composite.

Solved by 60 readers and the proposer.

An Application of Quadratic Reciprocity II

E 3012 [1983, 483]. *Proposed by J. L. Selfridge, Mathematical Reviews.*

Show that $2^a - 1$ does not divide $3^a - 1$ if $a > 1$. More generally, $2^a - 1$ does not divide $3^b - 1$ when $a > 1$ and a and b have the same parity.

Solution by Robert Breusch, Amherst College, Massachusetts. If a is even, then $3|(2^a - 1)$. Thus, $2^a - 1$ cannot be a divisor of $3^b - 1$ for any b, even or odd.

If a is odd and greater than 1, then $A = 2^a - 1 \equiv 1 \pmod 3$, A is a quadratic residue (mod 3), and $(A/3) = +1$, where $(A/3)$ is the Legendre symbol (as well as the Jacobi symbol). The quadratic reciprocity law holds for the Jacobi symbol, thus

$$\left(\frac{3}{A}\right) = \left(\frac{A}{3}\right) \cdot (-1)^{\frac{A-1}{2} \cdot \frac{3-1}{2}}.$$

Now $(A-1)/2 = 2^{a-1} - 1$ is odd and so is $(3-1)/2$. Therefore,

$$\left(\frac{3}{A}\right) = -\left(\frac{A}{3}\right) = -1,$$

and thus 3 is a quadratic nonresidue (mod A).

Assume now that $(2^a - 1)|(3^b - 1)$ with $b = 2n - 1$. It follows that $A|(3^{2n} - 3)$ and, therefore, that $(3^n)^2 \equiv 3 \pmod A$. But this implies that 3 is a quadratic residue (mod A), which is a contradiction. Therefore, $2^a - 1$ cannot divide $3^b - 1$, if a and b are odd.

Editorial comment. This problem is a generalization of E 2643 [1977, 217; 1978, 497], An Application of Quadratic Reciprocity.

Kee-wai Lau points out that the solution to this problem appears as Example 89 of the Chinese book *Hundred Examples in Elementary Number Theory* by Chao Ko and Chi Sun, Shanghai Education Press, 1980.

Ian Connell points out that this result has a connection with an open problem that arises in the solution of Waring's problem (cf. Hardy and Wright, *The Theory of*

Numbers, p. 337). Given integer quantities that satisfy the conditions $a \geqslant 3$, $3^a = 2^a A + B$, $0 < B < 2^a$, and $A + B = (2^a - 1)q + r$, $0 \leqslant r < 2^a - 1$, is q always 0?

Also solved by I. Connell (Canada), F. Dodd, L. L. Foster, I. Gerst, R. G. E. Pinch (Scotland), B. Powell, H. Schmidt, Jr., R. Stong, P. Y. Wu (Republic of China), and the proposer.

An Exponential Congruence

E 3014 [1983, 566]. *Proposed by Lorraine L. Foster, California State University, Northridge.*

Prove that the congruence $3^x \equiv 19 \pmod{2^n}$ is solvable for $n \geqslant 1$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We shall prove the stronger statement that, for $n \geqslant 1$, $3^x \equiv y \pmod{2^n}$ is solvable if and only if $y \equiv 1$ or $3 \pmod 8$. To this end we need the following lemma.

LEMMA. *For each* $k \geqslant 1$ *one has that* $2^{k+2} | (3^{2^k} - 1)$ *and no higher power of* 2 *divides* $3^{2^k} - 1$.

Proof. For $k = 1$ the assertion is trivial. The induction argument follows from

$$(3^{2a} - 1) = (3^a - 1)\{(3^a - 1) + 2\}.$$

It follows from the lemma that the multiplicative order of 3 (mod 2^n) is 2^{n-2}. So the numbers 3^x, $1 \leqslant x \leqslant (1/4)2^n$ are all different modulo 2^n. On the other hand, all powers of 3 are congruent to 1 or 3 (mod 8). So each number y, which is congruent to 1 or 3 (mod 8), occurs exactly once, as a power $3^x \pmod{2^n}$, $1 \leqslant x \leqslant 2^{n-2}$, and vice versa.

Also solved by forty-three other readers and the proposer.

Solutions to an Equation Involving Logs

E 3015 [1983, 566]. *Proposed by Ioan Tomescu, University of Bucharest.*

Let $h \geqslant 2$ be an integer and let $\alpha(h) \geqslant h$ be the unique solution of the equation

$$1 + \frac{x}{x-1} + \frac{x}{x-2} + \cdots + \frac{x}{x-h+1} = \ln \frac{(x)_h}{h!},$$

where $(x)_h = x(x-1)\cdots(x-h+1)$. Show that

$$\lim_{h \to \infty} \alpha(h)\left(2h + \frac{1}{2}\ln h\right)^{-1} = 1.$$

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. Let

$$f(y) := \frac{x}{x-y} - \ln\frac{x-y}{y+1} \text{ for } 0 \leqslant y < h < x.$$

Since $f'(y) > 0$ we see that f increases on $[0, h]$. Therefore,

$$f(0) + \int_0^{h-1} f(y)\,dy < \sum_{k=0}^{h-1} f(k) < \int_0^h f(y)\,dy$$

and, by definition, the middle term is 0 for $x = \alpha(h)$. For this value of x we find from the right-hand inequality by integration

$$h\ln(x-h) < (h+1)\ln(h+1),$$

i.e.,

$$x < h + (h+1)^{1+h^{-1}} = 2h(1+o(1)), \qquad (h \to \infty).$$

Similarly, from the left-hand side and this result we find

$$(h-1)\ln(x-h+1) > 1 - \ln x + h\ln h,$$

so

$$x - h + 1 > h^{h/(h-1)}(1+o(1)),$$

i.e.,

$$x > 2h(1+o(1)), \qquad (h \to \infty).$$

Also solved by A. Bondesen (Denmark), W. A. Newcomb, R. E. Shafer, R. Stong, A. Tissier (France), D. B. Tyler, and the proposer.

Eventually Linear Sets of Integers

E 3018 [1983, 567]. *Proposed by A. M. Nadel (student), Harvard University.*

A nonempty set S of positive integers is said to be eventually linear if there exist integers N and k such that for all $n > N$, $n \in S$ if and only if $k|n$. Show that any nonempty set of positive integers that is closed under addition is eventually linear.

Solution by Steven R. Weston, Randolph Air Force Base, Texas. Let $S = \{s_1, s_2, s_3, \dots\}$ be a nonempty set of positive integers that is closed under addition, with $s_1 < s_2 < s_3 < \cdots$. Let $D = \{s_{i+1} - s_i | i \text{ is a positive integer}\}$, and let k be the least element of D. Let $m \in S$ such that $m + k \in S$, and let $N = km^2$.

First, suppose $n > N$ and $k|n$. Then $n = N + qk$ for some q. By the Euclidean Algorithm, $q = ikm + j$ for some i and j, where $0 \leqslant j < km$. Hence,

$$n = (km - j)m + j(m+k) + ik^2m,$$

which is an element of S because it is a sum of multiples of m and $m + k$. Thus, if $n > N$ and $k|n$, then $n \in S$.

Now suppose $n > N$ and $n \in S$. By the Euclidean Algorithm,

$$n = N + qk + r$$

for some q and r, where $0 \leqslant r < k$. If $r > 0$, then we would have

$$N + qk < n < N + (q + 1)k,$$

but this would contradict the assumption that k is the least element of D. Hence, $r = 0$ and so $k|n$. Thus, if $n > N$ and $n \in S$, then $k|n$.

Claim: k is the greatest common divisor of all the elements of S.

Proof. Suppose $n \in S$. Then there exists a prime p such that $p > k$ and $pn > N$, where N is defined as above. Since k and p are relatively prime, and $k|pn$, it follows that $k|n$. Hence k divides every element of S. Also, k is the greatest such divisor since there can be no integer k' greater than k such that both $k'|N$ and $k'|N + k$.

Also solved by 46 other readers and the proposer. Robert Gilmer referred to results in: J. C. Higgins, Representing N-semigroups, Bull. Austral. Math. Soc., 1 (1969) 115–126; and L. Redei, The Theory of Finitely Generated Semigroups, Pergamon Press, Oxford-Edinburgh-New York, 1965, section 38. Thomas Moore referred to J. C. Higgins, Subsemigroups of the additive positive integers, The Fibonacci Quarterly, 10 (1972) p. 225; and M. Y. Sit and Man-Keung Siu, On the subsemigroups of N, Mathematics Magazine, 48 (1975) 225–227. See also this MONTHLY, E 2418 [1973, 560; 1974, 523–524].

The Tarry-Escott Problem Again

E 3032 [1984, 57]. *Proposed by J. O. Shallit, University of California, Berkeley.*

Let d, r be integers with $d \geqslant 2$, $r \geqslant 1$. Let p be a polynomial with real coefficients, and $\deg(p) < r$. Show how to partition the set

$$S = \{p(0), p(1), p(2), \ldots, p(d^r - 1)\}$$

into d disjoint subsets whose union is S, such that the sum of each subset is the same.

Comment by the editors. Several solvers pointed out that, strictly speaking, the problem is false as stated. It should have read "... partition the *list* of d elements

$$p(0), p(1), \ldots, p(d^r - 1)$$

into d disjoint *lists* whose union is exactly S. The element $p(i)$ is considered distinct from $p(j)$ whenever $i \neq j$."

With this rewording, the problem becomes a version of the so-called Tarry-Escott problem. A solution is given by putting $p(i)$ in the list numbered $s_d(i)$ (mod d), where $s_d(i)$ is the sum of the digits of the integer i when expressed in base d. A

proof can be given by induction on the degree r. Prouhet stated this result in 1851. For proofs of this and more general statements, see the references below.

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3. J. B. Roberts, A new proof of a theorem of Lehmer, Canad. J. Math., 10 (1958) 191–194.

Also solved by Anders Bager (Denmark); S. V. Kanetkar, Adnah G. Kostenbauder, Pierre Lalonde (Canada); David Lindsay, O. P. Lossers (Netherlands), Stan Philipp, and the proposer. R. D. Whittekin pointed out the problem's misstatement.

An Inequality with Sums and Powers

E 3041 [1984, 257]. *Proposed by J. Martin Borden, Worcester Polytechnic Institute, and David Mason, University of Delaware.*

For positive arguments define a function R by

$$R(s_1,s_2,\dots,s_n)=\frac{(s_1+s_2+\cdots+s_n)^{(s_1+s_2+\cdots+s_n)}}{s_1^{s_1}s_2^{s_2}\cdots s_n^{s_n}}.$$

(a) Show that $R(s_1,\dots,s_n)R(t_1,\dots,t_n)\leqslant R(s_1+t_1,\dots,s_n+t_n)$.

(b) Find a necessary and sufficient condition for equality to hold in the inequality of (a).

Solution I by Robert L. Young, Cape Cod Community College, West Barnstable, Massachusetts. Let $\mathbb{R}=R(s_1,\dots,s_n)R(t_1,\dots,t_n)/R(s_1+t_1,\dots,s_n+t_n)$. We will show that $\mathbb{R}\leqslant 1$ with equality if and only if $s_i=\lambda t_i$ for some positive constant λ and $i=1,2,\dots,n$.

Letting $S=\Sigma_{i=1}^n s_i$ and $T=\Sigma_{i=1}^n t_i$, we can write

$$\mathbb{R}=\frac{S^{s_1+\cdots+s_n}\cdot T^{t_1+\cdots+t_n}(s_1+t_1)^{s_1+t_1}\cdots(s_n+t_n)^{s_n+t_n}}{s_1^{s_1}\cdots s_n^{s_n}t_1^{t_1}\cdots t_n^{t_n}(S+T)^{s_1+t_1+\cdots+s_n+t_n}}$$

$$=\prod_{i=1}^{n}\left\{\frac{S(s_i+t_i)}{s_i(S+T)}\right\}^{s_i}\left\{\frac{T(s_i+t_i)}{t_i(S+T)}\right\}^{t_i}.$$

The Arithmetic Mean-Geometric Mean Inequality implies that

$$\prod\left\{\frac{S(s_i+t_i)}{s_i(S+T)}\right\}^{s_i}\leqslant\left\{\frac{1}{S}\sum\frac{S(s_i+t_i)}{S+T}\right\}^{S}=1$$

and

$$\prod\left\{\frac{T(s_i+t_i)}{t_i(S+T)}\right\}^{t_i} \leqslant \left\{\frac{1}{T}\sum\frac{T(s_i+t_i)}{S+T}\right\}^{T} = 1.$$

Hence, $R \leqslant 1$.

Equality holds if and only if

$$\frac{S(s_i+t_i)}{s_i(S+T)} = \frac{S(s_j+t_j)}{s_j(S+T)} \quad \text{and} \quad \frac{T(s_i+t_i)}{t_i(S+T)} = \frac{T(s_j+t_j)}{t_j(S+T)}$$

for all i, $j = 1, 2, \ldots, n$. This is clearly equivalent to $s_i = \lambda t_i$ for some constant $\lambda > 0$.

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We put $s_k = \sigma u_k$ where $\sigma = \sum_{k=1}^{n} s_k$, $t_k = \tau v_k$, where $\tau = \sum_{k=1}^{n} t_k$ and furthermore: $\lambda_1 = \sigma(\sigma + \tau)$ and $\lambda_2 = \tau/(\sigma+\tau)$. Then $\lambda_1 + \lambda_2 = 1$, whereas the proposed inequality is equivalent to:

$$\sum_{k=1}^{n}(\lambda_1 u_k \log u_k + \lambda_2 v_k \log v_k) \geqslant \sum_{k=1}^{n}(\lambda_1 u_k + \lambda_2 v_k)\log(\lambda_1 u_k + \lambda_2 v_k).$$

Since $x \log x$ is convex for $x > 0$, we have (Jensen's inequality):

$$\lambda_1 u_k \log u_k + \lambda_2 v_k \log v_k \geqslant (\lambda_1 u_k + \lambda_2 v_k)\log(\lambda_1 u_k + \lambda_2 v_k)$$

for $k = 1, 2, \ldots, n$ and with equality if and only if $u_k = v_k$. This proves the inequality of the problem with equality if and only if $s_k = ct_k$, where $c = \sigma/\tau > 0$ is a constant.

Also solved by A. Bondesen (Denmark), E. A. Escalona F. (Venezuela), R. Farwig (West Germany) T. Jager, H. Kappus (Switzerland), J.-M. Monier (France) S. Philipp, D. Sevcovic (Czechoslovakia), H. Zawadzki (Poland), and the proposers.

Random Walk Until No Shoes

E 3043 [1984, 310]. *Proposed by Gunnar Blom, University of Lund, Sweden.*

A has a house with one front door and one back door.

(a) A places n pairs of walking shoes at each door. For each walk, he chooses one door at random, puts on a pair of shoes, returns after the walk to a randomly chosen door and takes off the shoes at the door. Find the average number of finished walks until A discovers that no shoes are available at the door he has chosen for a further walk.

(b) A walks bare-footed if no shoes are available. Find the long run probability that A performs a given walk bare-footed, i.e., find the limit as $k \to \infty$ of the probability that A's kth walk is bare-footed.

Solution by Bennett Eisenberg, Lehigh University, Bethlehem, PA.

(a) $2n^2 + 2n$

(b) $1/(2n + 1)$

Proof. Let X_k be the minimum of the number of pairs of shoes outside the front and back doors after the kth walk. Let $T =$ the number of walks until A discovers that there are no shoes available. Let $f(k) = E(T|X_0 = k)$.

Then from standard probabilistic arguments it follows that $f(0) = \frac{1}{2} + \frac{1}{4}f(0) + \frac{1}{4}f(1)$. $f(j) = 1 + \frac{1}{4}f(j-1) + \frac{1}{2}f(j) + \frac{1}{4}f(j+1)$ for $1 \leqslant j \leqslant n-1$, and $f(n) = 1 + \frac{1}{2}f(n-1) + \frac{1}{2}f(n)$.

These equations can be rewritten as follows:

$$\begin{aligned} f(1) - f(0) &= 2f(0) - 2 \\ f(j+1) - f(j) &= f(j) - f(j-1) - 4 \quad \text{for } 1 \leqslant j \leqslant n-1 \\ f(n) - f(n-1) &= 2. \end{aligned}$$

These are easily solved to give $E(T|X_0 = n) = f(n) = 2n^2 + 2n$.

(b) Let X_k be the number of pairs of shoes in front of the front door after the kth walk. It is easily seen that $\{X_k\}$ is a Markov chain with transition probabilities $P(0,0) = P(2n,2n) = 3/4$, $P(0,1) = P(2n,2n-1) = P(j, j+1) = P(j, j-1) = 1/4$ for $1 \leqslant j \leqslant 2n-1$ and $P(j, j) = 1/2$ for $1 \leqslant j \leqslant 2n-1$. The transition matrix is doubly stochastic so

$$\lim_{k\to\infty} P(X_k = j) = \frac{1}{2n+1} \quad \text{for all } j.$$

Thus,

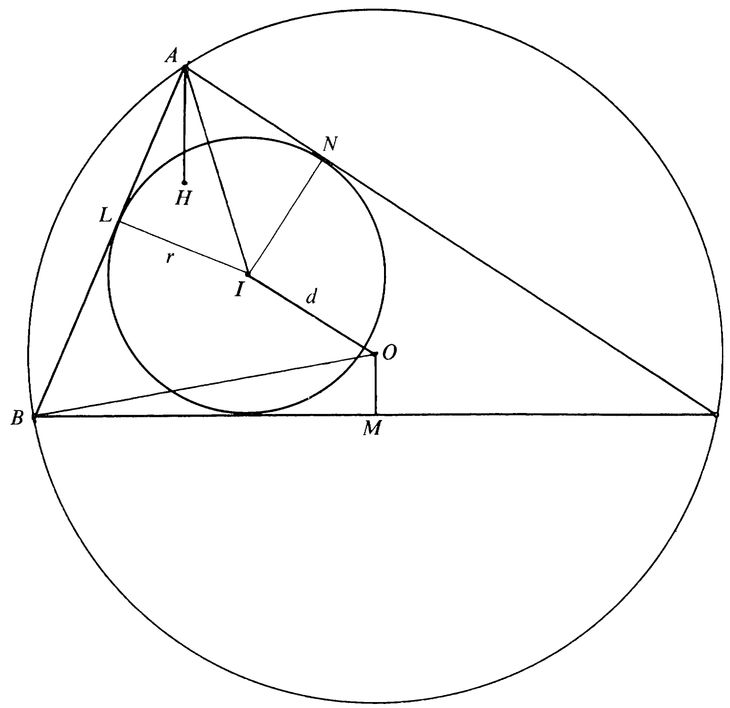
$$\begin{aligned} \lim_{k\to\infty} P((k+1)st) \text{ walk is barefoot} &= \frac{1}{2}\lim_{k\to\infty} P(X_k = 0) + \frac{1}{2}\lim_{k\to\infty} P(X_k = 2n) \\ &= \frac{1}{2n+1}. \end{aligned}$$

Also solved by P. -C. Chuang, E. Hertz, B. R. Johnson (Canada), L. R. King, O. P. Lossers (The Netherlands), W. A. Newcomb, M. Pachter (South Africa), R. Pinkham, A. J. Schwenk (Canada), L. Sennott, N. Vallee and J. -P. Carmichael (Canada), A. J. van Haagen, J. T. Ward, D. Wolfe, and the proposer.

Constructing a Triangle Given Three Lengths

E 3044 [1984, 310]. *Proposed by Jordi Dou, Barcelona, Spain.*

Construct a triangle ABC given: r, the inradius, $\overline{AI}$ where I is the incenter, and $\overline{AH}$ where H is the orthocenter.



Composite solution based on the solutions by The Chico Problem Group, Howard Eves, Jack Garfunkel, and Leila Nair (Canada).

CASE I. $AH \neq 0$. Since r and AI are given, the right triangles ANI and ALI can be constructed, where N and L are points of tangency to the incircle. This determines angle A.

Let M be the midpoint of BC. Since $OM = \frac{1}{2}AH$ and angle $BOM = A$, a right triangle BOM can be constructed, yielding $BO = R$. From the formula $d^2 = IO^2 = R(R - 2r)$, d can be constructed as the mean proportional between R and $R - 2r$.

With I as the center and $IO = d$ as the radius, describe a circle (I, d). With A as center and R as radius, describe an arc cutting the circle (I, d) at O and O'. (When $AI + d = R$, there will be only one point O.) With O as center and radius R, describe an arc cutting AL and AN extended at points B and C. Using O' also produces a solution.

CASE II. $AH = 0$. There are infinitely many solutions; for, if $AH = 0$, angle A must have measure $\pi/2$ and any line tangent to circle (I, r) which is not parallel to lines AL or AN determines a triangle which is a solution.

Note. No solution exists if any of the following inequalities fail:
(1) $r < AI$,
(2) $R \geq 2r$, where

$$R = \frac{AI^2 - 2r^2}{AI^2} \cdot \frac{AH}{2} \text{ if } \sqrt{2}\,r < AI \quad \text{or} \quad R = \frac{2r^2 - AI^2}{AI^2} \cdot \frac{AH}{2} \text{ if } r < AI < \sqrt{2}\,r.$$

(3) $AI - d \leq R \leq AI + d$, where $d^2 = R(R - 2r)$.

Also solved by A. Bondesen (Denmark), P.-C. Chuang, H. Demir (Turkey), J. Heuver (Canada), L. Kuipers (Switzerland), O. P. Lossers (The Netherlands), E. Morgantini (Italy), I. Paasche (West Germany), R. A. Simon (Chile), G. Velissarios (Greece), and the proposer.

ADVANCED PROBLEMS

For instructions about submitting solutions of these Advanced Problems, which should be mailed by May 31, 1987, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

6533. *Proposed by Boo Rim Choe (graduate student), University of Wisconsin-Madison.*

Let t be a fixed real number. Find all functions $f(z)$ holomorphic in a disc D centered at $z = 0$ that satisfy

$$|f(z)| = |f(x)| + t|f(y)|$$

for all $z = x + iy$ in D.

6534. *Proposed by F. S. Cater, Portland State University, Oregon.*

Let f and g be real-valued functions on the real line, $\mathbb{R}$, such that if B is any Borel subset of $\mathbb{R}$, then $f(B)$ and $g(B)$ are also Borel sets. For each number y, let $f^{-1}(y)$ be at most a finite set.

(a) Prove that there exist Borel sets A_1, A_2, B_1, B_2 such that

$$A_1 \cup A_2 = B_1 \cup B_2 = \mathbb{R}, \qquad A_1 \cap A_2 = B_1 \cap B_2 = \varnothing,$$
$$f(A_1) = B_1, \qquad g(B_2) = A_2.$$

(b)* Is the hypothesis on f^{-1} essential?

6535. *Proposed by Brockway McMillan, Sedgwick, Maine.*

Let $D = [0, \infty) \times [0, 1)$ and $f = f(x, y)$ be a real-valued function absolutely integrable on bounded measurable subsets of D. Assume that

$$\lim_{k \to \infty} \iint_{A_k \times B_k} f(x, y)\, dx\, dy = 0$$

for every pair of ascending sequences $A_1 \subseteq A_2 \subseteq \cdots, B_1 \subseteq B_2 \subseteq \cdots$ of bounded measurable sets such that

$$\bigcup_1^\infty A_k = [0, \infty) \quad \text{and} \quad \bigcup_1^\infty B_k = [0, 1).$$

Must f be absolutely integrable over D?

SOLUTIONS OF ADVANCED PROBLEMS

Differentiation Is Sometimes an Art

6492 [1985, 217]. *Proposed by Anatole Beck, University of Wisconsin-Madison.*

For which α is $f(x) = \int_0^x |t|^\alpha |\sin(1/t)|^{|1/t|}\, dt$ differentiable at $x = 0$?

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Since the integrand is an even function of t, it suffices to determine for which values of α the expression

$$x^{-1}\int_0^x t^\alpha |\sin(1/t)|^{1/t}\, dt$$

has a finite limit as $x \downarrow 0$.

Put $t = 1/u$, $x = 1/p$ and assume that $n\pi$ is the smallest multiple of π exceeding p. Then

$$\begin{aligned}\int_0^x t^\alpha |\sin(1/t)|^{1/t}\, dt &= \int_p^\infty u^{-\alpha-2}|\sin u|^u\, du \\ &= \int_p^{n\pi} u^{-\alpha-2}|\sin u|^u\, du + \sum_{k=0}^{\infty}\int_0^\pi (u + k\pi)^{-\alpha-2}(\sin u)^{u+k\pi}\, du.\end{aligned}$$

The first integral is $O(p^{-\alpha-2}) = O(x^{\alpha+2})$ for $x \downarrow 0$, and we observe that, for positive integers k,

$$\begin{aligned}&\int_0^\pi (u + k\pi)^{-\alpha-2}(\sin u)^{u+k\pi}\, du \\ &\quad = (k\pi)^{-\alpha-2}\left(1 + O\left(\frac{1}{k}\right)\right)\int_0^\pi (\sin u)^{u+k\pi}\, du.\end{aligned}$$

Since

$$\begin{aligned}\int_0^\pi (\sin u)^{k\pi}\, du &= B\left(\frac{1}{2}k\pi + \frac{1}{2}, \frac{1}{2}\right) \\ &= \sqrt{\pi}\,\frac{\left(\frac{1}{2}k\pi + \frac{1}{2}\right)}{\left(\frac{1}{2}k\pi + 1\right)} = \sqrt{2}\,k^{-1/2}\left(1 + O\left(\frac{1}{k}\right)\right),\end{aligned}$$

and the same estimate is valid for

$$\int_0^{\pi}(\sin u)^{(k+1)\pi}\,du,$$

we conclude that

$$\int_0^{\pi}(u+k\pi)^{-\alpha-2}(\sin u)^{u+k\pi}\,du=\pi^{-\alpha-2}\sqrt{2}\,k^{-\alpha-\frac{5}{2}}\left(1+O\left(\frac{1}{k}\right)\right).$$

Hence the infinite sum has the value

$$\pi^{-\alpha-2}\sqrt{2}\cdot\frac{2}{2\alpha+3}n^{-\alpha-\frac{3}{2}}+O\left(n^{-\alpha-\frac{5}{2}}\right)$$
$$=\frac{p^{-\alpha-\frac{3}{2}}}{\sqrt{\pi}}\,\frac{2\sqrt{2}}{2\alpha+3}+O\left(p^{-\alpha-\frac{5}{2}}\right).$$

Multiplication by p yields for $p\to\infty$ the limit zero provided that $\alpha>-1/2$, and the limit $\sqrt{(2/\pi)}$ in the case $\alpha=-1/2$. For other real values of α the limit does not exist.

I. E. Leonard adds that $f(x)$ is not defined on an open interval containing zero unless $\alpha>-3/2$. The proposer remarks that for the appropriate α the expression $f(x)$ "provides in closed form a strictly increasing function with derivative 0 at 0 that has unbounded derivatives in every neighborhood of 0."

Also solved by L. E. Clarke (England), I. E. Leonard, T. L. McCoy, William A. Newcomb, and Grzegorz Rzadkowski (Poland). One incorrect solution was received.

A Multiple Sine Integral

6493 [1985, 290]. *Proposed by M. L. Glasser, Clarkson College.*

Show that for $a_j>0$

$$\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_k\frac{\sin a_1x_1}{x_1}\cdots\frac{\sin a_kx_k}{x_k}\,\frac{\sin a_0(x_1+\cdots+x_k)}{x_1+\cdots+x_k}$$
$$=\pi^k\min\{a_0,\dots,a_k\}$$

(see 5314 [1965, 759]).

Solution I *by William A. Newcomb, Lawrence Livermore National Laboratory, Livermore, California.* We first show that for $s\neq 0$ and any positive numbers a and b we have

$$\int_{-\infty}^{\infty}\frac{\sin at}{t}\,\frac{\sin b(t+s)}{t+s}\,dt=\frac{\pi}{s}\sin ks,\qquad (*)$$

where

$$k=\min(a,b).$$

Indeed, from

$$\frac{1}{t} - \frac{1}{t+s} = \frac{s}{t(t+s)}$$

and elementary trigonometric identities, we see that the integral above equals $I + J$, where

$$I = \frac{1}{s}\int_{-\infty}^{\infty} \frac{\sin at}{t} \sin b(t+s)\, dt$$
$$= \frac{\sin bs}{2s}\int_{-\infty}^{\infty} \frac{\sin(a+b)t + \sin(a-b)t}{t}\, dt$$

and

$$J = -\frac{1}{s}\int_{-\infty}^{\infty} \sin at \frac{\sin b(t+s)}{t+s}\, dt = -\frac{1}{s}\int_{-\infty}^{\infty} \frac{\sin bt}{t} \sin a(t-s)$$
$$= \frac{\sin as}{2s}\int_{-\infty}^{\infty} \frac{\sin(b+a)t + \sin(b-a)t}{t}\, dt.$$

For $b > a$ this yields

$$I = \frac{\sin bs}{2s}(\pi - \pi) = 0, \quad \text{and} \quad J = \frac{\sin as}{2s}(\pi + \pi) = \frac{\pi}{s}\sin as.$$

Similarly, for $b < a$, we have

$$I = \frac{\pi}{s}\sin bs, \quad \text{and} \quad J = 0.$$

In either case, assertion $(*)$ is established.

Next, we show that the integral on the left of $(*)$ converges uniformly in s over any closed interval. Say $s \leqslant M$. For $2M < A < B$ we have

$$\int_A^B \left| \frac{\sin at \sin b(t+s)}{t(t+s)} \right| dt < 2\int_A^B \frac{dt}{t^2} < 2/A;$$

this approaches zero independently of s as $A \mapsto \infty$ (similarly for the integral from $-B$ to $-A$). Hence we have uniform convergence, and may take the limit as $s \to 0$ to obtain

$$\left(\int_{-\infty}^{\infty} \frac{\sin at \sin bt}{t^2}\, dt = \pi \min(a, b). \qquad (**)\right.$$

We now prove the result by induction on k. Let the k-fold sine integral on the left of the problem be denoted by S_k. For $k = 1$ the assertion of the problem is

equivalent to $(**)$. Next,

$$\begin{aligned}
S_{k+1} &= \int dx_1 \frac{\sin a_1 x_1}{x_1} \cdots \int dx_k \frac{\sin a_k x_k}{x_k} \\
&\quad \times \int dx_{k+1} \frac{\sin a_{k+1} x_{k+1}}{x_{k+1}} \frac{\sin a_0(x_1 + \cdots + x_k + x_{k+1})}{x_1 + \cdots + x_k + x_{k+1}} \\
&= \pi \int dx_1 \frac{\sin a_1 x_1}{x_1} \cdots \int dx_k \frac{\sin a_k x_k}{x_k} \frac{\sin(\min(a_{k+1}, a_0)(x_1 + \cdots + x_k))}{x_1 + \cdots + x_k} \\
&= \pi \cdot \pi^k \min(a_1, \ldots, a_k, \min(a_{k+1}, a_0)) \\
&= \pi^{k+1} \min(a_0, a_1, \ldots, a_k, a_{k+1})
\end{aligned}$$

and the result is established by induction.

Solution II *by A. A. Jagers, Technische Hogeschool Twente, Enschede, The Netherlands.* For $a > 0$ let B_a be the "block function" defined by

$$B_a(t) = \begin{cases} 1 & |t| < a, \\ \dfrac{1}{2} & |t| = a, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$2\frac{\sin as}{s} = \int_{-\infty}^{\infty} e^{ist} B_a(t)\, dt. \qquad (*)$$

By Fourier inversion (since $B_a(t)$ is even),

$$\pi B_a(t) = \int_{-\infty}^{\infty} e^{ist} \frac{\sin as}{s}\, ds. \qquad (**)$$

Substitute $(*)$ in the given multiple integral with $a = a_0$ and $s = x_1 + x_2 + \cdots + x_k$. Associate each exponential factor $\exp(ix_j t)$ with the corresponding $\sin a_j x_j / x_j$, and then use $(**)$ k times with $a = a_j$, $1 \leqslant j \leqslant k$. This reduces the multiple integral to

$$\frac{1}{2}\pi^k \int_{-\infty}^{\infty} \left(\prod_{j=0}^{k} B_{a_j}(t) \right) dt = \pi^k \min\{a_0, \ldots, a_k\}.$$

The proposer uses Fourier inversion to show, for $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, that

$$S = \int dx_1 \cdots \int dx_n \prod_{j=1}^{n} \frac{\sin a_j x_j}{x_j} f(x_1 + \cdots + x_n) = 2\pi^{n-1} \int_0^{\infty} dx \frac{\sin a_1 x}{x} f(x)$$

for a wide class of f; he has similar formulae for

$$C = \int dx_1 \cdots \int dx_n \prod_{j=1}^{n} \frac{1 - \cos a_j x_j}{x_j} f(x_1 + \cdots + x_n).$$

Thomas Delmer calls our attention to another formula of a similar nature,

$$\int dx_1 \cdots \int dx_k \frac{a_1}{\pi(x_1^2 + a_1^2)} \cdots \frac{a_k}{\pi(x_k^2 + a_k^2)} \frac{a_0}{\pi\left[a_0^2 + (x_1 + \cdots + x_k)^2\right]}$$
$$= \frac{1}{\pi(a_0 + \cdots + a_k)}.$$

Also solved by John A. Crow, Thomas Delmer, J. A. Grzesik, Victor Hernández (Spain), S. V. Kanetkar, Ignacy I. Kotlarski, L. Kuipers (Switzerland) and P. Szüsz (jointly), Kee-wai Lau (Hong Kong), I. E. Leonard, N. J. Lord (England), O. P. Lossers (The Netherlands), Syrous Marivani, Norbert Ortner (Austria), Lajos Takács, Pei Yuan Wu (Taiwan), and the proposer.

REVIEWS

EDITED BY ALAN L. EDMONDS AND JOHN H. EWING

Real Linear Algebra. By Antal E. Fekete. Marcel Dekker, 1985, xxi + 376.

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"Oh, no! Another undergraduate linear algebra text!" If you've been on a committee to pick a text, you undoubtedly understand that reaction. But Fekete would probably respond that his book is not a standard linear algebra text. He wrote it to fit into Steenrod's program to include a lot of geometric content in lower level mathematics courses (see N. Steenrod, "The Geometric Content of Freshman and Sophomore Mathematical Courses," in the Report of the Committee on Undergraduate Program in Mathematics, No. 17, Washington, DC: MAA, September, 1967). The "Real" in the title refers to the real numbers. In fact, most of the book deals with linear algebra in $\mathbb{R}^3$. In particular, Fekete stresses geometric meaning rather than matrix manipulations. This sounds good, but let's investigate the realities of teaching linear algebra today.

At most large schools there are different types of undergraduate linear algebra courses offered, i.e., for engineering students (with or without a calculus prerequisite), for mathematics and physics majors (proofs expected), or for general students (no calculus prerequisite). At small schools, usually only one course is offered and tries to be all things to all students.

Despite the various backgrounds and interests of the audience, all students need to learn the basic concepts (vector space, subspace, basis, linear transformation, etc.) and gain some competence in computation. Furthermore, there needs to be a clear distinction made between the mathematical concepts and the matrix manipula-

tions used to do the computations. Too many students think that matrix algebra and linear algebra are the same; they do not learn the most common and useful linear (or vector) spaces such as spaces of continuous or differentiable functions.

Although all students need to learn the same topics, the depth of comprehension and the applications depend on the audience. For example, mathematics and physics majors need to learn to write many simple proofs at this stage in their education; majors in engineering and other subjects do not. For the latter group, I give many applications chosen each term with the majors of that particular class in mind. Since I have never taught a class that mixed the two types, I can only speculate that I would try to give such a class some proofs and some applications.

Once one has decided on the content of the linear algebra course for a given audience, it is time to choose a book that covers all (or most) of the desired topics at the desired level. This is often hard to do. For example, consider a course for junior or senior engineers and computer science majors. They have had multivariate calculus and often have had a course in differential equations. The engineering college (rightly) wants applications included. But most of the books that contain appropriate applications do not assume a knowledge of calculus. Furthermore, they start off with matrices, systems of linear equations, determinants and vectors in 2- and 3-space, which the engineering students find very easy. They are then horrified when they get to the definition of a vector space and the course becomes difficult (usually after the last day to drop it). This order also gives them the idea that matrices are linear algebra which, as noted above, is not desirable.

For engineering students, I'd like to see a book which contains many applications and which starts off with the general definition of a vector space, introduces matrices as an example of a vector space, and teaches row reduction when it is needed. Thus, matrices and matrix manipulations are seen to be the tools and not the intrinsic objects.

There is a similar problem with texts for mathematics majors. What is needed is a book that introduces proofs, yet gets to decomposition of linear operators fast enough for a one-semester course and has lots of examples. Unfortunatly, Finkbeiner and Nomizu are out of print. Fekete might possibly work if one skipped the chapters on cross, dot and box products—i.e., start at Chapter 6—except that there is no material defining a vector space, a subspace, linear dependence in $\mathbb{R}^n$, $n > 3$, basis, or dimension.

This last omission is my biggest complaint with Fekete. Since he works in $\mathbb{R}^3$, he does not define the standard linear algebra concepts except in the special case of $\mathbb{R}^3$ and not always then. For example, the dimension of a vector space is never defined. It has a geometric meaning, but he uses, without comment, its algebraic equivalent. He also assumes knowledge of a great deal of plane, solid, and analytic geometry, which today's students do not learn in high school. Thus his motivating examples are new mathematics to the students.

However, he has many nice geometric applications of dot and cross product, which I plan to use when teaching third semester calculus. There are also a lot of

examples of 3×3 orthogonal matrices with rational coefficients, and I found his treatment of the classification of linear operators over $\mathbb{R}^3$ (and comments on operators over $\mathbb{R}^n$) refreshing.

Thus, while some of the book may be useful in various courses, it would not be appropriate as a text for the current linear algebra courses without redoing the entire lower level curriculum. Unfortunately, despite the plethora of linear algebra texts, I still have not found one to my liking for the courses I teach. I wish the publishers would realize that no one book can be used for the different types of linear algebra courses, and start publishing texts that correspond to the needs of the various audiences.

The Beauty of Doing Mathematics. By Serge Lang. Springer-Verlag, New York, 1985. xi + 127 pp.

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The lectures. Many of us have faced the problem of having to give a lecture on mathematics to a lay audience, sometimes called a "general" audience, and, quite probably, some of us have decided that the problem is unsolvable. The difficulty of popularizing the subject is great enough if the audience is homogeneous (high school juniors, graduate students in botany, or university administrators); the very idea of trying to explain current research on C^* algebras to a mixed group consisting of all these kinds of people and of others of unknown and unpredictable kinds fills me with fear and horror. If I knew that they came with their eyes open, free and willing, that would help—but not enough to make me volunteer.

Serge Lang is made of sterner stuff. The book under review is a transcribed and translated version of three lectures he gave to three thoroughly mixed audiences "ranging from young high school... students, to retired people, housewives, engineers, or just plain curious people". His intent was "to show what pure mathematics is by examples: by doing mathematics with the people in the audience". The mathematics, moreover, was to be "not artificial or superficial mathematics, but real mathematics, recognized as such by real mathematicians who do research...".

The times were Saturday afternoons in the middle of May, one each in 1981, 1982, and 1983; the place was the Palais de la Découverte in Paris; and the subjects were impressive indeed. The first lecture was about the twin prime conjecture, the prime number theorem, and the Riemann hypothesis, with an excursion into the uncountability of the real line. The second lecture was about the Fermat problem and Mordell's conjecture, with an excursion into undecidable propositions and the continuum hypothesis. The third lecture (which, with a couple of short intermis-

sions, went on for three hours) was about the classification of two-dimensional manifolds, the Poincaré conjecture, and Thurston's work on three-dimensional manifolds and their relation to hyperbolic geometry.

The lectures were conducted in a participatory manner, and either for that reason or, possibly unintentionally, as translation via a misleading cognate, they are referred to as conferences. (The subtitle of the book, by the way, is *Three Public Dialogues*.) The lecturer tried, apparently successfully, to draw the audience out. Riffling through the pages you get the idea that you are reading a play in which, to be sure, one character has a much bigger part than the others, but which is definitely not a soliloquy. The principal character is always called SERGE LANG; his full name appears several times on almost every page, alternating with GENTLEMAN, LADY, YOUNG MAN, SOMEONE, and SEVERAL VOICES IN THE AUDIENCE.

The asides. Interspersed with prime numbers and manifolds there are discourses on many of the usual things that mathematicians themselves like to talk about, such as: why do mathematicians do mathematics?, is pure mathematics useful?, and is mathematics properly taught nowadays?

Early on, for instance, there is the following quotation from André Weil. "According to Plutarch, it is a noble ideal to work to make one's name immortal. Ever since I was young, I hoped that my work would have a certain place in the history of mathematics. Is that not a motivation as noble as to try to get a Nobel prize?"

Lang doesn't quite agree. Says he: "I think rather that one does mathematics because one likes to do this sort of thing, and also, much more naturally, because when you have a talent for something, usually you don't have any talent for something else, and you do whatever you have talent for, if you are lucky enough to have it."

The pure versus applied debate keeps coming up, sometimes in answer to questions from the audience, and sometimes spontaneously, self-defensively. In the discussion period following the first lecture, for instance, Lang quotes a famous paragraph of von Neumann's, and then expresses his disagreement with it. Here are a few sentences from the von Neumann paragraph. "As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from 'reality', it is beset with very grave dangers.... At a great distance from its empirical source, or after much 'abstract' inbreeding, a mathematical subject is in danger of degeneration." Says Lang: "I don't feel any danger about doing mathematics for which I see no relation with the empirical world.... As for 'inbreeding', I don't understand what von Neumann means. Many of the most beautiful discoveries in mathematics come from the wedding of branches which a priori seem very far apart from each other. One of the characteristics of mathematical genius is the ability to bring together different branches, by what could be called 'inbreeding', or to bring together threads going off into many directions."

I think it is one of my privileges as reviewer, and perhaps even one of my duties, to record my own opinion, which is that Lang is wide of the mark—I think he misunderstands von Neumann. Immediately after warning of the dangers of mathematics becoming "more and more purely *l'art pour l'art*" von Neumann goes on to say that "this need not be bad... if the discipline is under the influence of men with an exceptionally well developed taste." He never says that the bringing together of different branches is inbreeding, is bad—that would be nonsense. If a mathematical structure brings together and intertwines algebra and geometry and analysis, it is becoming richer and not, like the kind of inbreeding that von Neumann means, more threadbare and artificial.

What von Neumann chooses not to say is that the mathematics that comes from reality can turn out to be ugly and soon forgotten, as the needs of reality change and new techniques for dealing with new problems are discovered. Similarly, of course, the mathematics that comes from the "baroque" inbreeding of the work of people with bad taste is often ugly and doomed to quick oblivion. The instinctive good taste of the geniuses who led the way is needed to guide the rest of us, and when we don't let it, we get generalization for the sake of generalization. We get what I have heard Zygmund call centipede mathematics (remove 99 legs and see what it can do); we get the elaborate elaborations of Baire category theory in fuzzy topological semigroups; we get inbreeding in the sense in which I believe von Neumann meant us to understand the word. I'd be willing to bet that Lang thinks as little of such inbreeding as von Neumann does. Lang, and I am guessing here, reacted strongly, emotionally, against von Neumann, because he didn't want to be told that he will be allowed to like what he likes only when its genealogy is "real".

Several discussions of teaching are begun by members of the audience. A high school student asks: "Do you think that mathematics should be taught... just for the beauty of it and not for applications to physics, or that at least until the end of high school, they should be turned towards physics, toward applications? SERGE LANG: "The way you phrase the question is too... exclusive. One does not prevent the other. It's obvious that the negation of one extreme does not imply an extreme on the opposite side. Do what... comes naturally. Of course, there should be applications when teaching mathematics. But from time to time, you must also be able to say: OK, let's look at $x^2 + y^2 = 1$ and let's find all the rational solutions."

A little later a physicist speaks. "It seems that in French schools, the main reason for the heavy-handedness and lack of understanding is that, behind the whole program, one tries to show, even to very young children, a logical construction which is completely irrefutable. Whether it is in physics or mathematics, a teacher can never allow himself to assert something without giving a clear proof of it." SERGE LANG: "I entirely agree with this evaluation, and I deplore it as much as you do. It is true that the textbooks tend toward a certain aridity and are pedantic."

The style. The style of the exposition is visibly oral—it often gives the impression of having been "dictated but not read" and of having been typed but not edited. The lectures (in French, of course) were taped, and the written version was

translated to English by the lecturer himself. The result is extremely informal, often bordering on the embarrassing. I don't want to belabor the point, but a few examples are in order.

"No, I don't mean finding millions of them, I mean prove that the sequence of prime numbers doesn't stop." [*Brouhaha in the audience* . . . .] (p. 7).

"Just before the conference, I was looking at a tenth grade textbook . . . and it's to vomit." (p. 31).

"I asked people what mathematics meant to them. One lady told me: 'It's to work with numbers.' Well, those answers are for the birds." (p. 73).

"I would like mathematicians not to intervene, because otherwise it's cheating. [*Laughter*.] Of course, mathematicians know the answer, but I am not giving this conference for them. [*Serge Lang throws the chalk at the gentleman. Laughter*.] (p. 85).

Several of the lecturer's comments that are followed by [*Laughter*.] are repeated several times. "Cheating" by mathematicians is one of them. Another one is this:

> "Who says it's equivalent . . . ?
> "[*Some hands go up*.]
> "Who says it's not equivalent?
> "[*Other hands go up*.]
> "Who keeps prudent silence? [*Laughter*.]" (p. 76).

Except for a few bobbles, the English is almost always good. There are several misprints (has there ever been a book without them?), but not an outrageous number; the book is well printed and looks attractive.

The level. The mathematical sophistication assumed on the part of the audience varies quite a lot. Lang does not assume (the audience makes it plain that he cannot assume) that everybody knows about logarithms, or even graphs. He explains graphs and quite frankly gives up on logarithms: "OK, there is something that's called the logarithm. It is denoted by $\log x$. You will find it on all the little hand calculators in the drug stores. I don't have time to explain it in greater detail." There is nothing wrong with that—we all have to do things like that in public lectures—but what is curious is how high the level gets at other times (with the same audience). Example: "Let $\pi_2(x)$ denote the number of twin primes $\leqslant x$. Then Hardy-Littlewood's conjecture is that $\pi_2(x) \sim (e^{\gamma})^2 F_2(x)x$."

Another indication of the level of the book is that it has two professional looking lists of somewhat intimidating references: they contain the names of authors such as Diophantus, Diderot, Hardy and Littlewood, Barry Mazur, and Carl Ludwig Siegel.

Here are a few lines from the second hour of the third lecture. "Take all compact, three-dimensional manifolds, without holes and without boundary. Can you describe all of them? The problem is unsolved. Poincaré's conjecture is that a three-dimensional manifold, compact, without holes, without boundary is equivalent to the sphere S^3." A little more than halfway into the third hour of the third lecture the following theorem appears. "Let F be a surface, compact, orientable, without

boundary, and not equivalent to the sphere or to the torus. Then there exists a discrete group Γ such that the surface F is equivalent to the hyperbolic plane on which we have identified points with respect to Γ. In other words,

$$F \sim \Gamma \backslash H^2."$$

Thurston's work is described as a partial generalization of this result about the 2-dimensional case to 3-manifolds.

The conclusion. I have been trying to describe the flavor and the level of the book. My own judgment is that the flavor is excellent, that the mathematics is certainly "real mathematics, recognized as such by real mathematicians", but that the level is inconsistent and, I think, unrealistic. The book would be very hard reading for anyone who doesn't know what a graph is or what a logarithm is, and, for that matter, I know many college teachers of mathematics who would find it too hard. That is not to say that they might not enjoy it. The author's sparkling and vivacious personality is almost certainly the feature that made the oral presentations the success they seem to have been, and that personality comes across, toned down, even in writing.

I wish I could have heard the live presentation. The book commits one well-known expository error: it tries to say too much, it ignores the dictum that less is more. One of the greatest techniques of clear exposition is omission; every speaker and writer must know what to leave out, or learn very soon if he has aspirations of being understood. The book doesn't leave out enough. Quite possibly the too much that's in the book didn't seem like too much at all to those who were part of the act and felt the bubbly enthusiasm that was on the stage before them.

My conclusion is that we have before us a noble attempt, but that it doesn't quite come off.

Superior Beings: If They Exist, How Would We Know? By Steven J. Brams. Springer-Verlag, New York, 1983. xix + 202 pp. $21.95, paper $11.95.

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We are currently in the midst of several revolutions in the mathematical sciences. Most conspicuous is the technological progress on digital computers and its general impact on mathematics. Accompanying this are many advances in discrete mathematics, and its synthesis, which is giving rise to a new abstract algebra, that may prove to be just as rich and probably more useful than traditional modern algebra. In geometry we have achieved profound new insights into higher-dimensional space,

and these will no doubt find applications to inherently multidimensional new fields such as operations research and in large scale computation.

Another revolution that is perhaps less obvious is the successful use of mathematics as a fundamental tool to study human beings: their behavior, values, interactions, conflicts, organizations, fair allocations, and decision making, as well as their interface with modern technology and systems. This latter revolution could, in time, prove to be just as influential as the turning of mathematics towards the study of physical objects and their dynamics some three centuries ago. Mathematics, as well as computers, may well become a major ingredient in social activity, management, and policy making. An indication of the extent to which the science of mathematics has penetrated into human activity is the suggestion by Rapaport (final chapter in [3]) that there should be a *profession* devoted to thinking mathematically about human affairs. And history shows that once mathematics enters a subject, it rarely withdraws or diminishes in use—even perhaps when it should, as with astrology.

The revolution in using mathematics to study human activities, along with other contemporary changes, should have great impact on the educational position of mathematics as well as in research directions and applications. It will surely allow for fresh combinations in teaching, learning, and research as envisioned by philosopher A. N. Whitehead, in which mathematics will play a major connecting role. In short, one will be able to bridge the great gulf between the two cultures of Baron C. P. Snow without the formidable barriers of prerequisite knowledge of the likes of physics and partial differential equations. When great educators like Morris Kline write about the need for applications in the teaching of mathematics, he, like Snow, clearly refers to the physical sciences and their use in classical engineering and technology. Other commentators and reviewers look toward recent mathematical advances with a sort of anticipation of possibly reliving those great success stories of the past, involving mathematics with the physical sciences. Other outstanding and innovative teachers have gently reminded us [5] that we must not forget yesterday's "hard" applied mathematics, while we bring in the new topics of the present. The point is that there are now a large number of new directions that can also be used to motivate students who may not get excited about the detailed workings of the physical world. This in no way diminishes the current value or past achievements of traditional applications, which led many of us into the study of mathematics.

But just how far will this movement of mathematics into human affairs go? Can it possibly extend to subjects such as literary criticism, ethics, theology, and individuals' personal relationships with God? First, Professor Brams gave us *Biblical Games: A Strategic Analysis of Stories in the Old Testament* (MIT Press, 1980), in which he used elementary matrix game theory to model several encounters between God and humans. Some people seem to get upset at the thought of using mathematics to analyze aspects of God or religion. See, for example, the exchange in *Mathematics Magazine* [1] related to this latter book. The author has followed with the book under review and even more recently, with still another one [2] that returns to mere human conflict. Is he really serious?

There are 78 different 2 by 2 general-sum (bimatrix) games when the payoffs are ordinal preferences [4]. It is well known that a few of these games lead to rather "parodoxical" situations when played between ordinary people. (The social scientists often use the term "parodox" to mean surprise or counterintuitive, rather than impossible as in logic.) These include the famous "prisoners' dilemma" described by A. W. Tucker in 1950, which is behind many apparently irrational escalations such as price wars, arms races, and the "tragedy of the commons". It also includes the game of chicken in which the two contestants are racing towards a head-on collision, and the one who swerves first "loses". This book analyzes in great detail several of the more interesting games from the class of 78 when played between an ordinary person (P) and a supreme being (SB), who may be endowed with capabilities such as omniscience, omnipotence, immortality, and incomprehensibility. Several intriguing questions and new parodoxes arise.

For example, in the revelation game SB can either reveal Himself R or not $\bar{R}$, and P can select to either believe in SB's existence B or not believe $\bar{B}$. In the knowability game SB can elect to be knowable or unknowable; whereas P can choose between investigating and expecting SB to be knowable, or else not investigate believing SB will be unknowable. Different games result depending upon how these two players rank order the resulting outcomes. The resulting ordinal playoffs in the revelation game may well be according to the following table where the first entry in each payoff pair is for the row player SB and the second is for the column play P. 4 is the most preferred payoff and 1 is the least desired one.

		P: B	P: $\bar{B}$
SB	R	(3, 4)	(1, 1)
SB	$\bar{R}$	(4, 2)	(2, 3)

The outcome (2, 3) is a Nash equilibrium in the sense that neither player can deviate from it unilaterally without doing worse. However, it is dominated by the more unstable payoff (3, 4) realized by the strategy pair (R, B).

In another case, one arrives at a parodox of omniscience when P is involved in a game of chicken with SB (page 69). If P is aware that SB does in fact know of P's ultimate decision in advance, then P can take advantage of this to obtain his highest payoff while SB must "swerve" at the last moment to avoid mutual disaster. But can P be sure of this? As a current bumper sticker says: God is coming, and is She mad! As Brams discusses in Chapter 6, SB may be incomprehensible (does not reveal His preferences and uses randomized strategies) and immortal (so willing to optimize His payoff over repeated games).

The final chapter discusses SB's undecidability.

Time will reveal whether the author's approach will prove valuable in studying the great questions of theology and philosophy. On the other hand, this book, as well as others by Brams, is most enjoyable to read. He has become a leading

popularizer of the interesting subject of game theory. His works can be highly recommended as collateral reading for introductory courses on mathematical modeling in the social, managerial and decision sciences—now perhaps even in theology. Surely the mathematics curriculum could benefit from an additional dose of "fun" that his books provide.

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TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatabilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, L. *The Joy of Science: Excellence and Its Rewards.* Carl J. Sindermann. Plenum Pr, 1985, xvii + 259 pp, $16.95. [ISBN: 0-306-42035-X] A fluffy popular description of successful careers of typical good scientists, mixing fictionalized accounts of individuals with lists of "how-to" advice for success in scientific careers—in research, teaching, administration, and government. Bits of true insight do occur, but form a nowhere dense subset of the whole. LAS

Mathematics Appreciation, T(13: 1). *Fundamentals of Mathematics, Fourth Edition.* William M. Setek, Jr. Macmillan, 1986, xv + 720 pp, $24. [ISBN: 0-02-409200-2] Introduces sets, logic, probability, statistics, metric system, mathematical systems, systems of numeration, sets of numbers, algebra, geometry, consumer mathematics and computers. Changes in this edition include the addition of a few topics and a glossary of terms. *Student Study Guide, Instructor's Manual* and *Test Package* are available. (*First Edition*, TR, November 1976.) JNC

Precalculus, T(13: 1). *Algebra and Trigonometry, Second Edition.* Marshall D. Hestenes, Richard O. Hill, Jr. Prentice-Hall, 1986, xiv + 626 pp, $30.95. [ISBN: 0-13-021866-9] Previously published as *Algebra and Trigonometry with Calculators* (TR, January 1982). *Second Edition* is more flexible regarding the use of calculators. Other changes include expanded exercise sets, restructured chapters, simplified and/or rewritten approaches, better exercise grading. Prerequisites are one and one-half years of high school algebra or geometry. Very readable. Attractive. Tables are included. JK

Precalculus, T(13: 1). *Algebra for College Students, Second Edition.* Bernard Kolman, Arnold Shapiro. Academic Pr, 1986, xiii + 643 pp, $23. [ISBN: 0-12-417900-2] Informal. Intuitive. Designed to encourage students to read the text and develop confidence. With progress checks and warnings. Review exercises with answers to many eliminate the need for a separate study guide. Some emphasis on calculator use, where appropriate. JK

Precalculus, T(13: 1). *Mathematics for Computer Technology.* Paul Calter. Ser. in Tech. Math. Prentice-Hall, 1986, xvi + 672 pp, $29.95. [ISBN: 0-13-562190-9] A technical mathematics text covering traditional topics from an algorithmic perspective; includes computer-related topics (e.g., numerical methods for solving equations, computer methods for evaluating determinants and solving systems of equations, binary, hexadecimal and octal numbers, Boolean algebra); programming exercises with complete programs in the answer key and on diskette. JNC

Education, P. *Professional Development for Teachers of Mathematics, A Handbook.* Ed: Ross Taylor. NCTM, 1986, vi + 66 pp, $7.50 (P). [ISBN: 0-87353-231-7] A detailed checklist of mechanics for conducting professional development activities ("Provide plenty of coffee, including decaffeinated...") emphasizing planning support, implementation and evaluation with very little discussion of mathematics itself or even of mathematics curriculum. LAS

History, P, L*. *IBM's Early Computers.* Charles J. Bashe, et al. MIT Pr, 1986, xviii + 716 pp, $27.50. [ISBN: 0-262-02225-7] A technological history of IBM's early computers, focussing on scientific and engineering issues, from Hollerith cards to transitors and ferrite core memories. Concludes with the opening in 1961 of the Watson Research Laboratory in Yorktown Heights, New York. A superb account of the people, issues, and scientific forces that shaped the early computer era. LAS

History, S, L*.** *Hilbert-Courant.* Constance Reid. Springer-Verlag, 1986, xv + 547 pp, $28 (P). [ISBN: 0-387-96256-5] Combined paperback edition of Reid's 1970 *Hilbert* (TR, June-July 1970; Extended Review, May 1971) and 1976 *Courant in Göttingen and New York* (TR, February 1977) with an integrated index of names and two albums of photographs from Königsberg, Göttingen, and New York. A superb account of the closely linked intellectual biographies of two giants who helped create modern mathematics. No one who teaches mathematics should fail to read this two-part classic. LAS

History, P, L*. *Intellectual Mastery of Nature: Theoretical Physics from Ohm to Einstein.* Christa Jungnickel, Russell McCormmach. U of Chicago Pr, 1986. *V. 1, The Torch of Mathematics 1800-1870*, xxiii + 350 pp, $55 [ISBN: 0-226-41581-3]; *Volume 2: The Now Mighty Theoretical Physics 1870-1925*, xvii + 435 pp, $65. [ISBN: 0-226-41584-8] A detailed study of the social, institutional, and political climate of physics in ninetenth century German universities that gave birth both to the discipline of theoretical physics and to its sponsor, the modern research institute. The theme of *Volume 1* is taken from Georg Ohm's unappreciated introduction of theoretical methods (the "torch of mathematics") in his theory of galvanic currents; the theme of *Volume 2* comes from Nobel laureate Wilhelm Wien's articulation around 1915 of how "the now mighty theoretical physicist" had joined experimental physics in a "higher unity." These two volumes provide a valuable contribution to the sociology of science and to the early history of mathematical physics. LAS

History, P, L. *The Mathematicians' Apprenticeship: Science, Universities and Society in England, 1560-1640.* Mordechai Feingold. Cambridge U Pr, 1984, viii + 248 pp, $39.50. [ISBN: 0-521-25133-8] A detailed description of the mathematical sciences (mathematics, astronomy, astrology) during "the Puritan hegemony" of pre-Newtonian England. Two theses dominate the work: that contrary to common belief, the two universities at Cambridge and Oxford contributed as much as did London to the emergence of mathematical science in England, and that the incubation provided by this Elizabethian period was an indispensible "mathematicians' apprenticeship" for the scientific ferment that was to follow. LAS

History, P, L*. *Writings of Charles S. Peirce: A Chronological Edition, V. 3, 1872-1878.* Ed: Christian J.W. Kloesel. Indiana U Pr, 1986, xxxvii + 633 pp, $40. [ISBN: 0-253-37203-8] 69 items, including 49 never before published, in this continuing chronological edition of Peirce. Includes his 1877-78 popular series "Illustrations of the Logic of Science," the first published formulation of pragmatism, as well as parts of his photometric research. Also includes manuscripts on the foundations of logic, geometry, algebra, non-associative multiplication, and Grassman's calculus of extension. (*V. 1*, TR, February 1983; *V. 2*, TR, February 1985.) LAS

Logic, P. *Selected Essays.* Jean Van Heijenoort. History of Logic III. Humanities Pr, 1985, 166 pp, $29.95. [ISBN: 88-7088-122-9] A collection of the author's short papers on logic emphasizing local rather than global frames of reference, followed by two more extensive but distinct papers: a survey of Jacques Herbrand's work in logic, and a devastating critique of Friedrich Engel's ignorance of and prejudice towards mathematics in the context of his development (with Karl Marx) of dialectical materialism. LAS

Logic, S*(15-18), P, L*.** *Philosophy of Logic, Second Edition.* W.V. Quine. Harvard U Pr, 1986, x + 109 pp, $6.95 (P). [ISBN: 0-674-66563-5] Revision of the brief introduction to the philosophy of deductive logic by the pre-eminent person in the field. Focus is on truth and grammar, the components of logic: logical truths are not true because of grammar or language. Set theory compared and contrasted with logic; alternative logics discussed. Important for all interested in philosophy and logic. RB

Foundations, P*, L*. *Harvey Friedman's Research on the Foundations of Mathematics.* Ed: L.A. Harrington, et al. Stud. in Logic & Found. of Math., V. 117. Elsevier Science, 1985, xvi + 408 pp, $55.50. [ISBN: 0-444-87834-3] A collection of papers celebrating the work of Harvey Friedman in consequence of his 1984 Waterman Award as "the outstanding young scientist in the United States." Survey papers touch on Friedman's astonishing contributions to set theory (especially Borel structures), computational complexity, intuitionism, proof theory, and recursion theory. Several papers are reprints of expository articles; others appear here for the first time. LAS

Foundations, P. *The Ethnomethodological Foundations of Mathematics.* Eric Livingston. Routledge & Kegan Paul, 1986, xiii + 241 pp, $49.95. [ISBN: 0-7102-0335-7] An approach to understanding the nature of mathematical proof by examining the "local production of social order" among mathematicians. The major part of the book consists of an extensive analysis of a proof of Gödel's incompleteness theorem, curiously devoid of any analysis of mathematicians' "social order." Extensive end notes, but no index. Little evidence of contact with significant mathematics or mathematicians beyond the realm of Gödel's theorem. LAS

Combinatorics, T(17: 1), S, P, L. *Theory of Matroids.* Ed: Neil White. Ency. of Math. & Its Applic. Cambridge U Pr, 1986, xvii + 316 pp, $39.50. [ISBN: 0-521-30937-9] A primer in the basic axioms and constructions of matroids. Contributions by leaders in

the field include chapters on axiom systems, lattices, basic exchange properties, orthogonality, graphs and networks, constructions, maps, semimodular functions, and an appendix on cryptomorphisms. Exercises and references are included for each topic. CEC

Combinatorics, P. *Matroid Theory.* Ed: L. Lovász, A. Recski. Elsevier Science, 1985, 439 pp, $74. [ISBN: 0-444-87580-8] A collection of sixteen papers presented at the Matroid Theory Colloquium held in Szeged, August 29-September 4, 1982. Includes program, list of participants. JS

Combinatorics, S(18), P. *The Linear Ordering Problem: Algorithms and Applications.* G. Reinelt. Res. & Expos. in Math., V. 8. Heldermann Verlag, 1985, xi + 158 pp, $38 (P). [ISBN: 3-88538-208-3] The central purpose here is the development of an algorithm, using the methods of polyhedral combinatorics, for the linear ordering problem. This is done in Chapter 3 and uses both cutting plane and branch and bound techniques. The remaining four chapters deal with applications. References, index, and tables. JS

Discrete Mathematics, T*(13-14: 1, 2), S, L. *Discrete Mathematics.* Paul F. Dierker, William L. Voxman. Harcourt Brace Jovanovich, 1986, xii + 589 pp, $35.95. [ISBN: 0-15-517691-9] A well-written introduction to discrete mathematics at the freshman-sophomore level. Chapter titles include algorithms, number systems, graph theory, boolean algebra, symbolic logic, difference equations, enumeration, probability, generating functions and automata. Lots of examples and exercises. CEC

Number Theory, P*. *The Selberg Trace Formula and Related Topics.* Ed: Dennis A. Hejhal, Peter Sarnak, Audrey Anne Terras. Contemp. Math., V. 53. AMS, 1986, ix + 554 pp, $42 (P). [ISBN: 0-8218-5058-X] In 1956, Atle Selberg introduced the "trace formula," a kind of super-Poisson-summation for Riemannian spaces. The profound implications of Selberg's ideas for number theory, automorphic forms, and group representations are still being explored, as shown by 28 papers from a summer conference at Bowdoin College in 1984. BC

Linear Algebra, T*(16-17: 1, 2), L*. *Matrix Analysis.* Roger A. Horn, Charles R. Johnson. Cambridge U Pr, 1985, xiii + 561 pp, $37.50. [ISBN: 0-521-30586-1] For a second course in linear algebra. Emphasizes topics important in applications. Among the special topics covered are Hermitian and complex symmetric matrices, eigenvalue location and perturbation theory, positive definite matrices, and component-wise non-negative matrices. AO

Group Theory, P. *Lecture Notes in Mathematics-1185: Group Theory, Beijing 1984.* Ed: Tuan Hsio-Fu. Springer-Verlag, 1986, v + 403 pp, $32.50 (P). [ISBN: 0-387-16456-1] Proceedings of an international symposium on group theory: eight eminent Western algebraists were invited to join numerous Chinese mathematicians in exchanges on finite groups and their connections with combinatorics, classical groups, algebraic groups and Lie groups (algebraic connections emphasized). Seven expository/survey contributions by invitees and nine papers by Chinese participants. RB

Group Theory, S(18), P. *Lecture Notes in Mathematics-1179: The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups.* Shi Jian-Yi. Springer-Verlag, 1986, x + 307 pp, $25. [ISBN: 0-387-16439-1] First three chapters are a summary of the Kazhdan-Lusztig theory of cells for a Coxeter group with the remainder of the book presenting original work of the author on affine Weyl groups. References, index. JS

Algebra, T(17: 1), S, P. *Lecture Notes in Mathematics-1175: Valuations of Skew Fields and Projective Hjelmslev Spaces.* Karl Mathiak. Springer-Verlag, 1986, vii + 116 pp, $11.60 (P). [ISBN: 0-387-16099-X] A fairly self-contained account of the theory of valuations of skew fields and their application to projective Hjelmslev spaces. Assumes basic notions of algebra, topology, and geometry. Includes exercises and a list of references. CEC

Algebra, P. *Group Rings, Crossed Products and Galois Theory.* Donald S. Passman. CBMS Reg. Conf. Ser. in Math., No. 64. AMS, 1986, viii + 71 pp, $12 (P). [ISBN: 0-8218-0714-5] Ten expository lectures emphasizing fairly recent results in these three fields, delivered at the CBMS Regional Conference, Mankato State University (Minnesota), June 1985. Group rings involving polycyclic-by-finite groups/Noetherian rings, arbitrary or infinite groups; crossed products and Galois theory of rings, group algebras; Galois theory of non-commutative rings. Summary, with few proofs. RB

Algebra, P. *Lecture Notes in Mathematics-1177: Representation Theory I, Finite Dimensional Algebras.* Ed: V. Dlab, P. Gabriel, G. Michler. Springer-Verlag, 1986, xv + 340 pp, $28.80 (P) [ISBN: 0-387-16432-4]; *Lecture Notes in Mathematics-1178: Representation Theory II, Groups and Orders.* xv + 370 pp, $28.80 (P). [ISBN: 0-387-16433-2] Proceedings of the international conference held August 16-25, 1984 at Carleton University in Ottawa. The second volume includes a recent comprehensive bibliography of representations of algebras covering the period 1979-1984. LAS

Calculus, T(13: 1, 2). *Technical Calculus, Second Edition.* Dale Ewen, Michael A. Topper. Ser. in Tech. Math. Prentice-Hall, 1986, xii + 564 pp, $29.95. [ISBN: 0-13-898164-7] Intuitive treatment. For students in engineering technology programs. Algebra and trigonometry are prerequisites. Changes

from *First Edition* (TR, May 1977) include several additional topics and exercises plus some realignment of material. JK

Calculus, P*, L. *Rethinking Mathematical Concepts.* Roger F. Wheeler. Ser. in Math. & Its Applic. Halsted Pr, 1981, 314 pp, $29.95 (P). [ISBN: 0-470-27116-7] A very helpful analysis of the use of mathematical language and of its impact on students who are learning to use this language. The focus is on calculus, but the examples range over much of the secondary and tertiary curricula. Mixes style (like Strunk and White) and pedantry (like R. Mitchell's *Underground Grammarian*) with sound advice on good teaching: makes one think about what mathematical language really says. Reprint of 1981 hardcover original (TR, March 1982). LAS

Calculus, T(13: 2). *Calculus of One Variable, Second Edition.* Stanley I. Grossman. Academic Pr, 1986, xx + 884 pp, $30. [ISBN: 0-12-304390-5] A reprinting of Chapters 1-14 of *Calculus, Third Edition* (TR, January 1985) omitting the chapters on vectors, multivariable calculus, and differential equations. JNC

Complex Analysis, P*. *New Constructions of Functions Holomorphic in the Unit Ball of C^n.* Walter Rudin. CBMS Reg. Conf. Ser. in Math., No. 63. AMS, 1986, xvi + 78 pp, $13 (P). [ISBN: 0-8218-0713-7] Notes from a series of expository lectures by the author at a 1985 CBMS Regional Conference at Michigan State University. Concerns methods and consequences of recent positive solutions, by Alexandrov and Low separately, of long-standing "existence of inner functions" problem. With a list of open problems. Lucid and readable. PZ

Complex Analysis, S(16-18), P*, L. *Kleinian Groups and Uniformization in Examples and Problems.* S.L. Krushkal, B.N. Apanasov, N.A. Gusevskiĭ. Transl. of Math. Mono., V. 62. AMS, 1986, vii + 198 pp, $66. [ISBN: 0-8218-4516-0] A unique, unified exposition of all main areas and methods in Kleinian groups and uniformization of manifolds, emphasizing examples, problems and unsolved problems (some new). The researcher, graduate student, or talented undergraduate is introduced quickly to contemporary problems. References to primary sources instead of proofs. 40-page exercise section; extensive bibliography. RB

Differential Equations, T(14-15: 1). *An Introduction to Differential Equations and Their Applications.* Stephen L. Campbell. Longman, 1986, x + 563 pp, $34.95. [ISBN: 0-582-98840-3] Flexible treatment of standard fare in a first course. Writing reflects author's concern for student. A bit less cookbookish than many. More than usual care given to applications in mechanics, electric circuits, and mixing and flow problems. Separate 25-page chapter on difference equations. JK

Differential Equations, T*(14-15: 1). *Elementary Differential Equations and Boundary Value Problems, Fourth Edition.* William E. Boyce, Richard C. DiPrima. Wiley, 1986, xvi + 703 pp, $32.95. [ISBN: 0-471-07895-6] Continues to be a solid, dependable introductory textbook. Much rewriting to improve exposition. Some minor rearrangement of topics. Answers/solutions are most trustworthy. Sets the standard for the genre. (*Second Edition*, TR, October 1969, Extended Review, October 1977; Second printing of *Second Edition*, TR, March 1977, Extended Review, October 1977.) JK

Numerical Analysis, L*. *Numerical Recipes: The Art of Scientific Computing.* William H. Press, et al. Cambridge U Pr, 1986, xx + 818 pp, $39.50 [ISBN: 0-521-30811-9]; *Example Book (FORTRAN)*, 1985, viii + 179 pp, $18.95 (P) [ISBN: 0-521-31330-9]; *Example Book (PASCAL)*, 1985, viii + 236 pp, $18.95 (P). [ISBN: 0-521-30956-5] A compendium of useful numerical techniques. Algorithms are clearly explained and alternative algorithms are compared. Pascal and FORTRAN implementations of the algorithms are given. A valuable reference. AO

Numerical Analysis, T(16-17: 1), P. *Numerical Methods in Fluid Dynamics: Initial and Intial Boundary-Value Problems.* Gary A. Sod. Cambridge U Pr, 1985, ix + 446 pp, $44.50. [ISBN: 0-521-25924-X] The first book of a planned two-volume series. An introduction to finite-difference methods for initial boundary-value problems that emphasizes the underlying concepts. Both classical and modern techniques are presented. AO

Functional Analysis, T(18: 1), S, P. *Characterizations of C*-Algebras: The Gelfand-Naimark Theorems.* Robert S. Doran, Victor A. Belfi. Pure & Appl. Math., V. 101. Dekker, 1986, xi + 426 pp, $69.75. [ISBN: 0-8247-7569-4] Starting with the two fundamental theorems of Gelfand and Naimark characterizing C* algebras, further characterizations and applications are developed. Special care is given to the historical development of the subject and its complexities for the beginner. Includes biographical sketches, exercises, notes and remarks, bibliography, appendices on Banach algebras, and numerous indexes. JS

Functional Analysis, P. *Nonlinear Functional Analysis and Its Applications.* Ed: Felix E. Browder. Proc. of Symp. in Pure Math., V. 45. AMS, 1986, $128 set [ISBN: 0-8218-1467-2]. *Part 1*, xii + 540 pp; *Part 2*, x + 577 pp. 100 papers—arranged alphabetically rather than by topic—from the AMS summer research institute held at Berkeley in July 1983. As compared to the similar volume issued 15 years ago, this volume represents a "much more forceful empha-

sis upon applications as opposed to general theory." Over 230 countries are represented by the authors of these papers—attesting to the strong international interest in the subject. LAS

Analysis, T*(15-16: 1). *Introduction to Analysis.* Maxwell Rosenlicht. Dover, 1986, 254 pp, $7 (P). [ISBN: 0-486-65038-3] Unabridged republication of the 1968 Scott Foresman edition (TR, April 1969; Extended Review, August-September 1969). Should be given serious adoption consideration by schools that still give a course in basic real analysis. JK

Geometry, T?(13: 1). *Introduction to Geometry.* James M. Stakkestad, Lin Wyant. Academic Pr, 1986, xi + 485 pp, $20 (P). [ISBN: 0-12-766140-9] Another traditional and unexciting Euclidean synthetic presentation with proofs in "statement-reason" format. JNC

Geometry, P. *Lecture Notes in Mathematics-1164: Geometry Seminar "Luigi Bianch" II—1984.* Mauro Meschiari, John H. Rawnsley, Simon Salamon. Springer-Verlag, 1985, vi + 224 pp, $14.40 (P). [ISBN: 0-387-16048-5] Three lectures (by the authors) given at the Scuola Normale Superiore, Pisa, Italy in 1984. JAS

Algebraic Topology, P. *Lecture Notes in Mathematics-1172: Algebraic Topology, Göttingen 1984.* Ed: L. Smith. Springer-Verlag, 1985, 209 pp, $14.40 (P). [ISBN: 0-387-16061-2] Proceedings of a conference held in Göttingen, November 9-15, 1984. JAS

Algebraic Topology, P. *Lecture Notes in Mathematics-1183: Algebra, Algebraic Topology and their Interactions.* Ed: J.-E. Roos. Springer-Verlag, 1986, xi + 396 pp, $32.50 (P). [ISBN: 0-387-16453-7] The Nordic Summer School and Research Symposium held in Stockholm in August 1983 explored striking analogies between local algebra and rational homotopy theory. This volume contains partial proceedings of the conference and subsequent developments, highlighting cases in which ideas from each field has influenced the other. RB

Topology, T*(15-17: 1), S*, L*. *Surface Topology.* P.A. Firby, C.F. Gardiner. Ser. in Math. & Its Applic. Halsted Pr, 1982, 216 pp, $29.95 (P). [ISBN: 0-470-27528-6] A geometric approach to algebraic topology in two and three-dimensions with significant side excursions into graphs, vector fields, and tessellations. An attractive and appealing introduction to topology that develops good insight into the standard examples. Paperback version of 1982 original hard cover edition (TR, April 1983). LAS

Statistics, P. *Selected Translations in Mathematical Statistics and Probability, Volume 16.* Ed: Lev J. Leifman. AMS, 1985, vi + 138 pp, $56. [ISBN: 0-8218-1468-0] Selected by a committee from the Institute of Mathematical Statistics and American Mathematical Society, these fourteen papers are translations from Russian of works published between 1971 and 1980. KK

Statistics, P. *Lecture Notes in Statistics-32: Generalized Linear Models.* Ed: R. Gilchrist, B. Francis, J. Whittaker. Springer-Verlag, 1985, vi + 178 pp, $13.20 (P). [ISBN: 0-387-96224-7] The published proceedings (consisting of eighteen papers) of the GLIM 85 Conference held in September 1985 at Lancaster University. KK

Statistics, P. *Cohort Analysis in Social Research: Beyond the Identification Problem.* Ed: William M. Mason, Stephen E. Fienberg. Springer-Verlag, 1985, viii + 400 pp, $38. [ISBN: 0-387-96053-8] A collection of fourteen papers on the use of cohort analysis in the social sciences. Papers are the result of an interdisciplinary conference held in 1979 in Snowmass, Colorado. KK

Statistics, P*. *Time Series in the Time Domain.* Ed: E.J. Hannan, P.R. Krishnaiah, M.M. Rao. Handbook of Stat., V. 5. Elsevier Science, 1985, xiv + 490 pp, $84.50. [ISBN: 0-444-87629-4] Fifth volume in this series of reference books on statistical methodology and applications (Series and Volume 1, TR, June-July 1981). Concerned with time domain methods, which involve parametric representations of the temporal relationships, as opposed to the frequency analysis approach surveyed in Volume 3 of this series, *Time Series in the Frequency Domain* (TR, December 1985). RSK

Statistics, S*(16-18), P, L*.** *A Celebration of Statistics: The ISI Centenary Volume.* Ed: Anthony C. Atkinson, Stephen E. Fienberg. Springer-Verlag, 1985, xv + 606 pp, $39. [ISBN: 0-387-96111-9] A collection of 25 (refereed) papers gathered from authors around the world to celebrate the centenary of the International Statistical Institute. Papers cover a great variety of topics, both important and interesting, including prediction and entropy, properties of wind-blown sand, robustness and many more. KK

Statistics, P. *Analyse de Données Chronologiques.* Guy Mélard. Pr U Montreal, 1985, 169 pp, $20 (P). [ISBN: 2-7606-0693-7] Lecture notes from NATO Advanced Study Institute on Analysis of Data held at the University of Montreal, July 26-August 13, 1982. Uses geometric representation in Hilbert space to analyze time series. KS

Statistics, T*(13-15), L*. *The Fascination of Statistics.* Ed: Richard J. Brook, et al. Popular Stat., V. 4. Dekker, 1986, xi + 433 pp, $24.75. [ISBN: 0-8247-7329-2] A *pot-pourri* of brief descriptions for a lay audience of the use of statistical methods in real applications: genetic evaluation of bulls, counting wild life, decoding DNA, etc. 30 independently authored vignettes are grouped into seven cat-

egories ranging from probability through prediction to modelling. An excellent resource for students and teachers of elementary statistics. LAS

Computer Literacy, S, P, L.** *The Cult of Information: The Folklore of Computers and the True Art of Thinking.* Theodore Roszak. Pantheon Books, 1986, xii + 238 pp, $17.95. [ISBN: 0-394-54622-9] A passionate discourse on deliterious effects of the computer culture (not of computers themselves), effects that "cheapen thought," and "distort the natural order of human priorities." Historian-author Roszak lays out a well-documented humanist manifesto in reaction to cheapness, shallowness, and deceitfulness, which he sees lurking behind the notion of an "information age." One should teach ideas and knowledge, not data and information: "The mind thinks with ideas, not with information." A worthy sequel to Weizenbaum's *Computer Power and Human Reason* (TR, June-July 1976; Extended Review, May 1978). LAS

Computer Literacy, T*(13-16: 1), L. *The Mystical Machine: Issues and Ideas in Computing.* John E. Savage, Susan Magidson, Alex M. Stein. Addison-Wesley, 1986, xvi + 407 pp, $22.95. [ISBN: 0-201-06462-6] A solid and well-written text for a course in computer literacy, put together by the chairman of the computer science department at Brown (Savage) and two undergraduate teaching assistants (Magidson, Stein). History, machine architecture, operating systems (including UNIX and MSDOS), programming concepts (in BASIC and Pascal with brief sections on Fortran, Prolog, Lisp, etc.), artificial intelligence, automation, education, privacy. One of the few books of this genre that avoids talking down to students. Concludes with a helpful glossary. LAS

Computer Literacy, S(13-14). *Guide to Using Lotus 1-2-3, Second Edition.* Edward M. Baras. Osborne McGraw-Hill, 1986, xii + 412 pp, $18.95 (P). [ISBN: 0-07-881230-5] An update to include version 2 of *Lotus 1-2-3.* A nice, straightforward how-to book. JAS

Computer Literacy, P. *The Practical Guide to Local Area Networks.* Rowland Archer. Osborne McGraw-Hill, 1986, xii + 283 pp, $21.95 (P). [ISBN: 0-07-881190-2] A handbook on local computer nentworks for the IBM PC (and compatibles), intended for the potential purchaser, installer, manager, user. Benefits and costs of networks; selection factors; planning and installation; network use. Detailed discussion of five products: IBM PC Network; 3Com EtherSeries; Corvus Omninet; Novell NetWare; Orchid PCnet. Glossary. No bibliography. RB

Computer Literacy, T(13: 1). *Computers and Data Processing Today with Basic, Second Edition.* Steven L. Mandell. West, 1986, xxvi + 693 pp, $27.95 (P). [ISBN: 0-314-96079-1] Covers the traditional topics standard in an introductory survey course. Major changes in this edition include new material on data structures and files, microcomputers, application software, and numbering systems. AO

Computer Programming, T(15-17). *File Techniques for Data Base Organization in COBOL, Second Edition.* L.F. Johnson, R.H. Cooper. Prentice-Hall, 1986, xxi + 410 pp, $28.95. [ISBN: 0-13-314717-7] Intended as a compromise between a second course in COBOL (with the VSAM environment) and a theoretical course, this book represents a number of substantial revisions to the 1981 *First Edition* (TR, October 1981). The orientation remains specific and practical. JAS

Computer Programming, T(13: 1, 2). *Pascal: Problem Solving and Programming with Style.* William C. Jones, Jr. Comput. Sci. & Tech. Ser. Harper & Row, 1986, xix + 532 pp, (P) [ISBN: 0-06-043409-0]; *Instructor's Manual to Accompany*, 56 pp (P). [ISBN: 0-06-363445-7] Early introduction to procedures and parameters. Many examples, program runs, review questions, programming exercises requiring programs of varying length and difficulty. Spiral approach. Separate subsections contrast Standard with UCSD and Turbo Pascal. Last chapter compares Modula-2 and Pascal. A 67-page Test Bank is available with the Instructor's Manual. DFA

Computer Programming, T(13: 1). *Problem Solving and Structured Programming with Pascal.* Ali Behforooz, Martin O. Hololen. Ser. in Comput. Sci. Brooks/Cole, 1986, xiii + 511 pp, (P). [ISBN: 0-534-05736-5] For the first course. The first chapter, which discusses problem solving and algorithm development, is programming language-free. A valuable later chapter is devoted entirely to preventing, locating and removing program errors. Lots to read: the first complete program appears on page 79; the Preface cites the "early introduction of subprograms," which begins on page 197. DFA

Computer Programming, T(13: 1, 2). *Introduction to Computing and Computer Science with Pascal.* Henry M. Walker. Little Brown, 1986, xxxi + 764 pp, $26.95. [ISBN: 0-316-91841-5] For a first course in computer science. Primary emphasis is on programming in Pascal, but also provides an introduction to problem-solving methodology, algorithm design, data structures, and social implications. AO

Computer Programming, T(13: 1). *FORTRAN 77 for Humans, Third Edition.* Rex Page, Rich Didday, Elizabeth Alpert. West, 1986, xiii + 462 pp, $26.95 (P). [ISBN: 0-314-93404-9] Introduction to computer programming, using FORTRAN 77. Has review exercises to test understanding, practice problems to hone routine programming skills, design

problems to develop creative problem solving ability. Complete presentation of the language. DFA

Computer Programming, S(14-16), L. *Programming in Common Lisp.* Rodney A. Brooks. Wiley, 1985, xv + 303 pp, $20.95 (P). [ISBN: 0-471-81888-7] An introduction to Lisp programming and, particularly, the Common Lisp dialect. Emphasizes good programming style. An appendix covers the differences between Common Lisp, Franz Lisp, and MacLisp. AO

Computer Programming, T(13-15: 1), S, L. *Programming in Ada: A First Course.* Robert G. Clark. Cambridge U Pr, 1985, x + 217 pp, $39.50; $17.95 (P). [ISBN: 0-521-25728-X; 0-521-27675-6] A relatively brief and gentle introduction to a subset of Ada. Includes some features such as packages and exceptions. For a complete course, it would need to be supplemented by larger examples, more exercises, and access to reference and system manuals. RWN

Computer Programming, T(13: 1). *Pascal, An Introduction to Methodical Programming, Third Edition.* William Findlay, David A. Watt. Pitman, 1985, xiv + 413 pp, $16.95 (P). [ISBN: 0-273-02188-5] Text for a first course in programming. Emphasizes stepwise refinement. Procedures introduced in chapter fourteen. Uses case studies to illustrate program development. (*First Edition*, TR, June-July 1979; *Second Edition*, TR, May 1983.) KS

Computer Programming, S(13-14). *LISP: An Interactive Approach.* Stuart C. Shapiro. Computer Science Pr, 1986, x + 150 pp, $19.95 (P). [ISBN: 0-88175-069-7] Designed to be used as a self-pace study guide. Organized as 32 brief sections with accompanying exercises. Emphasizes applicative programming techniques. AO

Computer Programming, T(13: 1). *A Guide to Programming in Turbo Pascal.* Bruce Presley, Tim Corica. Lawrenceville Pr (Distr: Delmar Pub), 1986, x + 353 pp, $19.95 (P). [ISBN: 0-931717-41-8] For the first course, high school or college. Includes a chapter on graphics. Concluding chapters introduce advanced topics: searching, sorting, linked lists, stacks, queues, binary trees. Includes demonstration programs and runs, major programming problems as examples, an ample supply of programming exercises of widely-varying difficulty. DFA

Computer Programming, T*(13-14: 1), L. *Oh! Pascal! Second Edition.* Doug Cooper, Michael Clancy. WW Norton, 1985, xxxi + 607 pp, $24.95 (P). [ISBN: 0-393-95445-5] An excellent introductory textbook. Well-written. Good exercises. This edition includes several new topics as well as expanded coverage of previously included topics. (*First Edition*, TR, October 1983.) AO

Computer Programming, S*(13-14), P, L*. *Pascal with Excellence: Programming Proverbs.* Henry F. Ledgard, John Tauer. Hayden Book, 1986, 242 pp, $16.95 (P). [ISBN: 0-8104-6490-X] A guide to better programming through stylistic guidelines, motivated by Strunk and White's *The Elements of Style.* Not a Pascal language manual, but a very appropriate supplement: a handbook for improving program quality for those with some Pascal familiarity. Help in becoming a new generation programmer, who knowingly writes correct programs. RB

Computer Programming, T(14-15: 1), S, P, L. *Advanced Turbo Pascal: Programming & Techniques.* Herbert Schildt. Osborne McGraw-Hill, 1986, x + 276 pp, $18.95 (P). [ISBN: 0-07-881220-8] Discussion with complete code of advanced algorithms (e.g., sorting, searching, trees, dynamical allocation), special applications (statistics, encryption, simulation, parsing), and practical problems (keyboard interface, porting, conversions from C and Basic, debugging). An excellent resource for those who must write real programs. A disk with source code of all examples is available for $29.95. LAS

Computer Programming, P. *Lecture Notes in Computer Science-221: Logic Programming '85.* Ed: Eiti Wada. Springer-Verlag, 1986, ix + 311 pp, $20.50 (P). [ISBN: 0-387-16479-0] English proceedings of the fourth annual Japanese conference on logic programming. Sponsored by ICOT, home of the famous ongoing "Fifth Generation" project in which Japanese corporations cooperate to produce super-performance hardware, expert system software (Prolog language), ultimately for general public use. Advances in Prolog; parallelism in architecture, programming; specific expert systems; knowledge engineering. RB

Software Systems, S, L. *The UNIX C Shell Field Guide.* Gail and Paul Anderson. Prentice-Hall, 1986, xxi + 374 pp, $19.95 (P). [ISBN: 0-13-937468-X] History; aliases; job control; shell programming; customization. A guide by extensive example to the powerful command interpreter known as "csh" available on most UNIX and XENIX systems. Assumes general familiarity with standard UNIX features. LAS

Software Systems, S(15-17). *Programming the UNIX System.* M.R.M. Dunsmuir, G.J. Davies. Halsted Pr, 1985, vii + 176 pp, (P). [ISBN: 0-470-20192-4] A guide to some of the more advanced features of the UNIX operating system (e.g., I/O programmming and interprocess communication). Presupposes prior experience with both UNIX and the C programming language. AO

Computer Science, T(13: 1, 2), L. *Principles of Computer Science: Concepts, Algorithms, Data Structures, and Applications.* M. Sandra Carberry, A. Toni Cohen, Hatem M. Khalil. Computer Science Pr, 1986, xvii + 636 pp, $34.95. [ISBN: 0-

914894-79-X] An introductory overview of computer science. Pascal is used to illustrate programming language concepts. Includes sections on computer architecture, operating systems, and programming languages. Particularly appropriate for students with some prior Pascal programming experience. AO

Computer Science, P. *Supercomputers: Algorithms, Architectures, and Scientific Computation.* Ed: F.A. Matsen, T. Tajima. U of Texas Pr, 1986, vii + 480 pp, $35. [ISBN: 0-292-70388-0] The proceedings of an international conference held in March 1985 at the University of Texas at Austin. Twenty-six papers review current work on the design and use of supercomputers. AO

Computer Science, P*. *Lecture Notes in Computer Science-164: Logics of Programs.* Ed: Edmund Clarke, Dexter Kozen. Springer-Verlag, 1984, vi + 527 pp, $22.50 (P). [ISBN: 0-387-12896-4] 34 papers on a wide variety of topics relating logics and programs: interval temporal logic, proof system for partial correctness, process logics, semantic models for communicating processes, and an algebra of hierarchies. RWN

Computer Science, P. *Lecture Notes in Computer Science-205: A Study in String Processing Languages.* Paul Klint. Springer-Verlag, 1985, viii + 165 pp, $12.80 (P). [ISBN: 0-387-16041-8] The author's new language named SUMMER: its motivation, formal definition, library of functions. General language design considerations and their application. RWN

Computer Science, P. *Computer-Generated Images: The State of the Art.* Ed: Nadia Magnenat-Thalmann, Daniel Thalmann. Springer-Verlag, 1985, x + 497 pp, $75. [ISBN: 0-387-70010-2] Forty-four papers selected from those presented at Graphics Interface '85, a conference held May 27-31, 1985 in Montreal, Canada. JAS

Computer Science, P. *Lecture Notes in Computer Science-217: Programs as Data Objects.* Ed: H. Ganzinger, N.D. Jones. Springer-Verlag, 1986, x + 324 pp, $20.50 (P). [ISBN: 0-387-16446-4] How can computer programs be generated and manipulated by computers? These workshop proceedings (Copenhagen, October 1985) deal with development of program manipulation tools: transforming programs for more efficient implementation; modular specification of compilers; lambda calculus flow analysis; "strict" functions; semantics-directed compiler generation. Primarily concerning functional programming languages (cf Lisp); 10 papers. RB

Computer Science, T(17-18: 1), P. *The Foundations of Program Verification.* Jacques Loeckx, Kurt Sieber, Ryan D. Stansifer. Ser. in Comp. Sci. Wiley, 1984, ix + 230 pp, $29.95. [ISBN: 0-471-90323-X] Concentrates on the standard methods: Floyd's inductive assertions method, Hoare's axiomatic method, and Scott's fixed point induction. Mathematically rigorous but does not presume prior knowledge of logic or semantics. AO

Computer Science. *High-Speed Computation.* Ed: Janusz S. Kowalik. NATO ASI Ser. F, V. 7. Springer-Verlag, 1984, ix + 441 pp, $46.60. [ISBN: 0-387-12885-9] Proceedings of a NATO advanced research workshop on applied parallel computing held in Julich, West Germany, in June 1983. Begins with a policy-oriented keynote address by Ken Wilson; contains 27 papers on hardware, algorithms, and applications. LAS

Computer Science, P. *Lecture Notes in Computer Science-214: CAAP '86.* Ed: P. Franchi-Zannettacci. Springer-Verlag, 1986, vi + 306 pp, $20.50 (P). [ISBN: 0-387-16443-X] Proceedings of the 11th European Colloquium on Trees in Algebra and Programming held in Nice in March 1986. Twenty papers and two invited lectures oreganized into 7 sections: language theory; algebraic theory of semantics; graphs and grammars; program schemes and programming; tree-automata and transducers; probability on trees; logic for computing. RB

Computer Science, P. *Lecture Notes in Computer Science-216: LUCAS, Associative Array Processor: Design, Programming and Application Studies.* Christer Fernstrom, Ivan Kruzela, Bertil Svensson. Springer-Verlag, 1986, xii + 323 pp, $20.50 (P). [ISBN: 0-387-16445-6] A compilation of the design principles, programming tools and experiences from the LUCAS (Lund University [Sweden] Content Addressable System) project, in which a computer with 128 processors was built (in 1982) and evaluated. Modified Pascal compiler and microprogramming language facilitate expression of parallel algorithms. Applications to image processing, signal processing, relational databases. RB

Computer Science, S(14-16), P. *Multidimensional Clustering Algorithms.* Fionn Murtagh. Compstat Lect., V. 4. Physica-Verlag, 1985, 131 pp, $20.50 (P). [ISBN: 3-7051-0008-4] Collection of recent results in algorithmic treatment of cluster analysis. Includes pattern recognition, information storage and retrieval. Emphasis is on practical problems. Little background in the area is required. KK

Computer Science, P. *Lecture Notes in Computer Science-193: Logics of Programs.* Ed: Rohit Parikh. Springer-Verlag, 1985, vi + 424 pp, $22.80 (P). [ISBN: 0-387-15648-8] Proceedings of a conference held at Brooklyn College, June 17-19, 1985. Thirty research papers on logics designed for reasoning about computer programs. KS

Computer Science, T*(15-16: 1), L.** *Computation and Automata.* Arto Salomaa. Ency. of Math. & Its Applic. Cambridge U Pr, 1985, xiii + 284 pp, $39.50. [ISBN: 0-521-30245-5] Clearly written,

well-motivated introduction to mathematical topics in theoretical computer science. Covers computability and recursive functions, formal languages and automata, computational complexity, cryptography. Short exercise set at end of each chapter. KS

Computer Science, T(15-17: 1). *Fundamentals of Logic Design and Switching Theory.* Arthur D. Friedman. Computer Science Pr, 1986, ix + 262 pp, $26.95. [ISBN: 0-88175-098-0] A good introduction including the necessary mathematics with no specific prerequisites other than general analytic skills (algebra and intuitive logic). The emphasis is on logic essentially to the exclusion of electronics. Substantial problem sets, few that classify as elementary exercises—supplementary exercises would be needed for the most unsophisticated students. The index seems too brief. A reader with a weak memory would have difficulty finding basic vocabulary. JAS

Computer Science, P. *Computer Graphics: Visual Technology and Art.* Ed: Tosiyasu L. Kunii. Springer-Verlag, 1985, xi + 382 pp, $49.50. [ISBN: 0-387-70009-9] Proceedings of the conference, Computer Graphics Tokyo '85. JAS

Applications, P. L. *Actuarial Mathematics.* Ed: Harry H. Panjer. Proc. of Symp. in Appl. Math., V. 35. AMS, 1986, xi + 127 pp, $26. [ISBN: 0-8218-0096-5] Lecture notes from the AMS 1985 Summer Short Course at Laramie, Wyoming, including papers on risk theory, "credibility" theory (revision of rates), "graduation" (smoothings), as well as analyses of Social Security and TIAA/CREF. LAS

Applications, P, L*. *Annual Review of Fluid Mechanics, V. 18, 1986.* Ed: Milton van Dyke, J.V. Wehausen, John L. Lumley. Annual Reviews, 1986, x + 522 pp, $32. [ISBN: 0-8243-0718-6] 16 survey essays on such topics as strange attractors, vortex flows, and boundary layers, each well written with careful references to the research literature. Includes author and title indices to all previous volumes, grouped by subject. There should be such a series for mathematics! LAS

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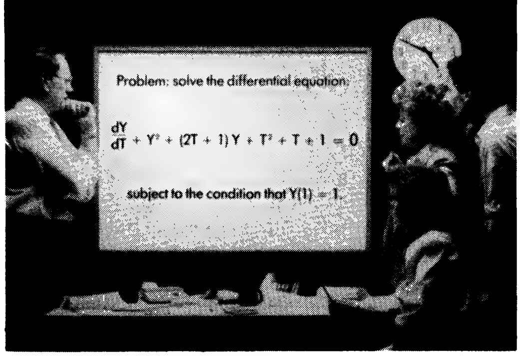

Symbolically...

```
(C1) DEPENDS(Y,T)$
(C2) DIFF (Y,T) + Y^2 + (2*T + 1)*Y + T^2 + T + 1;
```

(D2) $\frac{dY}{dT} + Y^2 + (2T + 1)Y + T^2 + T + 1$

```
(C3) SOLN:ODE(D2,Y,T);
```

(D3) $Y = -\frac{\%C\,T\,\%E^T - T - 1}{\%C\,\%E^T - 1}$

```
(C4) SOLVE(SUBST([Y = 1, T = 1],D3),%C),NUMER;
(D4)     [%C = 0.5518192]
(C5) SPECIFIC SOLN:SUBST(D4,SOLN);
```

(D5) $Y = -\frac{0.5518192\,T\,\%E^T - T - 1}{0.5518192\,\%E^T - 1}$

and Numerically.

```
(C6) FORTRAN(D5)$
        Y = - (0.5518192*T*EXP(T) - T - 1)
     1      /(0.5518192*EXP(T) - 1)
```

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THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 2 February 1987

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(ISSN 0002-9890)

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Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

A Tale of Two Coins

F. K. HWANG
AT & T Bell Laboratories, Murray Hill, NJ 07974

1. Introduction. A common phenomenon in combinatorial search theory is that while it is often straightforward to find an optimal procedure searching for one object, it is immensely more difficult to search optimally for two objects. For example, Bellman and Glass **[1]** studied the problem of identifying two irregular coins in a set of n coins with a balance scale. They wrote: "A small amount of analysis discloses the enormous difference in complexity between the one-coin and the two-coin problem." Cairns **[2]** and Tošić **[10]** also proposed procedures for the two-coin problem, but an optimal procedure remains elusive.

While a balance scale provides information about the irregulars by comparing the weights of two subsets of coins, we can also detect irregulars by a spring scale which provides information by weighing a subset of coins. Instead of talking about the balance scale and the spring scale, we use the more general terms of the comparison-type device and the test-type device. In this paper we survey results from various models of the test-type device in identifying two irregulars (not necessarily identical).

2. Classification of Models under the Test-Type Device. Let x and y denote the two irregulars in a set of n coins (so that there are $n(n-1)$ possibilities of such a pair). The feedback of an ideal device on testing a set S would yield one of the four possible types:

$$F(S) = \varnothing \quad \text{meaning } x \notin S,\ y \notin S,$$
$$F(S) = X \quad \text{meaning } x \in S,\ y \notin S,$$
$$F(S) = Y \quad \text{meaning } x \notin S,\ y \in S,$$
$$F(S) = XY \quad \text{meaning } x \in S,\ y \in S.$$

Depending on the actual testing device in use and the nature of the two irregulars, the partition of feedback may be coarser than the partition $\varnothing\,|X|\,Y|XY$. In this paper we will be concerned with testing devices whose feedbacks can be described by partitions of the set $\{\varnothing, X, Y, XY\}$. Among the fifteen possible partitions the trivial partition $\varnothing, X, Y, XY$ yields no information to identify either X or Y. The partition $\varnothing, X|\,Y, XY$ ($\varnothing, Y|X, XY$) yields no information to identify $X(Y)$. These three partitions are ruled out immediately. We now discuss the other twelve partitions.

Suppose that the device is a spring scale and the weight of a regular is known. If the two irregulars have the same weight, then each weighing tells us exactly how

F. K. HWANG: I have been working at AT & T Bell Laboratories as a research mathematician since I got my Ph.D. from North Carolina State in 1967. The title of the current paper reflects, I hope, my background of B.A. in English literature.

many irregulars are in the set weighed. Therefore the partition is $\varnothing \mid X, Y \mid XY$. If the two irregulars have known different weights, then we can further differentiate whether x or y is present when the set weighed contains only one irregular. So the partition is $\varnothing \mid X \mid Y \mid XY$ in general except for the case that one irregular is heavier and the other lighter by the same amount than a regular, then the partition is $\varnothing, XY \mid X \mid Y$. This exceptional case was picturesquely described by Christen [4] as the candy factory problem in which workers of a candy factory pack boxes containing a fixed number of equally heavy candy pieces but one mischievous worker shifts a piece of candy from one box to another. So one box becomes too heavy and the other too light, but their total weight remains constant. If the weighing device is set to detect only when a set of boxes is underweight, then the partition is $Y \mid \varnothing, X, XY$.

The partition $\varnothing \mid X, Y, XY$ corresponds to the standard group testing model for separating defectives from good items. Dorfman [6] originated group testing in his blood testing problem where blood samples can be grouped for testing such that one negative response clears a whole group. Group testing has been shown to be useful in many other industrial applications like leakage test, flow test, pattern recognition, screening, ... When the two defectives are not equally defective, for example, one completely blocks the flow and the other merely reduces the flow in a test of flow devices, then the flow is blocked whenever the severe defective is presented regardless of the presence of the intermediate defective. The partition is thus $\varnothing \mid X \mid Y, XY$ (interpreting y as the severe defective).

Finally, the partition $\varnothing, XY \mid X, Y$ can occur in pattern recognition when the problem is to identify the presence of a few 1's (irregulars) among 0's (regulars), and the device consists of a check bit which adds up the input bits with modulo 2 arithmetic (if it uses Boolean logic, then it is the group testing model).

We list the seven partitions discussed above and give each a name for easy reference:

$\varnothing \mid X \mid Y \mid XY$	model B (Basic),
$\varnothing \mid X \mid Y, XY$	model D (various Degrees in irregularity),
$\varnothing, XY \mid X \mid Y$	model C (Candy factory),
$\varnothing \mid X, Y, XY$	model G (Group testing),
$Y \mid \varnothing, X, XY$	model U (Underweight),
$\varnothing, XY \mid X, Y$	model P (Parity),
$\varnothing \mid X, Y \mid XY$	model Q (Quantity).

A partition is said to be symmetric to another if it can be obtained from the other by interchanging X and Y. Clearly, all results for one partition can be translated into results for the other. Therefore we need only to discuss one partition from every pair of symmetric partition. In this sense we will ignore partitions $\varnothing, X \mid Y \mid XY$, $\varnothing \mid Y \mid X, XY$ and $X \mid \varnothing, Y, XY$ since they are symmetric to the partition $\varnothing, Y \mid X \mid XY$ and models D and Q, respectively.

A partition is said to be dual to another if one partition can be obtained from the other by interchanging each type T of feedback by the type $XY \setminus T$. Let A be a procedure for the partition U which has a dual partition V. Then we can obtain a dual procedure of A for V by testing the complementary set $\bar{S}$ whenever A tests S. Thus all results for one partition are again immediately translated into results for the dual partition and we need only discuss one of them. We will ignore the partitions, $XY | \varnothing, X, Y$ and $\varnothing,\ X | Y | XY$ since they are dual to models G and D, respectively (the partition $X | \varnothing, Y,\ XY$ is dual to model U but already ignored due to reason of symmetry).

3. Lower Bounds for the Number of Tests. The *resolution number* of a model is the number of parts in the partition associated with that model. The *test history* consists of all tests performed and their feedback. Associated with a test history is a *solution space* consisting of all possible pairs $\{(u, v)\}$ such that $x = u$ and $y = v$ are consistent with the test history, i.e., one cannot deduce from the test history that it is impossible to have $x = u$ and $y = v$. The solution space before any test is done is called the *original solution space*. Suppose that the cardinality of the original solution space is s and the resolution number of the model m is r. Let $T_m(n)$ denote the number of tests required to identify the two irregulars among n coins under an optimal procedure for model m. By repeatedly using the fact that if we divide q things into p parts, then one part must contain at least $\lceil q/p \rceil$ things, where $\lceil z \rceil$ denotes the smallest integer not less than the number z, we find that $T_m(n) \geqslant \lceil log_r s \rceil$. This lower bound is usually referred to as the information-theoretic bound. If the two irregulars are not distinguishable as in models G, P, Q, then $s = \binom{n}{2}$. If they are distinguishable as in models B, U, F, D, then $s = n(n-1)$. We list the information-theoretic bounds for the various models:

Model	B	G, P	U	C, D	Q
Bound	$\lceil \log_4\{n(n-1)\} \rceil$	$\left\lceil \log_2 \binom{n}{2} \right\rceil$	$\lceil \log_2\{n(n-1)\} \rceil$	$\lceil \log_3\{n(n-1)\} \rceil$	$\left\lceil \log_3 \binom{n}{2} \right\rceil$.

The information-theoretic bound above is achievable only if we can always find a subset to test which partitions all pairs in the solution spaces evenly into the r parts, by which we mean that if $n \leqslant r^k$, then the ith partition contains no part larger than r^{k-i}. Sometimes this can't be done. For example, the first test of a subset of size s in model P partitions the $\binom{n}{2}$ pairs into $s \times (n-s)$ pairs and $\binom{s}{2} + \binom{n-s}{2}$ pairs. For $n = 91$ we have $\binom{n}{2} = 4095 = 2^{12} - 1$. But there does not exist an s leading to a 2^{11} and $2^{11} - 1$ partition. However, it is not easy to characterize the set of n for which no even partitions exist. For model G, Chang, Hwang, and Lin **[3]** were able to show that no even partition exists for those n which satisfy

$$\binom{n}{2} \leqslant 2^{\alpha} < \binom{n+1}{2}$$

for some α and thus improved the lower bound to $\left\lceil \log_2 \binom{n+1}{2} \right\rceil$.

For model C the first test of a subset of size s partitions the $n(n-1)$ ordered pairs into two piles of $s \times (n-s)$ ordered pairs each and one pile of $n(n-1)-2s(n-s)$ ordered pairs, a considerably larger number. Christen [4] took full advantage of this discrepancy by looking at the cumulative effect of the first t tests. He was able to obtain an achievable lower bound which approaches $\{(1+\log_2 3)/\log_2 3\}\log_2 n$ (see Sect. 4 for its exact form).

4. Upper Bounds for the Number of Tests. For model B we show that the lower bound of $T_B(n)$ can be achieved. Let the n coins be represented by the n binary numbers from 0 to $n-1$. The ith subset to be tested consists of all coins whose ith bit is one. Let $\{i_1, i_2, \ldots, i_k\}$ be the set of labels such that F (the ith subset) $= X$ or XY. Then x is the coin which has 1 in bits $i_1, i_2, \ldots, i_k$ and 0 elsewhere. Since every two coins have distinct binary representations, x is uniquely determined. Similarly we can determine y. We will refer to this procedure as *procedure B*.

The number of tests required for procedure B is

$$\lceil \log_2 n \rceil = \left\lceil \log_2\{n(n-1)\}^{1/2} \right\rceil = \lceil \log_4\{n(n-1)\} \rceil.$$

EXAMPLE 1. For eight coins $\{000, 001, 010, 011, 100, 101, 110, 111\}$ the first test is on the four coins $\{100, 101, 110, 111\}$, the second test on $\{010, 011, 110, 111\}$ and the third test on $\{001, 011, 101, 111\}$. Suppose that $x = 001$ and $y = 101$. Then

$$F(\{100, 101, 110, 111\}) = Y,$$
$$F(\{010, 011, 110, 111\}) = \varnothing,$$
$$F(\{001, 011, 101, 111\}) = XY.$$

From the feedback $(Y, \varnothing, XY)$ we identify $x = 001$ and $y = 101$.

For model U we test all the subsets in procedure B as well as their complementary sets. It is easily seen that for every ordered pair of coins there exists a subset containing the first coin but not the second coin. Therefore the set of subsets with feedback Y is not empty. Furthermore, for each coin j belonging to a subset with feedback Y, there exists a unique coin k such that $y = j$ implies $x = k$. To see this, note that each coin j appears in $\lceil \log_2 n \rceil$ subsets and any ordered pair of coins (j, k) generates a distinct feedback pattern over the $\lceil \log_2 n \rceil$ subsets tested. Now take any subset with feedback Y (preferably the smallest such set). Using the halving procedure we can identify y, hence x, in $\lceil \log_2 n \rceil - 1$ tests. The total number of tests required is thus at most $3\lceil \log_2 n \rceil - 1$. Note that if the feedback of the first $2\lceil \log_2 n \rceil - 1$ tests is $\varnothing \cup X \cup XY$ throughout, then we can skip the next test since we know its feedback must be Y. This slight improvement reduces $T_N(6)$ from eight to seven.

EXAMPLE 2. For eight coins the first six subsets to be tested are

$$S_1 = \{100, 101, 110, 111\}, \quad S_2 = \{010, 011, 110, 111\}, \quad S_3 = \{001, 011, 101, 111\},$$
$$S_4 = \{011, 010, 001, 000\}, \quad S_5 = \{101, 100, 001, 000\}, \quad S_6 = \{110, 100, 010, 000\}.$$

Suppose that $F(S_1) = F(S_3) = Y$ and no other subset yields outcome Y. Then either $y = 101$, $x = 000$ or $y = 111$, $x = 010$. One more test on the coin 101 differentiates the two cases.

For model P we do exactly as in model U except that we do not have to test the $\lceil \log_2 n \rceil$ complementary sets (their feedback can be deduced from the test feedback of the $\lceil \log_2 n \rceil$ original sets) and that we halve a subset with feedback $X \cup Y$ instead of feedback Y. Therefore the total number of tests required is

$$2\lceil \log_2 n \rceil - 1 = \left\lceil \log_2 \binom{n}{2} \right\rceil,$$

which is at most one more than the lower bound.

For model Q Christen [5] proved $T_Q(n) \leqslant t$ if $n \leqslant f_{t+2}$ where f_k is the kth Fibonacci number ($f_0 = 0$ and $f_1 = 1$). The procedure which classifies f_{t+2} coins in t tests can be described by the following steps:

(i) If the two irregulars are contained in a set of f_k coins, test f_{k-1} coins.

(ii) If one irregular is contained in A and the other in B, where A and B are two diagonal sets of f_k and f_{k-1} coins, respectively, test a set consisting of f_{k-1} coins from A and f_{k-2} coins from B.

(iii) If either one regular is contained in A and the other in B, or one is in C and the other in D, where A, B, C, D are disjoint sets of f_k, f_{k-2}, f_{k-1} and f_{k-1} coins, respectively, test a set consisting of f_{k-1} coins from A, f_{k-2} coins from B, f_{k-3} coins from C and none from D.

It is straightforward to verify that the tests specified in (i), (ii) and (iii) always result in situations as described in (i), (ii) and (iii). $T_Q(f_{t+2}) = t$ can then be proved by induction.

EXAMPLE 3. For $f_6 = 8$ coins we first test a set S of $f_5 = 5$ coins. Suppose that the feedback is one. Test a set $A \cup C$, where A consists of $f_4 = 3$ coins from S and C consists of $f_3 = 2$ coins from $\bar{S}$. Suppose that the feedback is again one. Then either one irregular is in A and the other in $B \equiv \bar{S} - C$, or one is in C and the other in $D \equiv S - A$. Test a set consisting of $f_3 = 2$ coins from A, $f_1 = 1$ coin from both B and C. Suppose that the feedback is one again. Then either one irregular is the tested coin in B and the other is the untested coin in A; or one irregular is the tested coin in C and the other is one of the two untested coins in D. Test a set consisting of the tested coin in C and one of the two untested coins in D and we identify the two irregulars.

Model G has been very well studied under the context of "group testing." First note that by using the halving procedure twice we can identify the two irregulars separately in at most

$$\lceil \log_2 n \rceil + \lceil \log_2 (n-1) \rceil \leqslant \lceil \log_2 (n+1) \rceil + 1$$

tests which is two more than the lower bound. So the interesting problem here is a purely academic one—to see how hard it is to obtain an optimal procedure. Several

procedures [3], [8], [9] have been proposed which improve the twice-halving procedure mentioned above. But the final gap between the construction and the lower bound remains unbridged. To understand the progress made by these procedures we need to have a new criteria which is equivalent to the number of test criteria but enlarges its details. Define $N_m(t)$ to be the largest n for model m such that t tests suffice to identify the two irregulars under an optimal procedure. Since $N_m(t)$ is nondecreasing in t and $T_m(n)$ is nondecreasing in n, the solving of one completely solves the other.

The currently best bound g_t for $N_G(t)$ was given by Chang, Hwang, and Lin [3] to be:

$$g_t = \begin{cases} \left\lceil 43 \cdot 2^{\frac{t}{2}-5} \right\rceil - 1 & \text{for } t \text{ even}, \\ \left\lceil 31 \cdot 2^{\frac{t-1}{2}-4} \right\rceil - 1 & \text{for } t \text{ odd}. \end{cases}$$

The procedure goes as follows. For even t, partition the l_t coins into four piles A, B, C, D with cardinalities

$$a = 2^{\frac{t}{2}-3}, \quad b = c = 2^{\frac{t}{2}-2}, \quad d = g_{t-1} - 2^{\frac{t}{2}-2}$$

First test $A \cup B$. If $F(A \cup B) = \varnothing$, then the two irregulars in $C \cup D$ can be identified in $t-1$ more tests by induction. If $F(A \cup B) \neq \varnothing$, test $A \cup C$ next. If $F(A \cup C) = \varnothing$, then B must contain an irregular which can be identified in $(t/2) - 2$ tests using the halving procedure. Then we use the halving procedure again to identify the other irregular in $B \cup D$ in $t/2$ tests (since $b + d - 1 < 2^{t/2}$). If $F(A \cup C) \neq \varnothing$, we test $B \cup C$ next. If $F(B \cup C) = \varnothing$, then A must contain an irregular which can be identified in $(t/2) - 3$ tests using the halving procedure. Then we use the halving procedure again to identify the other irregular in $A \cup D$ in $t/2$ tests. If $F(B \cup C) \neq \varnothing$, then D contains no irregular. We next test A. If $F(A) = \varnothing$, then B and C must contain one irregular each which can be identified by the halving procedure in $t-4$ tests. If $F(A) \neq \varnothing$, then A and $B \cup C$ must contain one irregular each which can be identified by the halving procedure in $t-4$ tests. The odd t case is similar but slightly more complicated.

EXAMPLE 4. For $t = 8$, $g_8 = 21$, $a = 2$, $b = c = 4$, $d = 11$ ($g_7 = 15$). For $n = 21$ coins we first test $A \cup B$ of six coins. Suppose that $F(A \cup B) \neq \varnothing$. We test $A \cup C$ of six coins next. Suppose that $F(A \cup C) \neq \varnothing$ again. We then test $B \cup C$ of eight coins. Suppose that $F(B \cup C) = \varnothing$. We then identify one irregular from the set A of two coins in one test and identify the other from $A \cup D$ of twelve coins in four tests. The total number of tests is $1 + 1 + 1 + 1 + 4 = 8$.

Christen [4] gave an optimal procedure for model C which achieves the upper bound

$$N_C(t) = \begin{cases} 7 & \text{if } t = 4, \\ (3^{h(t)} - 1 + 2^{1+t-h(t)})/2 & \text{if } 1 + 2^{t-h(t)} < 3^{h(t)}, \\ \lfloor 3(3^{h(t)} + 2^{t-h(t)})/4 \rfloor & \text{otherwise,} \end{cases}$$

where $h(t) = \lfloor (1+t)/(1+\log_2 3) \rfloor$ and $\lfloor z \rfloor$ denotes the integral part of the number z. To describe this procedure we need to introduce some terminology. Suppose that a set S of u coins is tested and $F(S) = X$; then the complementary set $\bar{S}$ must contain y. We call this situation the *$u:n-u$ configuration*. In a $u:n-u$ configuration, suppose V from S and W from $\bar{S}$ with v and w coins, respectively, are tested and the feedback is $F(V \cup W) = \varnothing$. Then either $x \in V$ and $y \in W$, or $x \in S - V$ and $y \in \bar{S} - W$. We call this situation the *$v:u-v+w:n-u-w$ configuration*; the two alternatives are the *summands* of the configuration. Similarly we can define a configuration $z_1 : z_1' + z_2 : z_2' + \cdots + z_k : z_k'$ with k summands, where the ith alternative is $x \in z_i$ and $y \in z_i'$. To *halve an $x:y$ summand* means to take $\lceil x/2 \rceil$ coins from the first component and $\lfloor y/2 \rfloor$ coins from the last. Christen's procedure then essentially halves all summands or subsets in the solution spaces as long as a component or a subset of at least four coins remains. Some ad hoc rules are given to take care of cases with fewer than four coins.

EXAMPLE 5. For $t = 6$, $h(6) = 2$ and $1 + 2^{6-h(6)} = 17 > 3^{h(6)}$. Hence $N_C(6) = \lfloor 3(3^2 + 2^4)/4 \rfloor = 18$. For $n = 18$ coins we halve the set and test a set of nine coins. Suppose that the feedback is $\varnothing \cup XY$. Then we have the situation where the two irregulars are either both in the set tested or both in its complementary set. We halve both subsets and test a set of $\lfloor 9/2 \rfloor + \lfloor 9/2 \rfloor = 8$ coins. Suppose that the feedback is X. Then we have the $4:5+5:4$ configuration. We halve both summands and test a set of $\lceil 4/2 \rceil + \lfloor 5/2 \rfloor + \lceil 5/2 \rceil + \lfloor 4/2 \rfloor = 9$ coins. Suppose that the feedback is $\varnothing \cup XY$. Then we have the configuration $2:2+2:3+3:2+2:2$. Since none of the four summands has a component of size at least 4, we halve no more and instead use the ad hoc rules provided by Christen to solve the configuration in three more tests.

Recently, Hwang and Xu [7] gave a procedure for model D which yields an upper bound approximately $\frac{4}{3}\log_2 n$ for $T_D(n)$. They considered two situations $(a \times b)$ and $[n, b]$. The former denotes the case that y is known to be contained in a set A of a coins and x is known to be contained in a set B of b coins, where A and B are disjoint. The latter denotes the case that y is known to be contained in a set of n coins of which a subset B of b coins contains x. Note that $b = n$ denotes the canonical situation that a set of n coins contains both x and y. We can certainly identify y and x in the situation $(a \times b)$ by using the halving procedure on the two sets A and B separately. But surprisingly, the $\lceil \log_2 a \rceil + \lceil \log_2 b \rceil$ result is not optimal. Given t and a with $\alpha = [\log_2 a]$, the largest b such that the situation

$(a \times b)$ can be solved in t tests is

$$b^o = \sum_{i=\alpha}^{t} \binom{t}{i}$$

and the first subset to be tested should consist of $\lfloor a/2 \rfloor$ coins from A and

$$\max\left\{ b - \sum_{i=\alpha-1}^{t-1} \binom{t-1}{i}, 0 \right\}$$

coins from B. The validity and optimality of this procedure can be proved by induction.

To describe the procedure for the situation $[n, b]$, we need to define a function $d(u, v)$:

$$\begin{aligned}
d(0,0) &= d(0,1) = 1, \\
d(4w,0) &= 2^{3w-1} + 1, \qquad \text{for } w \geqslant 1, \\
d(4w,k) &= d(4w-1, k-1), \quad \text{for } w \geqslant 1, k = 1,2,\ldots, w+1,
\end{aligned}$$

$$d(4w+i, k) = \min\left\{ \sum_{j=k}^{w+1} d(4w+i-1, j), 2^{3w+i-1} + \varepsilon(k) \right\},$$

$$\text{for } w \geqslant 0,\ i = 1,2,3,\ k = 0,1,\ldots, w+1,$$

where $\varepsilon(k) = 1$ if $k = 0$ and $\varepsilon(k) = 0$ otherwise. We can prove by induction that the $[2^{3w+i-1} + 1, d(4w + i, k)]$ situation can always be solved in $4w + i - k$ tests. This implies that if we start with $n = d(4w + i, 0)$ coins, we can always identify x and y in $4w + i$ tests. Note that $d(4w, 0) = 2^{3w+i-1}$. Hence an upper bound of $T_D(n)$ is approximately $\frac{4}{3}\log_2 n$.

EXAMPLE 6. Suppose that $n = d(7,0) = 22$. We first test a subset S of seven coins. If $F(S) = \varnothing$, then the remaining $d(6,0) = 15$ coins can be done in six more tests by induction. If $F(S) = Y$, we have the new (15×7) situation. Since $\alpha = \lceil \log_2 15 \rceil = 4$ and $t = 6$, we have

$$b^o = \sum_{i=4}^{6} \binom{6}{i} = 22 > 7.$$

Hence the (15×7) situation can be solved in six more tests (note that seven tests are required if we test the 15 coins and the 7 coins separately). If $F(S) = X \cup XY$, we have the new $[22, 7]$ situation. Write $t = 4w + i = 7$; then we have $w = 1$ and $i = 3$. Set $k = 1$. Since $2^{3w+i-1} > 22$ and $f(4w + i, k) = 7$, the $[22, 7]$ situation can be solved in $4w + i - k = 6$ tests.

5. Conclusion. Our main objective in studying the two-irregular problems is to explore the boundaries between easy problems and hard problems in combinatorial search theory. A systematic study of all possible models under the test-type device

provides a spectrum of problems with varying degrees of difficulty. So far, optimal procedures have been found for two models, i.e., models B and C.

Near-optimal procedures, which can differ from optimal procedures by at most one test, have been found for model P and G. We suspect that the determination of an optimal procedure for model G is much harder than for model P. For the other three models we list the gaps between the lower bounds and the upper bounds (in approximate terms) in the following table:

Model	U	Q	D
lower bound	$2\log_2 n$	$2\log_3 n$	$2\log_3 n$
upper bound	$3\log_2 n$	$\log_r n$	$\frac{4}{3}\log_2 n$

where r is the golden ratio $(\sqrt{5}+1)/2 \cong 1.62$.

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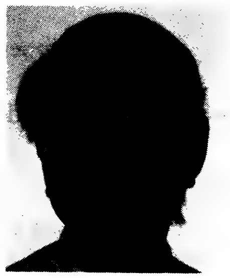
Gauss-Jordan Reduction: A Brief History

STEVEN C. ALTHOEN: I received my B.A. at Kenyon College and my Ph.D. in 1973 at The City University of New York under the direction of Eldon Dyer. I would like to thank Professor Stephen B. Maurer of Swarthmore College for preventing me from crediting the wrong Jordan in my texts by relaying Tucker's observation. My primary research interest is the classification of finite-dimensional real division algebras.



RENATE MCLAUGHLIN: I did my undergraduate work in Germany and received my Ph.D. in complex analysis (1968) at the University of Michigan under the direction of Peter L. Duren. My primary research interest is still complex analysis. But with two children in elementary school, I have also become interested in the mathematics that goes on (or better: does not go on) in elementary school classrooms.

A. W. Tucker [**28**] has pointed out that it was the geodesist Wilhelm Jordan (1842–1899) and not the mathematician Camille Jordan (1838–1922) who introduced the Gauss-Jordan method of solving systems of linear equations. There is a natural tendency to attribute the method to Camille Jordan, who is justly credited with another linear algebra topic, the Jordan normal form. Recall that in Gaussian Elimination, row operations are used to change the coefficient matrix to an upper triangular matrix. The solution is then found by back substitution, starting from the last equation in the reduced system. In Gauss-Jordan Reduction, row operations are used to diagonalize the coefficient matrix, and the answers are read directly.

This article is about Wilhelm Jordan and the introduction of his method.

1. How do we know it was Wilhelm Jordan? There is little doubt as to the identity of the Jordan referred to in "Gauss-Jordan reduction." It seems that the names were attached to the method by numerical analysts. For example, Householder [**13**, p. 141] states,

> The Gauss-Jordan method, so-called, seems to have been described first by Clasen [7]. Since it can be regarded as a modification of Gaussian elimination, the name of Gauss is properly applied, but that of Jordan seems to be due to an error, since the method was described only in the third edition of his *Handbuch der Vermessungskunde* prepared after his death.

This reference is clear, but so is the fact that the third edition of Jordan's book was prepared by Jordan himself, well before his death. The foreword to the third edition

is dated May 1888 and signed by Jordan. Furthermore, the book review by C. Müller [**23**] states that Jordan himself prepared the fourth edition, which appeared in 1895. Incidentally, the frontispiece of that particular edition is a picture of Gauss, since he was famous for his geodesy as well as his mathematics and physics. It should also be noted that since Clasen's article [**7**] appeared in the same year as Jordan's third edition, it seems that their discoveries were independent.

Johnson [**15**, p. 66] states:

> So far as I know, this method of viewing the reduced normal equations did not appear explicitly in any treatise upon Least Squares prior to the third edition of W. Jordan's *Vermessungskunde*,... .

It should be pointed out that although the first edition contains no hint of Gauss-Jordan reduction, the germ of the idea is already present in the second edition (1877) in the example on pages 34 and 35.

Kunz [**18**, p. 221] has a section whose title "Gauss-Jordan Method" has the footnote: W. Jordan, "Handbuch der Vermessungskunde." Both Bartlett [**5**] and Leland [**19**] mention Jordan's *Handbuch der Vermessungskunde* in their bibliographies. In Gewirtz, Sitomer, and Tucker [**10**, p. 246] we find:

> Wilhelm Jordan (1819-1904 [sic]) devised the pivot reduction algorithm, known as Gauss-Jordan elimination, for geodetic reasons.

Writers of current linear algebra texts generally apply the name "Gauss-Jordan reduction" without reference to its origins. However, it was through Wilhelm Jordan's *Handbuch der Vermessungskunde* that Gauss-Jordan reduction was introduced to the world. Today Clasen, whom we discuss below, is largely forgotten.

2. Who was W. Jordan? We know about Wilhelm Jordan through several obituaries [**26**], [**16**], [**24**] (see also [**11**] and [**25**]). He was born March 1, 1842, in the town of Ellwangen in southern Germany. Following high school, he attended what today would be called an engineering college in Stuttgart. He worked for two years as an engineering assistant on preliminary work for railroad construction and as a "trigonometer" on measuring elevations. He spent two further years as an "Assistant" in geodesy at the college in Stuttgart, and in 1868, when he was only 26 years old, he became full professor of geodesy at the technical college in Karlsruhe.

Jordan was actively involved in surveying several areas of Germany. In 1873 he became editor of the *Zeitschrift für Vermessungswesen* (*Journal for Surveying*) and he remained in this capacity until his death. Jordan was a prolific writer. His major work started in 1873 as *Taschenbuch der Praktischen Geometrie* (*Pocket Book of Practical Geometry*). In later editions this became the *Handbuch der Vermessungskunde* (*Handbook of Geodesy*). By the time of Jordan's death, five editions of this book had appeared, and it had been translated into French, Italian, and Russian.

Jordan's ability to present abstract ideas in lively ways was credited with the wide distribution of this book.

In 1882 Jordan left Karlsruhe and went to the Technical University in Hanover. He continued to be active in surveying and in publishing his works. Apparently he was a first-rate teacher who had a particular talent for bringing out the connections between theory and the real-world problems confronting him. His field trips with students were praised as examples of how one ought to teach.

Although apparently physically fit, Jordan had suffered for years from heart disease and other problems. In 1899, at the age of 57, he died in a state of depression.

3. What was the Problem? Gauss invented the Method of Least Squares to find a best linear function to approximate observed data. The method was naturally attractive to geodesists. Here is a quick description of the mathematics involved, along with an interesting theorem, which was first explicitly stated by Gauss.

Suppose we make m observations, each of which depends on n inputs. Following Jordan, we call the observed values $-l_1, \dots, -l_n$, but to undertake a modern analysis, it will be useful to label the input vectors

$$(a_{i1}, \dots, a_{in}), \quad \text{for } i = 1, \dots, m.$$

Our m observations l_i are a function of the input vectors. It is desirable to have a good linear approximation for that function. Let

$$y = L(y_1, \dots, y_n) = y_1 x_1 + \cdots + y_n x_n$$

denote a general linear function with coefficients $x_1, \dots, x_n$. Let v_i denote the error that arises from the linearity assumption:

$$\begin{aligned} v_i &= L(a_{i1}, \dots, a_{in}) - (-l_i) \\ &= a_{i1}x_1 + \cdots + a_{in}x_n + l_i, \quad \text{for } i = 1, \dots, m. \end{aligned} \tag{1}$$

We select $x_1, \dots, x_n$ to minimize the sum of the squared errors

$$E = v_1^2 + \cdots + v_m^2.$$

Thus, the Method of Least Squares reduces to a simple problem of multivariate calculus in which we solve the system of *normal equations*:

$$\partial E / \partial x_j = 0, \qquad \text{for } j = 1, \dots, n.$$

Let $A = (a_{ij})$, $v = \operatorname{col}(v_1, \dots, v_m)$, $l = \operatorname{col}(l_1, \dots, l_m)$, and $x = \operatorname{col}(x_1, \dots, x_n)$, where col means *column vector*. System 1) can then be written as

$$v = Ax + l.$$

Direct multiplication yields:

$$\begin{aligned} E &= v_1^2 + \cdots + v_m^2 \\ &= \sum a_{i1}a_{i1}x_1^2 + 2\sum a_{i1}a_{i2}x_1x_2 + \cdots + 2\sum a_{i1}a_{in}x_1x_n \\ &\quad + 2\sum a_{i1}l_ix_1 + \cdots, \end{aligned}$$

where we have listed the terms involving x_1 and all the summations run from $i = 1$ to $i = m$. Then

$$\partial E/\partial x_1 = 2\sum a_{i1}a_{i1}x_1 + 2\sum a_{i1}a_{i2}x_2 + \cdots + 2\sum a_{i1}a_{in}x_n + 2\sum a_{i1}l_i = 0,$$

or

$$\sum a_{i1}a_{i1}x_1 + \sum a_{i1}a_{i2}x_2 + \cdots + \sum a_{i1}a_{in}x_n + \sum a_{i1}l_i = 0.$$

Gauss denotes $\Sigma a_{ij}a_{ik}$ by $[a_j a_k]$ and $\Sigma a_{ij}l_i$ by $[a_j l]$. Thus, he would write this equation as

$$[a_1a_1]x_1 + \cdots + [a_1a_n]x_n + [a_1l] = 0.$$

The remaining $n - 1$ equations are obtained similarly.

It is now easy to verify that the entire system of normal equations can be represented in matrix form as

$$A^tAx = -A^tl$$

(where A^t denotes the transpose of A) with solution

$$x = -(A^tA)^{-1}A^tl.$$

Note that A^tA is a symmetric matrix.

Here is a modern statement and proof of a theorem that occurs in Article 13 of Gauss's *Disquisitio de Elementis ellipticis Palladis* [**8**].

GAUSS'S THEOREM. *If Gauss-Jordan reduction is used to diagonalize the symmetric matrix*

$$\begin{bmatrix} A^tA & -A^tl \\ (-A^tl)^t & l^tl \end{bmatrix},$$

then the number in the lower right corner is the sum of the squared errors incurred by using the coefficients of the solution vector x as coefficients for the linear approximation and the given data points. That is, after triangularization, the lower right corner will contain

$$E = v_1^2 + \cdots + v_m^2 = v^tv.$$

Proof. View the system

$$\begin{bmatrix} A^tA & -A^tl \\ (-A^tl)^t & 0 \end{bmatrix}$$

as the linear programming problem:

$$\text{Maximize } w = (-A^tl)^tx \tag{2}$$

subject to

$$A^tAx = -A^tl.$$

Then the number in the lower right after triangularization will be the negative of the value of the objective function $(A^tl)^tx$ at the solution to the system. So if we begin

with $l^t l$ in that corner, we will obtain

$$l^t l + (A^t l)^t x,$$

where $x = -(A^t A)^{-1} A^t l$.
On the other hand,

$$\begin{aligned}
v^t v &= (Ax + l)^t (Ax + l) \\
&= (x^t A^t + l^t)(Ax + l) \\
&= x^t A^t A x + x^t A^t l + l^t A x + l^t l \\
&= x^t A^t A\left[-(A^t A)^{-1} A^t l\right] + x^t A^t l + l^t A x + l^t l \\
&= -x^t A^t l + x^t A^t l + l^t A x + l^t l \\
&= l^t l + (A^t l)^t x. \blacksquare
\end{aligned}$$

Thus, if we triangularize the symmetric system

$$\begin{bmatrix} A^t A & -A^t l \\ (-A^t l)^t & l^t l \end{bmatrix},$$

we obtain not only the solution to the normal equations, but also the value of the sum of the squared errors for the particular observations.

4. What was Gauss's Method and Notation? In 1810, in his *Disquisitio de Elementis ellipticis Palladis*, Gauss [**8**], [**9**, p. 123] sets out to determine details ("elliptical elements") about the orbit of Pallas, the second-largest asteroid of the solar system. He obtains a system of linear equations in six unknowns, where not all equations can be satisfied simultaneously. Hence he needs to determine values for the unknowns that will minimize the total squared error. Instead of merely solving the problem at hand, Gauss digresses and introduces a method for dealing with such systems of linear equations in general. This is where his characteristic notation appears for the first time.

To save space, we use the symbols of Section 3 above (Gauss lets $p, q, r, \ldots$ denote the variables and writes $\Omega = w^2 + w'^2 + w''^2 + \cdots$ for the sum of the squared errors, etc.). Gauss's starting point is the system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n + l_1 = v_1, \\ \quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + l_m = v_m. \end{cases}$$

He merely states that "it is easy to see" [*facile quidem perspicitur*] that in order for the total squared error $E = v_1^2 + \cdots + v_m^2$ to be a minimum, the following conditions must be satisfied:

$$\begin{cases} a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m = 0, \\ \quad \cdots \\ a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m = 0. \end{cases}$$

Gauss does not use the words "method of least squares" in the section where he considers this system of equations, but earlier in the paper [**8**, p. 16] he indicates that he has long used "certain tricks" [*quaedam artificia practica*] that make the application of the method of least squares more convenient. At any rate, the last system is (except for a common factor of 2) identical to the system of normal equations $\partial E/\partial x_1 = 0, \ldots, \partial E/\partial x_n = 0$. Now Gauss introduces his characteristic notation

$$[a_1 l] = a_{11}l_1 + a_{21}l_2 + \cdots + a_{m1}l_m,$$
$$[a_i a_k] = a_{1i}a_{1k} + a_{2i}a_{2k} + \cdots + a_{mi}a_{mk},$$

and so forth. Hence the unknowns $x_1, \ldots, x_n$ need to be determined from the equations

$$\begin{cases} [a_1a_1]x_1 + \cdots + [a_1a_n]x_n + [a_1l] = 0, \\ \cdots \\ [a_na_1]x_1 + \cdots + [a_na_n]x_n + [a_nl] = 0. \end{cases} \tag{3}$$

Gauss now uses a procedure (but not matrix notation) that is essentially equivalent to what today is known as Gaussian elimination. He expresses E explicitly in terms of $x_1, \ldots, x_n$ and shows that $E - R_1^2/[a_1a_1]$ is independent of x_1. Here R_1 denotes the left-hand side of the first row of the system above. Next he eliminates x_2 from $E^{(1)} = E - R_1^2/[a_1a_1]$, and so forth. In this way he obtains a representation

$$E = \frac{(A_1(x_1, \ldots, x_n))^2}{a_1} + \frac{(A_2(x_2, \ldots, x_n))^2}{a_2} + \cdots + \frac{(A_n(x_n))^2}{a_n} + A,$$

where $a_1, \ldots, a_n$ are positive numbers. Here A represents the minimum value of E, and the unknowns are determined by back substitution from the equations

$$A_1(x_1, \cdots, x_n) = 0,$$
$$A_2(x_2, \cdots, x_n) = 0,$$
$$\cdots$$
$$A_n(x_n) = 0.$$

This is what we called Gauss's Theorem in Section 3 above.

5. What was Jordan's Method and Notation? On page 83 of the third edition of his *Handbuch der Vermessungskunde* [**17**], Jordan presents a numerical example, derived from a least squares application in geodesy, to illustrate the method that has come to be known as Gauss-Jordan reduction. The particular system he considers would now be written as

$$\begin{cases} 17.50x - 6.50y - 6.50z = 2.14, \\ -6.50x + 17.50y - 6.50z = 13.96, \\ -6.50x - 6.50y + 20.50z = -5.40, \\ -2.14x - 13.96y + 5.40z = w - 100.34, \end{cases} \tag{4}$$

where the w in the last line is from equation (2). However, since all systems of

normal equations are symmetric, Jordan adopts an abbreviated representation:

$$
\begin{array}{rrrrl}
\underline{17.50x} - & 6.50y - & 6.50z - & 2.14 & = 0 \\
+ & \underline{17.50y} - & 6.50z - & 13.96 & = 0 \\
 & + & \underline{20.50z} + & 5.40 & = 0 \\
 & & + & \underline{100.34}, &
\end{array}
$$

where the number $l'l$ just "floats" at the lower right. The modern method for solving system (4) uses row operations as follows.

$$
\begin{bmatrix}
17.50 & -6.50 & -6.50 & 2.14 \\
-6.50 & 17.50 & -6.50 & 13.96 \\
-6.50 & -6.50 & 20.50 & -5.40 \\
-2.14 & -13.96 & 5.40 & -100.34
\end{bmatrix}
$$

$$
\begin{array}{l}
\\
R_2 + (6.5/17.5)R_1 \\
R_3 + (6.5/17.5)R_1 \\
R_4 + (2.14/17.5)R_1
\end{array}
\begin{bmatrix}
17.50 & -6.50 & -6.50 & 2.14 \\
0 & 15.09 & -8.91 & 14.75 \\
0 & -8.91 & 18.09 & -4.61 \\
0 & -14.75 & 4.61 & -100.08
\end{bmatrix}
$$

$$
\begin{array}{l}
R_1 + (6.5/15.09)R_2 \\
\\
R_3 + (8.91/15.09)R_2 \\
R_4 + (14.75/15.09)R_2
\end{array}
\begin{bmatrix}
17.50 & 0 & -10.34 & 8.49 \\
0 & 15.09 & -8.91 & 14.75 \\
0 & 0 & 12.83 & 4.10 \\
0 & 0 & -4.10 & -85.66
\end{bmatrix}
$$

$$
\begin{array}{l}
R_1 + (10.34/12.83)R_3 \\
R_2 + (8.91/12.83)R_3 \\
\\
R_4 + (4.10/12.83)R_3
\end{array}
\begin{bmatrix}
17.50 & 0 & 0 & 11.79 \\
0 & 15.09 & 0 & 17.60 \\
0 & 0 & 12.83 & 4.10 \\
0 & 0 & 0 & -84.35
\end{bmatrix}
$$

$$
\begin{gathered}
x = 11.79/17.50 = 0.67 \\
y = 17.60/15.09 = 1.17 \\
z = 4.10/12.83 = 0.32 \\
\text{Total squared error} = 84.35
\end{gathered}
$$

Jordan's presentation uses the same arithmetic in a different arrangement. In the first place, he takes advantage of the symmetry that remains as the reduction proceeds. Secondly, he uses different size type so that numbers to be added can be placed conveniently one above the other without confusion. Finally, to achieve this convenience, the entries above the diagonal are placed to the right of the matrix. Here then is Jordan's layout of his method. We have inserted letters next to several of the numbers so that they can be identified in the notes that follow. Of course these letters were not included by Jordan.

a	b	c	l			
$+17.50$	-6.50	-6.50	-2.14			
	$+17.50$	-6.50	-13.96			
	$-2.41a$	$-2.41a$	$-0.79a$			
		$+20.50$	$+5.40$			
		$-2.41a$	$-0.79a$			
			$+100.34$			
			$-0.26b$			
	$+15.09c$	$-8.91c$	$-14.75c$	$-6.50d$		
		$+18.09e$	$+4.61e$	$- +.50f$		
		$-5.26g$	$-8.71g$	$-3.84h$		
			$+100.08i$	$-2.14j$		
			$-14.42k$	$-6.35h$		
		$+12.83l$	$-4.10l$	$-10.34m$	$-8.91n$	
			$+85.66o$	$-8.49m$	$-14.75p$	
			$-1.31q$	$-3.30r$	$-2.85s$	
			$+84.35t$	$-11.79u$	$-17.60v$	$-4.10w$
			$= [vv]$	-17.50	-15.09	-12.83 neg. denominator
				$+0.67$	$+1.17$	$+0.32$
				$= x$	$= y$	$= z$

(a) $\frac{6.5}{17.5}R_1$

(b) $\frac{2.14}{17.5}R_1$

(c) $R_2 + \frac{6.5}{17.5}R_1$

(d) $(1,2)$-entry of original system

(e) $R_3 + \frac{6.5}{17.5}R_1$

(f) $(1,3)$-entry of original system

(g) $\frac{8.91}{15.09}R_2$

(h) $\frac{6.5}{15.09}R_2$

(i) $R_4 + \frac{2.14}{17.5}R_1$

(j) $(1,4)$-entry of original system

(k) $\frac{14.75}{15.09}R_2$

(l) $R_3 + \frac{8.91}{15.09}R_2$

(m) $R_1 + \frac{6.5}{15.09}R_2$

(n) $(2,3)$-entry after first reduction

(o) $R_4 + \frac{14.75}{15.09}R_2$

(p) $(2,4)$-entry after first reduction

(q) $\frac{4.10}{12.83}R_3$

(r) $\frac{10.34}{12.83}R_3$

(s) $\frac{8.91}{12.83}R_3$

(t) $R_4 + \frac{4.10}{12.83}R_3$

(u) $R_1 + \frac{10.34}{12.83}R_3$

(v) $R_2 + \frac{8.91}{12.83}R_3$

(w) $(3,4)$-entry after second reduction

To describe problems in general, Jordan uses and extends the notation of Gauss. He attempts, in the fashion of the times, to give recursive formulae without the help of an index or subscripts. Thus, the input data are labeled a, b, c, etc. and l. In [**17**, p. 77] he writes the general normal equations as:

$$
\begin{aligned}
\underline{[aa]x} + [ab]y + [ac]z + [al] &= 0,\\
\underline{[bb]y} + [bc]z + [bl] &= 0,\\
\underline{[cc]z} + [cl] &= 0,\\
\underline{[ll]}&.
\end{aligned}
$$

Then, Gauss-Jordan reduction is given by the tableau [**17**, p. 82]

$$
\begin{array}{ccccc|ccccc}
[aa]_3 & [ab]_0 & [ac]_0 & [ad]_0 & [al]_0 & & & & & \\
 & [bb] & [bc] & [bd] & [bl] & & & & & \\
 & & [cc] & [cd] & [cl] & & & & & \\
 & & & [dd] & [dl] & & & & & \\
 & & & & [ll] & & & & & \\
\hline
 & [bb.1]_3 & [bc.1]_1 & [bd.1]_1 & [bl.1]_1 & [ab]_0 & & & & \\
 & & [cc.1] & [cd.1] & [cl.1] & [ac]_0 & & & & \\
 & & & [dd.1] & [dl.1] & [ad]_0 & & & & \\
 & & & & [ll.1] & [al]_0 & & & & \\
\hline
 & & [cc.2]_3 & [cd.2]_2 & [cl.2]_2 & (ac.1) & [bc.1]_1 & & & \\
 & & & [dd.2] & [dl.2] & (ad.1) & [bd.1]_1 & & & \\
 & & & & [ll.2] & (al.1) & [bl.1]_1 & & & \\
\hline
 & & & [dd.3]_3 & [dl.3]_3 & (ad.2) & (bd.2) & [cd.2]_2 & & \\
 & & & & [ll.3] & (al.2) & (bl.2) & [cl.2]_2 & & \\
\hline
 & & & & [ll.4] & (al.3) & (bl.3) & (cl.3) & [dl.3]_3 & \text{Numer.} \\
 & & & & & [aa]_3 & [bb.1]_3 & [cc.2]_3 & [dd.3]_3 & \text{Denom.} \\
\cline{6-10}
 & & & & & -x & -y & -z & -t & \text{Quotient}
\end{array}
$$

The entries in this tableau are determined by formulae that yield the usual Gauss-Jordan reduction. For example,

$$[bb.1] = [bb] - \frac{[ab]}{[aa]}[ab],$$

$$(ac.1) = [ac] - \frac{[bc.1]}{[bb.1]}[ab].$$

6. What about Clasen? In the quotation from Householder given above, we find the claim that the so-called Gauss-Jordan method seems to have been first described by Clasen in [7]. In [**22**], Thomas Muir also writes about this same article by Clasen:

> In the solution of a set of linear equations with arithmetical coefficients considerable latitude is available in the choice of the series of derived equations which is to end up with the value of the unknowns. The choice made by the writer ... is made with real skill and deserves attention.

Clasen's name appears as "Clasen (abbé B.-I.), curé-doyen d'Echternach (Grand-Duché de Luxembourg)" in the membership lists of The Société Scientifique de Bruxelles from 1887–1901 [**2**]. His name appears on the list of deceased members in 1902 [**3**] and it would seem from the historical volume [**4**, p. 62] that he died that year. He published only one article in the Annales de la Société Scientifique de Bruxelles [**4**, pp. 121 and 234]. In this section we take a closer look at that article.

First, it is worth noting that Clasen gives no references, and the name of Gauss (or Jordan) is never mentioned. The only other mathematician in the article is Paul Mansion, who apparently served as referee. Clasen thanks him in a brief footnote. In today's notation, Clasen's method applied to the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

becomes

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix} \to \begin{bmatrix} m & 0 & a_{13}^{(1)} & \cdots & b_1^{(1)} \\ 0 & m & a_{23}^{(1)} & \cdots & b_2^{(1)} \\ a_{31} & a_{32} & a_{33} & \cdots & b_3 \\ & & \cdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & b_n \end{bmatrix} \to$$

$$\begin{bmatrix} m & 0 & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & b_1^{(1)} \\ 0 & m & a_{23}^{(1)} & a_{24}^{(1)} & \cdots & b_2^{(1)} \\ 0 & 0 & R & a_{34}^{(1)} & \cdots & b_3^{(1)} \\ & & \cdots & & & \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & b_n \end{bmatrix} \to \begin{bmatrix} R & 0 & 0 & a_{14}^{(2)} & \cdots & b_1^{(2)} \\ 0 & R & 0 & a_{24}^{(2)} & \cdots & b_2^{(2)} \\ 0 & 0 & R & a_{34}^{(1)} & \cdots & b_3^{(1)} \\ & & & \cdots & & \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & b_n \end{bmatrix}, \text{ etc.}$$

Here,

$$m = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

and

$$R = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The arrows indicate the obvious row operations.

Clasen points out repeatedly that his method does not involve division—a definite sign of precalculator days. He also inserts extra equations as checks.

Clasen himself does not use matrix notation, rather he writes the successively occuring equations in a sequence that seems rather laborious. For example, if he were to solve a 3 × 3 system, he would proceed as follows.

$$\begin{aligned}
X_1 &: a_{11}x + a_{12}y + a_{13}z + b_1 = 0 \\
Y_1 &: a_{21}x + a_{22}y + a_{23}z + b_2 = 0 \\
Z_1 &: a_{31}x + a_{32}y + a_{33}z + b_3 = 0 \\
\hline
X_1 &: a_{11}x + a_{12}y + a_{13}z + b_1 = 0
\end{aligned}$$

$$\begin{array}{rlrr}
a_{11}Y_1: & a_{11}a_{21}x + a_{11}a_{22}y & +a_{11}a_{23}z & +a_{11}b_2 = 0 \\
-a_{21}X_1: & -a_{21}a_{11}x - a_{21}a_{12}y & -a_{21}a_{13}z & -a_{21}b_1 = 0 \\
\hline
Y_2: & my & +(a_{11}a_{23} - a_{21}a_{13})z & +a_{11}b_2 - a_{21}b_1 = 0
\end{array}$$

$$\begin{array}{rlrr}
mX_1: & ma_{11}x + ma_{12}y & +ma_{13}z & +mb_1 = 0 \\
-a_{12}Y_2: & -a_{12}my - a_{12}(a_{11}a_{23} - & a_{21}a_{13})z & -a_{12}(a_{11}b_2 - a_{21}b_1) = 0 \\
& ma_{11}x + [ma_{13} - a_{12}a_{11}a_{23} + a_{12} & a_{21}a_{13}]z & +mb_1 - a_{12}(a_{11}b_2 - a_{21}b_1) = 0 \\
& ma_{11}x & +a_{11}a'_{13}z & +a_{11}b'_1 = 0 \\
X_2: & mx & +a'_{13}z & +b'_1 = 0
\end{array}$$

$$\begin{array}{rllr}
mZ_1: & ma_{31}x & +ma_{32}y + ma_{33}z + & mb_3 = 0 \\
-a_{32}Y_2: & & -a_{32}my - a_{32}a'_{23}z & -a_{32}b'_2 = 0 \\
-a_{31}X_2: & -a_{31}mx & -a_{31}a'_{13}z & -a_{31}b'_1 = 0 \\
\hline
Z_3: & & Rz & +b'_3 = 0
\end{array}$$

$$\begin{array}{rllr}
RY_2: & Rmy & +Ra'_{23}z & +Rb'_2 = 0 \\
-a'_{23}Z_3: & & -a'_{23}Rz & -a'_{23}b'_3 = 0 \\
\hline
Y_3: & Rmy & & +b''_2 = 0
\end{array}$$

$$\begin{array}{rllr}
RX_2: & Rmx & +Ra'_{13}z + & Rb'_1 = 0 \\
-a'_{13}Z_3: & & -a'_{13}Rz - & a'_{13}b'_3 = 0 \\
X_3: & Rmx & & +b''_1 = 0
\end{array}$$

At the end of the article, Clasen emphasizes that his method requires fewer calculations than a method using determinants, and he shows how his method can be used to compute determinants.

The same volume of the Annales de la Société Scientifique de Bruxelles that contains Clasen's article also has a review by P. Mansion [**20**]. Mansion (1844–1919) was a well-known Belgian mathematician and, from 1865, professor at the University of Ghent [**1**], [**27**]. Mansion calls Clasen's method "the method of equal coefficients," and he gives a fairly detailed description of Clasen's work. Mansion also gives no reference to Gauss or anyone else.

Later, in 1930, R. Mehmke [**21**] describes the "traditional and still customary" method of solving systems of linear equations—it is clearly Gaussian elimination, although the name Gauss is never mentioned. Mehmke does observe that back substitution can be troublesome, especially in large systems. He states that Clasen's method is superior to Gaussian elimination, even though the world has completely ignored the former.

The purpose of Mehmke's article is to remind the readers of Clasen's work. But Mehmke introduces a more formalized notation (e.g. arrays of numbers where each variable has a column) that does not appear in Clasen's work. To aid with the computations, Mehmke even uses paper strips! Finally, Mehmke "improves" Clasen's method by adapting it to accelerated elimination, where several variables are eliminated at each step. Mehmke mentions neither Gauss nor Jordan.

7. What about other methods of solving linear systems? In 1947, E. Bodewig [**6**, p. 931] writes concerning large systems of linear equations that none of the available methods will remove the inherent difficulties and that calculating machines need to be used. He continues, "The newly famous electronic calculating machines solve such systems automatically, so that the method used does not matter. However, the price of such machines is so high that only one specimen of each of the two types exists (or will ever exist). Everyone else in the world must be satisfied with the usual calculating machines for which the method does matter."

The lack of adequate calculators made every little shortcut important. Even though the solution of linear systems was understood in principle, the process was unpleasant. Different methods for dealing with this unpleasantness were popular at various times. At least two survey articles ([**6**] and [**14**]) list by name many long forgotten methods of solving linear systems.

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Merlin's Magic Square

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In this article, we present a mathematical analysis of a game played on a popular electronic toy for children. To discuss the nature of the mathematics involved at the very beginning would bias the reader's attention and would, for some, spoil any element of surprise. So we proceed directly to a description of the game.

1. The Game.

The game is played on a battery operated toy known as MERLIN manufactured by Parker Brothers. [See Figure 1.]

One of the six different games that MERLIN can play is called Magic Square, hence the title of this article. The "keyboard" of MERLIN consists of nine buttons which form a 3×3 array and are numbered from 1 to 9 as in the accompanying diagram. MERLIN has other buttons as well but they are not relevant to the game of Magic Square. [See Figure 2.]

The buttons are translucent and through each of them can be seen an individual light; each light is in one of two possible states, "OFF" or "FLASHING." For convenience, we will say that the corresponding button is OFF or FLASHING. To begin the game, the circuitry in the toy selects, at random, an initial pattern of flashing lights. [See Figure 3.]

At this point, by pressing any one of the buttons, the player can alter the states of a certain subset of the buttons according to the "rules of the game" that will be explained in detail below. To win the game, the player must, by pressing buttons one after another, transform the toy from its initial random pattern of flashing lights to a particular "winning" pattern; the winning pattern is the one in which all lights, except #5, are flashing. [See Figure 4.]

In a way that is predetermined by the circuitry of the game, the effect of pressing a button is to toggle its state and those of certain neighboring buttons to the opposite state. Although the predetermined selection of effected neighbors varies from button to button, there is a great deal of symmetry built into this selection so that the rules of the game can be easily learned by a child. The chart that follows

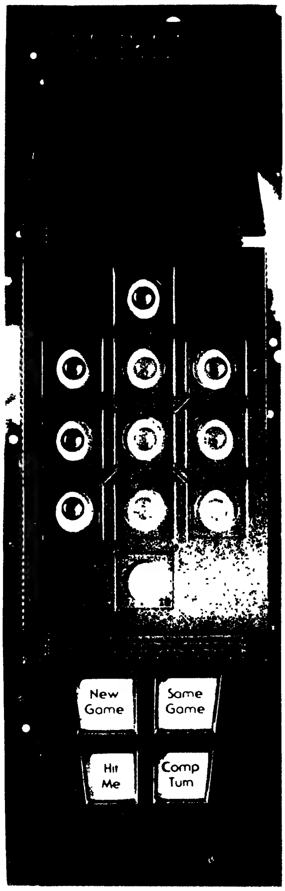

FIG. 1.

1	2	3
4	5	6
7	8	9

FIG. 2. Keyboard of Merlin's Magic Square.

OFF 1	FLASH 2	OFF 3
FLASH 4	FLASH 5	OFF 6
FLASH 7	OFF 8	OFF 9

FIG. 3. A typical initial pattern.

FLASH 1	FLASH 2	FLASH 3
FLASH 4	OFF 5	FLASH 6
FLASH 7	FLASH 8	FLASH 9

FIG. 4. The winning pattern.

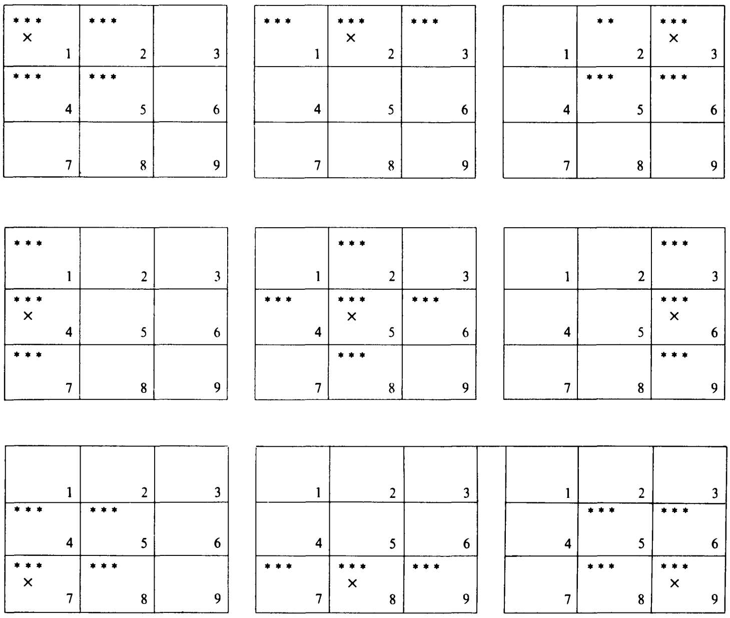

FIG. 5. Pressing the button marked X toggles the states of buttons marked $* * *$.

describes in detail the effects of pressing the various buttons:

Rules of Magic Square

Pressing button	1	toggles the states of buttons	1, 2, 4, and 5
	2		1, 2, and 3
	3		2, 3, 5, and 6
	4		1, 2, and 7
	5		2, 4, 5, 6, and 8
	6		3, 6, and 9
	7		4, 5, 7, and 8
	8		7, 8, and 9
	9		5, 6, 8, and 9

See Figure 5 for a visual summary of the rules.

As one observes a child playing this game, there are several questions that come to mind. Some games seem to last a very long time, over fifty moves; others are abandoned in hopeless frustration.

Question 1. Can the game always be won no matter what initial pattern the toy selects at the beginning?

Surely one would hope, in fairness to the young player, that the answer to this question is "yes"; but it is conceivable that there exist some initial patterns from which, with the stated rules of the game, there is no path to the winning pattern.

To an observer, the behavior of a player newly introduced to this game appears rather random. Also, since the effect of pressing a button can be immediately undone by pressing it again, the novice player will frequently "take back a move" by pressing a button twice in succession. After many games, the player acquires some skill in recognizing how certain patterns lead to certain other patterns, but in the back of the player's mind there is still the general question of anticipating the effect of pressing a long sequence of buttons. A particular instance of this question might be the following:

Question 2. Consider two plays of the game determined by pressing the sequences S and S' of buttons:

$$S: \qquad 3, 5, 8, 9, 2, 1, 8, 3, 7, 4, 8, 5$$
$$S': \qquad 9, 7, 2, 6, 8, 3, 3, 5, 2, 4, 7, 1, 5, 6, 2, 7$$

Given the same initial pattern, how will the patterns produced by the two plays S and S' differ?

We may say that the *length* of a play of the game is the number of moves (= number of buttons pressed) that it involves. Let us say that a play of the game is *optimal* if it is a play of shortest length among all plays that lead from the given initial pattern to the winning pattern, assuming that one exists. An *optimal* player is one whose play, in any game that can be won, is optimal. Because there are only a finite number ($2^9 = 256$) of possible patterns, it is clear that if the game can always be won from any initial state, then there is a maximum to the lengths of possible optimal plays.

Question 3. Assuming the game can always be won, what is the maximum length of the game for an optimal player?

Question 4. For which initial pattern(s) will an optimal play of this maximum length be required?

Question 5. Is there an algorithm for deciding, given an initial pattern, whether the game can be won and for finding an optimal play?

The reader may wish to seriously ponder some of these questions before reading the analysis in the next section. In fact, a reader with access to MERLIN is advised to play a few games before continuing.

2. The Mathematics.

A mathematical model of this game is obtained using the theory of vector spaces. As the field of scalars, we will take the binary field, $\mathscr{B} = \{0, 1\}$, whose only elements are 0 and 1, with binary arithmetic as the field operations. So addition, $\oplus$, and multiplication, $\odot$, for $\mathscr{B}$ are given in the following tables:

$\oplus$	0	1
0	0	1
1	1	0

$\odot$	0	1
0	0	0
1	0	1

Consider the space $\mathscr{B}^9$ of nine-tuples of zeros and ones as a vector space over the field $\mathscr{B}$. An arbitrary pattern P of flashing lights can be represented by a vector $\mathbf{v}_P = (v_1, v_2, \ldots, v_9) \in \mathscr{B}^9$ where

$$v_i = \begin{cases} 0 & \text{if in the pattern } P \text{ light } i \text{ is off} \\ 1 & \text{if in the pattern } P \text{ light } i \text{ is flashing.} \end{cases}$$

Thus, for example, the winning pattern, in which all lights except #5 are flashing, is represented by the vector

$$\mathbf{v}_W = (1,1,1,1,0,1,1,1,1).$$

Because vector addition in $\mathscr{B}^9$ is addition modulo 2 in each coordinate, we can use this to represent the effect of pressing a button. For example, the effect of pressing button #1 is to toggle the states of buttons 1, 2, 4, and 5, leaving the states of the remaining buttons unchanged. Thus if P is the current pattern and P' is the pattern that results when button #1 is pressed, then

$$\mathbf{v}_{P'} = \mathbf{v}_P \oplus \mathbf{u}_1 \text{ where } \mathbf{u}_1 = (1,1,0,1,1,0,0,0,0).$$

Example. If P is the pattern in which buttons 2, 3, 5, 8, and 9 are flashing,

$$[\text{i.e., } \mathbf{v}_P = (0,1,1,0,1,0,0,1,1)]$$

and button 1 is pressed, then in the resulting pattern P', buttons 1, 3, 4, 8, and 9 will be flashing

$$\begin{aligned}[\text{i.e., } \mathbf{v}_{P'} = (1,0,1,1,0,0,0,1,1) &= (0,1,1,0,1,0,0,1,1) \\ &\oplus (1,1,0,1,1,0,0,0,0).]\end{aligned}$$

Similarly, for $i = 2, 3, \ldots, 9$, if P' results from P by pressing button $\#i$, then $\mathbf{v}_{P'} = \mathbf{v}_P \oplus \mathbf{u}_i$ where

$$\begin{aligned}
\mathbf{u}_2 &= (1,1,1,0,0,0,0,0,0) \\
\mathbf{u}_3 &= (0,1,1,0,1,1,0,0,0) \\
\mathbf{u}_4 &= (1,0,0,1,0,0,1,0,0) \\
\mathbf{u}_5 &= (0,1,0,1,1,1,0,1,0) \\
\mathbf{u}_6 &= (0,0,1,0,0,1,0,0,1) \\
\mathbf{u}_7 &= (0,0,0,1,1,0,1,1,0) \\
\mathbf{u}_8 &= (0,0,0,0,0,0,1,1,1) \\
\mathbf{u}_9 &= (0,0,0,0,1,1,0,1,1).
\end{aligned}$$

In this language, the ability to win the game, given an initial pattern P, is equivalent to the existence of a finite sequence of buttons $i_1, i_2, \ldots, i_k$ satisfying

$$\mathbf{v}_W = \mathbf{v}_P \oplus \mathbf{u}_{i_1} \oplus \mathbf{u}_{i_2} \oplus \cdots \oplus \mathbf{u}_{i_k}. \tag{†}$$

Now, individual buttons may occur more than once in the sequence $i_1, i_2, \ldots, i_k$ but since vector addition is commutative and associative, we can regroup like terms

in (†); also, $\mathbf{u}_i \oplus \mathbf{u}_i = \mathbf{0}$ in $\mathscr{B}^9$. Thus the existence of a winning play, given the initial pattern P, is equivalent to the existence of scalars $s_1, s_2, \ldots, s_9 \in \mathscr{B}$ such that

$$\mathbf{v}_W = \mathbf{v}_P \oplus s_1\mathbf{u}_1 \oplus s_2\mathbf{u}_2 \oplus \cdots \oplus s_9\,\mathbf{u}_9 \qquad (\dagger\dagger)$$

The physical interpretation is that

$$s_i = \begin{cases} 0 & \text{if button } \#i \text{ is not pressed} \\ 1 & \text{if button } \#i \text{ is pressed} \end{cases}$$

3. The Conclusions.

The analysis given thus far enables us to answer Question 2 and to partially answer Question 3. It is obvious that pressing a button twice *in succession* has no effect on the outcome of the game. What is not obvious, and what follows from (††), is that pressing the same button twice, or an even number of times, *during an entire play of the game* has no effect on the final pattern; pressing the same button an odd number of times during a play of the game has the same effect on the final pattern as pressing it once. So the answer to Question 2 is that the two sequences S and S' result in the same pattern: both are equivalent to pressing the six buttons 1, 2, 4, 7, 8, 9 in any order.

Because no button need ever be pressed more than once, the answer to Question 3, assuming that every game can be won, is less than or equal to 9; any play is equivalent to a play of length at most 9. The longest possible optimal play would arise from an initial pattern P which would require, for a win, that each button be pressed exactly once [i.e., when $s_1 = s_2 = \cdots = s_9 = 1$].

Based on (††), the answer to Question 1 is also near at hand. We see from (††) that the game can be won from initial pattern P if and only if $\mathbf{v}_W - \mathbf{v}_P$ can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_9$. Since every element of $\mathscr{B}^9$ can be written in the form $\mathbf{v}_W - \mathbf{v}_P$ for some pattern P, we conclude that the game can be won from any initial pattern if and only if the set $\mathscr{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_9\}$ spans $\mathscr{B}^9$. Since $\mathscr{B}^9$ is nine-dimensional, this is equivalent to the condition that $\mathscr{B}$ is a basis for $\mathscr{B}^9$.

To answer Question 1 then, we can compute the determinant of the 9×9 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

whose columns are the vectors from $\mathscr{B}$. We find $\det A = 5 \equiv 1 \pmod 2$, so A is

invertible [as an element of the ring of 9×9 matrices over the field $\mathscr{B}$] and the answer to Question 1 is YES.

At this point, we can appreciate the problems faced by the designer of the game. The winning pattern and the consequences of pressing the various buttons [i.e., the Rules of the Game] must be chosen to involve enough symmetry so that children can easily remember them, can formulate goals, and can anticipate the effects of their actions during the play of the game; yet, the set of 9-tuples, determined as above by the rules of the game, must also be chosen to form a basis for $\mathscr{B}^9$.

The problem of finding an algorithm for optimal plays of the game becomes that of solving a certain system of linear equations over the field $\mathscr{B}$. To win the game, given an initial pattern P, we must find scalars $s_1, s_2, \ldots, s_9 \in \mathscr{B}$ such that

$$\mathbf{v}_W = \mathbf{v}_P \oplus s_1\mathbf{u}_1 \oplus s_2\mathbf{u}_2 \oplus \cdots \oplus s_9\mathbf{u}_9.$$

This is equivalent to solving the matrix equation $A\mathbf{x} = \mathbf{v}_W - \mathbf{v}_P$, where A is, as above, the matrix whose columns are $u_1, u_2, \ldots, u_9$. Because $-1 = 1$ in $\mathscr{B}$, it is convenient to rewrite this as the equivalent system $A\mathbf{x} = \mathbf{v}_W \oplus \mathbf{v}_P$.

Since A is invertible, the unique solution to this system is

$$\mathbf{x} = A^{-1}(\mathbf{v}_W \oplus \mathbf{v}_P).$$

Of course, A^{-1} here denotes the inverse of A as a binary matrix. This can be computed by finding the ordinary inverse of A as a real matrix and reducing this modulo 2.

The reader can verify that the inverse of A, as a real matrix, is

$$\left(\frac{1}{5}\right)\begin{bmatrix} -1 & 2 & -1 & 2 & 2 & -3 & -1 & -3 & 4 \\ 3 & -1 & 3 & -1 & -1 & -1 & -2 & 4 & -2 \\ -1 & 2 & -1 & -3 & 2 & 2 & 4 & -3 & -1 \\ 3 & -1 & -2 & -1 & -1 & 4 & 3 & -1 & -2 \\ -1 & 2 & -1 & 2 & -3 & 2 & -1 & 2 & -1 \\ -2 & -1 & 3 & 4 & -1 & -1 & -2 & -1 & 3 \\ -1 & -3 & 4 & 2 & 2 & -3 & -1 & 2 & -1 \\ -2 & 4 & -2 & -1 & -1 & -1 & 3 & -1 & 3 \\ 4 & -3 & -1 & -3 & 2 & 2 & -1 & 2 & -1 \end{bmatrix},$$

so the inverse of A as a binary matrix is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Example. Suppose the initial pattern P has lights 1, 2, 3, 5, 8, and 9 flashing. So

$$\mathbf{v}_P = (0,1,1,0,1,0,0,1,1).$$

To discover which buttons to press to win, add $\mathbf{v}_W$ to $\mathbf{v}_P$ to get

$$\begin{aligned}\mathbf{v}_W \oplus \mathbf{v}_P &= (1,1,1,1,0,1,1,1,1) \oplus (1,1,1,0,1,0,0,1,1)\\ &= (0,0,0,1,1,1,1,0,0).\end{aligned}$$

Then compute (always modulo 2)

$$\mathbf{x} = A^{-1}(\mathbf{v}_W \oplus \mathbf{v}_P) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Conclude that pressing buttons 2, 3, and 4, in any order, will result in a win.

Here is a more mechanistically minded presentation of the algorithm. Given an initial pattern P, select a subset, S_P, of the columns of A^{-1} as follows: $5 \in S_P$ iff button 5 is flashing; for $i \neq 5, i \in S_P$ iff button i is off. Let A_P^{-1} denote the submatrix of A^{-1} obtained by retaining only the columns of A^{-1} whose indices belong to S_P. Then the optimal play consists in pressing button i iff the sum, modulo 2, of the entries in the ith row of A_P^{-1} is 1.

If the matrix A^{-1} is visible [displayed on screen by an overhead projector, for example], it is not difficult with MERLIN in hand to implement this algorithm mentally, to the great astonishment of most of the audience.

Finally, we will answer Questions 3 and 4: are there any initial pattern(s) which require that all 9 buttons be pressed to win? In mathematical terms, the question becomes: for which P, if any, does

$$A^{-1}(\mathbf{v}_W \oplus \mathbf{v}_P) = (1,1,\ldots,1)?$$

Because A is invertible, we know that such a P does exist and is unique. The reader can verify that the unique solution to the equation above is

$$\mathbf{v}_P = (0,1,0,1,1,1,0,1,0).$$

In other words, if the initial pattern has buttons 2, 4, 5, 6, and 8 flashing, then it will require pressing all 9 buttons once to win.

UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

Neighborly Families of Congruent Convex Polytopes

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A family of convex d-polytopes in E^d is called **neighborly** [**5**], [**6**], [**10**]–[**16**] if every two members meet in a $(d-1)$-dimensional set (which lies, therefore, on a hyperplane; this hyperplane separates them and contains a facet of each one of them).

A neighborly family in E^d, $d \geqslant 3$, can be infinite [**3**], [**9**]; however, if every member in a neighborly family has at most k facets, then there can be at most 2^k members [**8**]. We [**16**], [**15**] have recently solved Bagemihl's conjecture ([**1**], see also [**2**], [**5**], [**6**]), which states that the maximum number of neighborly tetrahedra in E^3 is 8; our proof depends heavily on Baston [**2**] and on a few searches by computer (Baston showed that the maximum is at most 9). The best known upper bound for neighborly families of d-simplices in E^d for $d > 3$ is 2^{d+1} [**8**]; the best known lower bound is 2^d [**11**].

Concerning neighborly families of translates of a given convex polytope, there is an established maximum of 5 in E^3 [**4**] and an upper bound of $2d - 1$ for all $d \geqslant 4$, which is conjectured to be also the maximum [**4**]. If all the members of a neighborly family are translates of a d-cube in E^d, then the maximum number of members is $d + 1$ [**12**]; but the following problem is open [**7**, #55]; [**13**].

Problem 1. What is the maximum number of congruent convex polytopes in a neighborly family in E^3? It seems probable that it is finite.

The analogs of Problem 1 in higher dimensions are no longer open [**13**]: there is no upper bound to the number of congruent convex polytopes in a neighborly family in E^d, $d \geqslant 4$; the same is true for congruent symmetric ones in E^d, $d \geqslant 4$, and for just symmetric ones in E^d, $d \geqslant 3$.

The maximum number of neighborly d-cubes is 3 squares in the plane and 6 cubes in E^3 and is conjectured to be $3 \cdot 2^{d-2}$ for all $d \geqslant 4$ [**2**], [**14**]. If we are concerned only with combinatorial types, then the maximum is 4 in the plane, and there is the upper bound 2^{2d} for all $d \geqslant 3$ [**8**]. There is an example of 24 neighborly convex polytopes in E^3, all combinatorially equivalent to a cube [**10**, p. 282–283]. Thus we have Problem 2.

Problem 2. What is the maximum number of neighborly convex polytopes in E^3, all combinatorially equivalent to a cube? This maximum is between 24 and 64; the known bounds for the corresponding maximum for prisms are 16 [**10**, p. 282] and 32 [**8**].

It has been remarked in [7, #55] that there can be seven congruent neighborly convex polytopes in E^3. We present the following two examples of *eight* congruent neighborly convex polytopes in E^3.

First example. (Based on [**1**], [2].) Let P be the triangular prism with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,5)$, $(1,0,5)$, and $(0,1,6)$; thus the top makes an angle of $\pi/4$ with the base. Let F_1 and F_2 denote the facets of P which lie in the xz- and yz-planes, respectively. Put around P three congruent copies of P, each having one facet on the xy-plane and all lying above that plane, as shown in Figure 1.

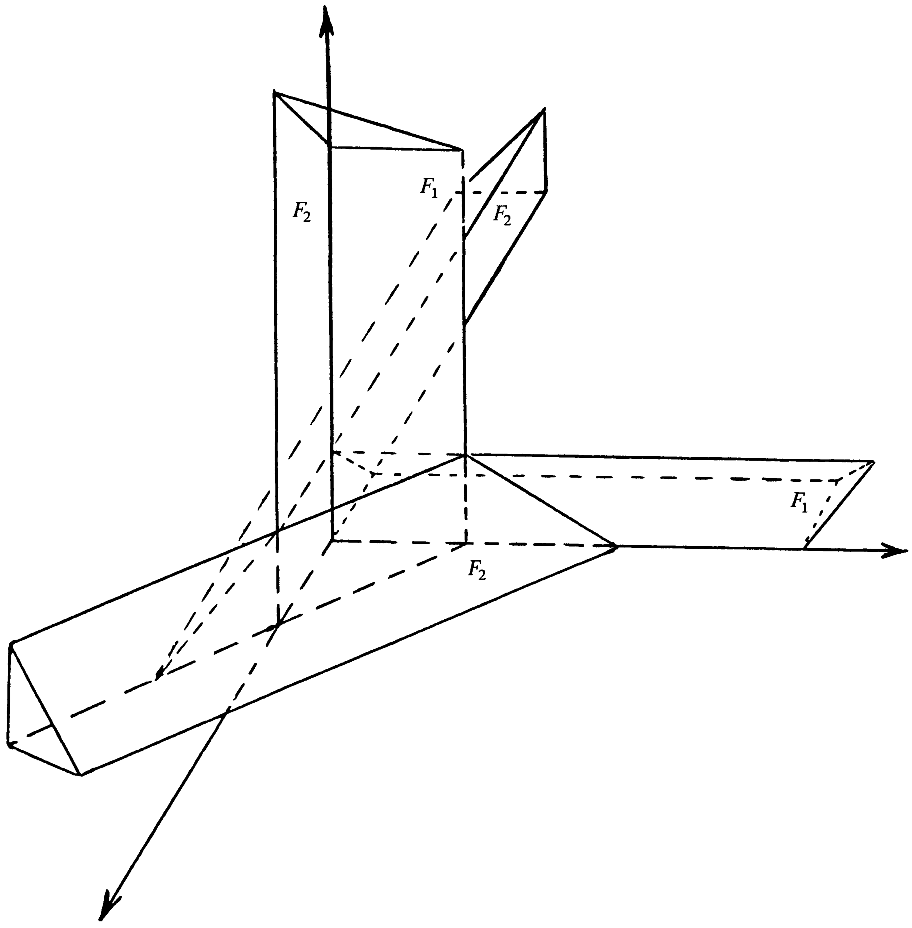

FIG. 1.

Four other copies of P lie below the xy-plane and they all have one facet on this plane, similar to the example of eight neighborly tetrahedra, due to Bagemihl [**1**] and Baston [**2**]. The eight facets which are in the xy-plane are displayed in Figure 2.

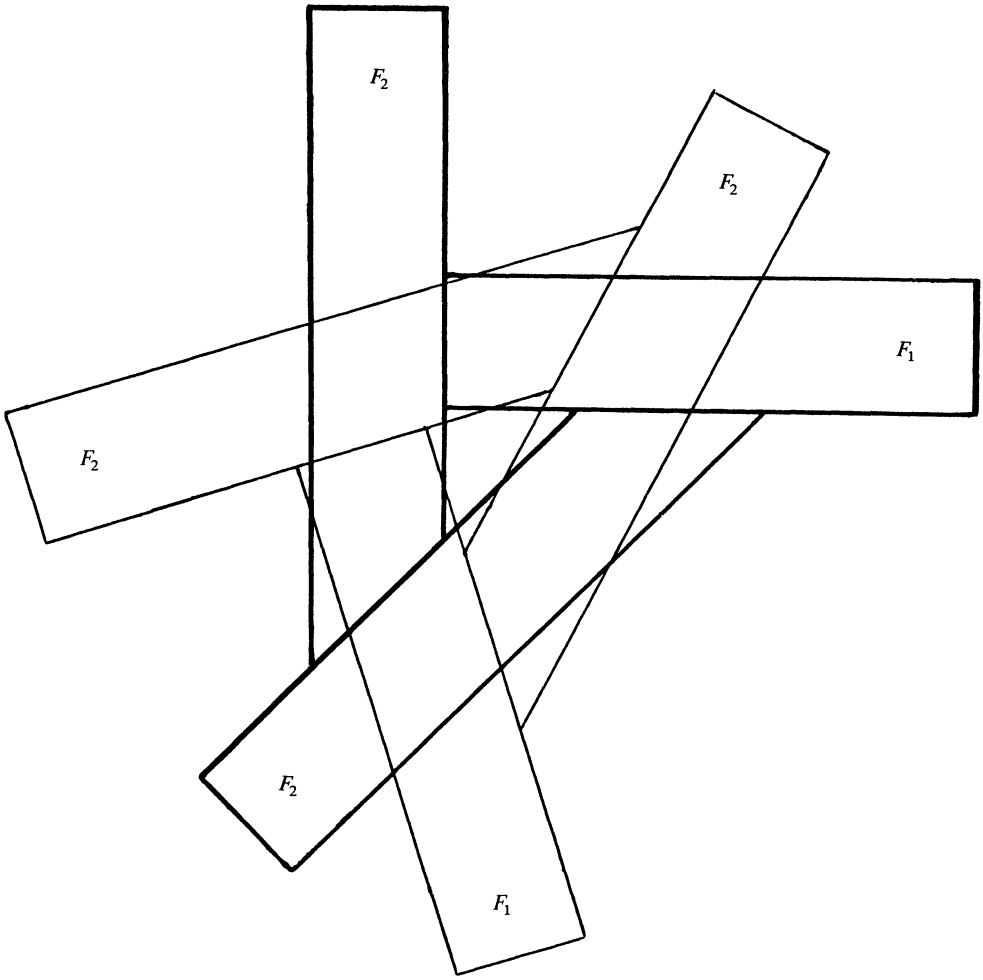

FIG. 2.

Second example. (Based on [**10**].) Let Q_1 be the triangular prism, having its vertices at the points $(4,0,0)$, $(5,0,0)$, $(-6,-1,0)$, $(-5,-1,0)$, $(-6,0,1)$, and $(-5,0,1)$. Let $Q_2 = Q_1 + \mathbf{i}$, i.e., Q_2 is the translation of Q_1 by the vector $(1,0,0)$. Let $f\colon E^3 \to E^3$ be the rotation of E^3 around the z-axis through an angle π; f is given also by $f(x, y, z) = (-x, -y, z)$; take $Q_3 = f(Q_1)$ and $Q_4 = f(Q_2)$. Let $g\colon E^3 \to E^3$ be given by $g(x, y, z) = (-y, x, -z)$; g is a rotation around the z-axis through an angle $\pi/2$, followed by a reflexion in the xy-plane. Define Q_n, for $5 \leqslant n \leqslant 8$, by $Q_n = g(Q_{n-4})$. It follows easily that $\{Q_1, \dots, Q_8\}$ is a neighborly family of prisms, all congruent to Q_1. A similar example is in [**10**, p. 279–280].

Problem 3. Is there a neighborly family in E^3, consisting of more than eight congruent convex polytopes?

Our second example can be used, by extending each prism into a parallelepiped, to get eight neighborly parallelepipeds in E^3: applying the idea of our paper **[11]**, we get the following

THEOREM $\frac{1}{2}$. *For each $d \geqslant 3$, there exists a neighborly family in E^d, consisting of 2^d d-parallelotopes.*

On the other hand, it follows from **[8]** and from the fact that at most one of any two parallel facets of a member of a neighborly family can contain intersections of pairs of members, that a neighborly family of d-parallelotopes in E^d contains at most 2^d members; thus we conclude

THEOREM 1. *The maximum number of neighborly d-parallelotopes in E^d is 3 for $d = 2$ and it is 2^d for all $d \geqslant 3$.*

A facet of a member of a neighborly family is called **free** if it contains no intersections of pairs of members of the family; the following is still open **[10**, p. 295, #4].

CONJECTURE. *Every finite nonempty neighborly family in E^d has a free facet, for all $d \geqslant 3$.*

A family of convex d-polytopes in E^d is called **nearly neighborly** **[10]** if for every two members P_1 and P_2 of the family there exists a hyperplane that separates P_1 and P_2 and which contains a facet of each; thus, the intersection $P_1 \cap P_2$ can be of arbitrary dimension, including the possibility that it is empty. Clearly, every neighborly family is also nearly neighborly.
Of the many properties and problems concerning nearly neighborly families, we mention the following **[8]**.

> If each member of a nearly neighborly family has at most k facets, then there can be at most 2^k members.

Here is a close relative to the theorem on the nonexistence of nine neighborly tetrahedra in E^3.

Problem 4. How many tetrahedra can there be in a nearly neighborly family in E^3? The maximum number is at most 16 **[8]** and at least 8 **[1]**, **[2]**; is it 8?

We conclude by presenting a solution to the analog of Problem 1 for nearly neighborly families, as follows.

THEOREM 2. *There exist arbitrarily large neighborly families in E^d, consisting of symmetric congruent members, for all $d \geqslant 2$.*

Proof. For an arbitrary integer n, $n \geqslant 3$, let B_i denote the unit ball in E^d, centered at the point $(3^{i-1}, 0, \ldots, 0)$, $1 < i < n$. Let T_1 be a set of $2\binom{n}{2}$ points on the boundary of B_1, such that if T_j is defined as the set $T_1 + (3^{j-1} - 1, 0, \ldots, 0)$ on the boundary of B_j, then the following property holds: for every pair (i, j) $1 \leqslant i < j \leqslant n$,

there exists a hyperplane H_{ij} which separates B_i and B_j and which supports B_i and B_j at points of T_i and T_j, respectively. T_i contains $\binom{n}{2}$ pairs of antipodal points; let ε be the minimum distance determined by points of T_1; then $\varepsilon > 0$, since T_1 is finite. For each $i < j$, tilt the hyperplane H_{ij} so that the new hyperplane will cut two caps away, one cap from B_i and one cap from B_j, such that both caps are of diameter $\varepsilon/2$. Cut away the antipodal caps as well, from both B_i and B_j; next, for all k, $1 \leqslant k \leqslant n$, $k \neq i, j$, cut away from B_k the two (antipodal) caps, obtained by translating the corresponding caps of B_i by a distance $|3^{k-1} - 3^{i-1}|$.

The result is a collection of n congruent symmetric convex bodies in E^d which form a nearly neighborly family. Each member has only a finite number of $(d-1)$-dimensional faces, used for the nearly neighborly property; thus, one can easily inscribe a convex polytope in each member, yielding a nearly neighboring family of n congruent symmetric convex polytopes.

Concerning Problem 4, we can show that there can be at most 15 nearly neighborly tetrahedra in E^3 (to appear in *J. Comb. Theory A*).

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LETTERS TO THE EDITOR

Editor:

It is well known that Évariste Galois' *Premier Mémoire* on group theory and related topics was rejected by M. Poisson (see, e.g., H. M. Edwards, *Galois Theory*, Springer-Verlag, New York, 1984) possibly on Lacroix's recommendation alone. For a project on mathematical pedagogy, I would appreciate information on representative samples of the five memoirs most highly recommended by Poisson in the same general time-frame: the authors, titles and summary results of the memoirs as well as Poisson's criticism and favorable reaction to the memoirs.

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[*So we editors have to be careful, don't we? H. S. W.*]

Editor:

I would like to compliment the MONTHLY and Professor Friedler for his review of *Actuarial Mathematics* (June-July MONTHLY, Page 489) and his message to the mathematics community concerning the actuarial profession. His thoughts need to be supplemented with the information that the Society of Actuaries is changing the format, but not the content, of the associateship exams (Parts 1–5) beginning in 1987.

The associateship exams will be offered in 11 pieces (to be called courses) beginning with the May 1987 sitting. Course 100 (old Part 1, calculus and linear algebra) and course 110 (old Part 2, probability and statistics) will remain as single exams. The old Part 3 exam (operations research, numerical analysis, and applied statistics) will become three separate exams (courses 120, 130, and 135). The old Part 4 will become two separate courses, 140 on compound interest and 150 on contingencies. The old Part 5 exam will be divided into 4 courses. One additional change which should be especially helpful to undergraduates who get a late start on pursuing an actuarial career, the Course 100 exam will now be offered three times per year, with a February sitting added to the usual May and November sittings.

In the spirit of Professor Friedler's review, I urge members of the mathematics community to become familiar with the new exam arrangement so that you may convey this information to your students. Recent information on the exams and actuarial careers may be obtained from the Society of Actuaries and the Casualty Actuarial Society.

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Dear Friends and Colleagues:

Both Bee (Bertha Halley Ross) and I have always felt that in our complex society working with eager and talented youth is not only deeply satisfying but is critically important as well.

The success of an enterprise such as ours is due to the warm response of our eager young charges and to the encouragement and often actual help of many accomplished colleagues. Among such colleagues who are no longer with us one should mention Max Dehn, Louis Mordell, Yuri Rainich, Waclaw Sierpinski, H. Steinhaus, Ivo Thomas and Paul Turan.

All of us—the young participants, the faculty and the young participant colleagues acting as counsellors—owe a very deep debt to Bee. Her sensitivity to academic subtleties, her affection for young people and her dedication contributed immeasurably to the quality of what we were able to accomplish. Her commitment sustained me in the years when the prospect of continuing our work seemed hopeless. Her own interests were in the Arts and Letters and the Social Sciences. She was the first lay woman research assistant in Sociology at Notre Dame.

Today's mark of approbation by our friends and colleagues of the MAA is very heartwarming but it is mixed with sadness caused by my inability to give adequate credit to Bee for her part in our achievement.

Thank you for your kindness.

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NOTES

EDITED BY CAROL G. CRAWFORD, RICHARD LIBERA, AND ANITA E. SOLOW

Constructing a Fair Border

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In 1983, in this MONTHLY [2], Professor Theodore P. Hill investigated the following problem: n countries $A_1, \ldots, A_n$ lie adjacent to a disputed territory D. They agree on equal division, which means that each country must receive $1/n$ of the total value of D. Since each country has its own idea of what is valuable, this means that each must receive $1/n$ of the total value of D *by its own evaluation*. In addition, the portion of D received by each country must be adjacent to that

*The research for this paper was supported by the Wisconsin Alumni Research Foundation.

country, and connected. In this particular, this problem is different from the other "fair division" problems cited by Hill. It is essential that each country value each point as having value 0.

Professor Hill shows that the division is possible, but declares himself innocent of any division algorithm. Every such situation cries out for correction, and I here offer an algorithm, of the usual kind, to divide the disputed territory. In the course of making the division, each country makes several representations, and if each of the representations is true, then a fair division will indeed take place.

To begin with, we formalize the problem by making some definitions. The disputed territory D is an open, connected piece of the plane (or sphere) whose boundary has value 0 according to each country. For each country A_j, there is an arc γ_j lying in $\partial(D)$ and a neighborhood N_j of γ_j such that $N_j \subset \gamma_j \cup A_j \cup D$. This does not disallow points in $\partial(D)$ which are limit points of two countries, or of none. This is the meaning of *adjacent* for the first part of the paper. Later, I will consider an expanded version of this concept. The territory awarded to each country must also lie adjacent to that country by the same definition, and we will secure this by assuring that a portion of γ_j will have a neighborhood consisting only of points of A_j and of the awarded territory B_j.

We begin with a special case of the problem in which the territory D is simply connected. It will turn out that all the interesting part of the study lies in this "easy" case. The rest will follow by an induction on the connectivity of D.

To solve the problem, we need actually only make a single territorial award, of a territory B_j to a country A_j. We will have B_j evaluated at least at $1/n$ of D by A_j, and at most $1/n$ by all the other countries. The identity of A_j is an artifact of the process. The rest of the apportionment then follows inductively for the remaining countries.

Actually, the countries will be working with a homeomorphic map $h(D)$ of D, to which they assign values according to the values of the regions represented. The fair division of the map will produce a fair division of the territory. The map will be chosen so that $h(D)$ is the unit disc $|z| < 1$, and so that every radius or concentric circle is valued at 0 by all the countries. We will now show that such a map exists.

THEOREM. *Let $D \neq \mathbb{R}$ be a simply-connected region in $\mathbb{R}$, and $\mu_1, \dots, \mu_n, \dots$ be countably many measures defined in D with $\mu_j(\{z\}) = 0$, $\forall z \in D$, $\forall j \in \mathbb{N}$. Then $\exists h$ is a conformal mapping of D onto $U = \{z \,|\, |z| < 1\}$ such that*

$$\text{for every radius } R_\theta = \{z \,|\, |z| < 1 \quad \text{and} \quad \arg z = \theta\},$$

$$\text{and every circle } C_r = \{z \,|\, |z| = r\},$$

$$\mu_j(h^{-1}(R_\theta)) = \mu_j(h^{-1}(C_r)) = 0, \forall j \in \mathbb{N}.$$

Proof. Let h_1 be any conformal homeomorphism of D onto U and $j \in \mathbb{N}$. We denote by $\mathscr{K}_1$ the set of all diameters L of U with $\mu_j(h_1^{-1}(L)) > 0$. We denote by

$\mathscr{K}_2$ the set of all circles K orthogonal to the unit circle C_1 with $\mu_j(h_1^{-1}(K \cap U)) > 0$. We denote by $\mathscr{K}_3$ the set of all circles $K \subset U$ with $\mu_j(h_1^{-1}(K)) > 0$. Since any two of the members of $\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3$ intersect in at most two points, and since the sum of their measures is thus the measure of their union, and thus less than 1, $\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3$ is a countable collection of lines and circles.

For each circle $K \in \mathscr{K}_3$, we identify a point of U which we call the *conformal center* of K (written $c(K)$). $a = c(K)$ is the unique point for which the conformal mapping h_a defined by $f_a(z) = (z - a)/(1 - \bar{a}z)$ gives $f_a(K)$ as a circle with center 0. We denote by $\mathscr{K}_4$ the set of all the conformal centers of the circles in $\mathscr{K}_3$. Then $\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_4$ is a countable set of lines, circles, and points. Taking the union of these for all $j \in \mathbb{N}$ still gives a countable set of lines, circles, and points. The union V of these lines, circles and points is a first-category set of Lebesgue measure 0 in U. If we take any point $a \notin V$, and $\mathscr{S}$ a line in $\mathscr{K}_1$, a circle in $\mathscr{K}_2$, or a circle in $\mathscr{K}_3$, for any $j \in \mathbb{N}$, then $f_a(\mathscr{S})$ is not a diameter of U or a circle with center 0. Thus, $h = f_a h_1$ is a conformal homeomorphism meeting the requirements of the theorem. QED

We may assume without loss of generality that $D = U$, and that every radius and concentric circle in U has value 0 for all n countries.

We will now begin a series of **auctions**, so designed that there can be no more than n of them, and at the end of the series, an award will be made. We begin the process by requiring each country to submit its bid for the smallest radius r for which $\mu_j(D_r) = 1/n$, where $D_r = \{z||z| < r\}$. Note that the condition on concentric circles assures that $\mu_j(D_r)$ is a continuous function of r, so that the question has a unique answer for each A_j.

In the highly likely case that the low bid is submitted by only one country, we are almost through. In that case, each of the other countries values D_r at less than $1/n$. The winning country A_j joins the arc γ_j to 0 by a sector of U, and each of the other countries in turn sharpens the sector to a subsector so that the union of the subsector with D_r has value less than $1/n$, according to its evaluation. When every country has had the opportunity to assure that the union of sector and disc is less than $1/n$, these are awarded to A_j in conformity with the rules of the problem, and the other $n - 1$ countries go about dividing what remains, which is more than $(n - 1)/n$, according to each.

The improbability of having two countries bid the same radius does not excuse us from the necessity of dealing with that case, and that is the bulk of the process. Assume that k of the countries bid r_1; rename these as $A_1, \ldots, A_k$, with $2 \leqslant k \leqslant n$. We denote these countries as *players*; the other $n - k$ are *waiters*. The waiters will be given the opportunity to assure, each of them, that the country which gets an award of territory, receives no more than $1/n$ by its (the waiter's) valuation, thus assuring that there is at least $(n - 1)/n$ to be divided in the next round, when only

$n - 1$ countries will be claimants. Among the players, one is designated as the *declarer*; rename it as A_1. The next series of moves is designed to either make an award of territory to A_1 in accordance with the requirements of the theorem, or else to move A_1 into the status of a waiter. As in the previous case, A_1 joins its territory to the disc D_{r_1} by a sector S_1, small enough so that every waiter agrees that $S_1 \cup D_1$ has value less than $1/n$. This is clearly possible if each country in turn is permitted to reduce S_1 to a size consistent with its requirements. At the same time, each player sharpens S_1 so that its value is less than $1/n$.

When S_1 has been designated, we are ready for our **second auction**. Each player chooses a new radius r so that $S_1 \cup D_r$ has value $1/n$, and is the smallest radius giving this result. Of course, each of these radii is no greater than r_i. The minimum of these radii is denoted $r_2 \leqslant r_1$, and the corresponding disc D_{r_2}. If A_1 is one of the countries that has bid r_2 the second time, then $B_1 = S_1 \cup D_{r_2}$ is A_1's share of D, and all players and waiters agree that it is of value no more than $1/n$, while A_1 agrees that it is worth exactly $1/n$. If A_1 has bid more than r_2, then A_1 agrees that $S_1 \cup D_{r_2}$ is worth less than $1/n$, and joins the waiters, all of whom must be of this opinion.

We now look at all the players who hold that the value of $S_1 \cup D_{r_2}$ is $1/n$, and all the others, with A_1, join the waiters. From among the remaining players, we select a new declarer; rename it as A_2. We connect A_2 to D_{r_2} by a sector S_2 such that $\mu_j(W_2) < 1/n$ for each player A_j; where $W_2 = S_1 \cup S_2$ and $\mu_j(V_2) < 1/n$ for each waiter A_j, where $V_2 = W_2 \cup D_{r_2}$. This situation is much like the one when A_1 was declarer, except that V_2 is connected to both A_1 and A_2, and this is a problem. In the next paragraph, we dispose of this problem.

A_2 will designate a point $x_2 \in U \setminus V_2$ which has the property that every neighborhood of x_2 has positive value to A_2. This can be done by using heavy artillery, and choosing x_2 as any *measure point* of U/V_2. A measure point x of a set S is one with

$$\lim(\mu(S \cap N(x, \varepsilon))/\mu(N(x, \varepsilon))) \to 1 \quad \text{as} \quad \varepsilon \to 0.$$

The existence of these (they are a set, i.e., equal with S) follows from standard theorems on convolution (see [**1**, p. 268]). Alternatively, A_2 can select any square Q lying in U/V_2 with $\mu_2(Q) > 0$ and $\overline{Q}$ disjoint from both $\overline{V}_2$ and C_1. Then quartering Q will give at least one square Q_1 with $\mu_2(Q_1) > 0$, quartering Q_1 will give $\mu_2(Q_2) > 0$, etc. If $x = \bigcap_n Q_m$, then x has the desired property. A_2 chooses a partial sector P_2 which is a sector of a disc $\tilde{D}$ of radius less than 1 and containing x_2. P_2 is chosen so that

$$\mu_2(S_1 \cup S_2 \cup P_2) < 1/n,$$

and refined by each player A_j so that

$$x_2 \in P_2 \quad \text{and} \quad \mu_j(W_2 \cup P_2) < 1/n.$$

For the waiters A_j, P_2 is taken with $x_2 \in P_2$, but P_2 so small that $\mu_j(V_2 \cup P_2) <$

$1/n$. Now that P_2 has been added to V_2, $\mu_2(V_2 \cup P_2) > 1/n$, and A_2 must yield up all of S_1 beyond some radius, which A_2 is free to choose, and it must choose it so that the remaining set still has value more than $1/n$. The truncated part of S_1 we call T_1.

Now A_2 holds an option on $D_{r_2} \cup T_1 \cup S_2 \cup P_2$, which is valued by A_2 at more than $1/n$, and by each waiter at less. The remaining players agree that $T_1 \cup S_2 \cup P_2$ has value less than $1/n$. It is now time for the **third auction**. Each player A_j selects the smallest radius r for which

$$\mu_j(T_1 \cup S_2 \cup P_2 \cup D_r) = 1/n.$$

Let the smallest bid be r_3. Every country A_j which has bid r_3 remains a player; otherwise it becomes a waiter. If A_2 is still a player, then A_2 takes

$$B_2 = T_1 \cup S_2 \cup P_2 \cup D_{r_3}$$

as its settlement. Otherwise A_2 is a waiter, and a new declarer is chosen; call it A_3. A sector S_3 is chosen so that $\mu_j(W_3) < 1/n$ for each player A_j, where

$$W_3 = T_1 \cup S_2 \cup P_2 \cup S_3,$$

and $\mu_j(V_3) < 1/n$ for each waiter A_j, where $V_3 = W_3 \cup D_{r_3}$. A point x_3 is chosen in $U \cup V_3$ so that every neighborhood N of x_3 has $\mu_3(N) > 0$, and P_3 is chosen analogous to P_2, with S_2 then trimmed to T_2, and it is time for a new auction.

Thus we proceed. If a declarer ever wins a second auction, the process ends. If not, then that declarer becomes a waiter, and a new declarer is chosen from among the surviving players. Eventually, the process must end, because of the finite supply of players. At that point, some A_j has acquired a territory B_j, with $\mu_j(B_j) \geqslant 1/n$, and $\mu_i(B_j) \leqslant 1/n, \forall i \neq j$. This gives the desired algorithm.

We can weaken the concept of adjacency by asserting that A_j is adjacent to D if there is a point $x \in \partial(D)$ and a neighborhood N of x so that $N \subset A_j \cup \overline{D}$. In that case, the conformal homeomorphism h constructed in Theorem 4 will have an arc in C_1 corresponding to prime ends lying in N. Thus, for some subarc of C_1 which corresponds to part of $\partial(B_j) \cap \partial(\underline{D})$, its preimage in $\partial(D)$ will have a neighborhood N_j which is a subset of $A_j \cup \overline{D}$. Then $A_j \cup N_j \cup D$ will be an open connected set, which is what we require in this problem. Since this paper does not deal essentially in the properties of prime ends, the proof of this fact is omitted.

It is hard to see how much weaker we can make the concept of adjacency. For example, we could construct a model in which a region D is bordered by n countries in such a way that every point of $\partial(D)$ is also a boundary point of all n countries. In that case, no subregion of D can be adjoined to any country in such a way as to give an open connected region.

Note that it is essential that the boundary have value 0. If not, consider D to be a square, with $A_1, \dots, A_4$ bordering it on the four sides. If each country puts all its value on a portion of the opposite side of the square, then any award of territory to A_1 must cut off A_2 and A_4 from their desired goals.

We now turn to the connectivity of D. If D is finitely connected, then we can solve the problem by induction. We know that the algorithm works for connectivity 1. Suppose it works for $k-1$, and D is k-connected. Then D has an outer boundary Y_1 and inner boundaries $Y_2, \ldots, Y_k$. Let $x \in Y_1$ and $y \in Y_k$. Then there are uncountably many circles which pass through x, y and some other point of D, and they are pairwise disjoint in D. Thus, only countably many can have nonzero value to any A_j, and it follows that one of them, say one passing through $z \in D$, has 0 value to all the A_j. Let γ be an arc of that circle which joins Y_1 to another Y_i. We can now solve the apportionment problem for the region $D \setminus \gamma$, which is of connectivity $k-1$. In that solution, some country will be left at the end with all the territory not assigned to the other $n-1$. Since none of the awards will bound γ, we can throw in γ with this last award.

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An Anticommutativity Consequence of a Ring Commutativity Theorem of Herstein

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One of the most beautiful results in ring theory is the following theorem of Herstein [**1**] which gives a condition on the commutators $xy - yx$ of a ring that is necessary and sufficient for commutativity:

THEOREM 1. *Let R be a ring such that for each $x, y \in R$ there exists a natural number $n(x, y) > 1$, depending on x and y, with*

$$(xy - yx)^{n(x,y)} = xy - yx.$$

Then R is commutative.

A ring R is said to be *anticommutative* if $xy + yx = 0$, for all $x, y \in R$. Such rings are common. For example, a ring of characteristic 2 is anticommutative if and only if it is commutative. Any ring in which $x^2 = 0$ for all x is anticommutative. For example, consider the ring of those 4×4 matrices $A = (a_{ij})$ over any commutative ring R with $\operatorname{char}(R) \neq 2$, which have $a_{ij} = 0$ if $j \leqslant i$, $a_{23} = 0$, $a_{24} = a_{13}$ and $a_{34} = -a_{12}$.

In this note, we use Herstein's theorem to prove an analog in which commutators are replaced by *anticommutators* $xy + yx$ and commutativity by anticommutativity.

MAIN THEOREM. *Let R be a ring such that for each $x, y \in R$ there exists an even natural number $n(x, y)$ with $(xy + yx)^{n(x,y)} = xy + yx$. Then R is anticommutative.*

We note that this result does not hold if we drop the condition that $n(x, y)$ is even. Consider for example Z_p, the ring of integers modulo an odd prime p. Here, every anticommutator satisfies $(xy + yx)^p = xy + yx$, but Z_p is not anticommutative, since in particular $1 \cdot 1 + 1 \cdot 1 \neq 0$.

While one can prove the theorem by elementary means if $n(x, y)$ is assumed to have a small constant value, the general proof requires the Jacobson structure theory. We begin with the simplest possible case in which $n(x, y) = 2$ for all $x, y \in R$, in order to convey a sense of the details which is suppressed in the later appeal to the structure theory. Our first lemma is well known and can be found for example in [**2**]. We include a short proof for convenience.

LEMMA 1. *Let R be a ring in which $xy = 0$ implies $yx = 0$. If e is an idempotent in R, then $e \in Z(R)$, the center of R.*

Proof. For all $x \in R$, $0 = ex - e^2x = e(x - ex) \Rightarrow (x - ex)e = 0$. Thus $xe = exe$. Similarly, $0 = xe - xe^2 \Rightarrow ex = exe$. Hence $e \in Z(R)$.

LEMMA 2. *If R is a ring in which $x^2 \in Z(R)$, then $(xy)^2 = (yx)^2$, for all $x, y \in R$.*

Proof. We have

$$\begin{aligned}(xy)^2 &= x(yxy) = x\left[(xy)^2 + y^2 - (xy - y)^2 - xy^2\right] \\ &= \left[(xy)^2 + y^2 - (xy - y)^2 - xy^2\right]x = (yxy)x = (yx)^2.\end{aligned}$$

THEOREM 2. *Let R be a ring in which $(xy + yx)^2 = xy + yx$ for all $x, y \in R$. Then R is anticommutative.*

Proof. Suppose that $xy = 0$. Then

$$yx = xy + yx = (xy + yx)^2 = (yx)^2 = y(xy)x = 0.$$

Hence, by Lemma 1, $xy + yx \in Z(R)$, for all x, y, so

$$x(xy + yx) = (xy + yx)x$$

and thus $x^2 \in Z(R)$ for all x. By Lemma 2, $(xy)^2 = (yx)^2$, for all x, y.

Next, $(-xy - yx) = (-xy - yx)^2 = (xy + yx)^2 = xy + yx$ so $2(xy + yx) = 0$, for all x, y.

Finally,

$$\begin{aligned}(xy + yx)^2 &= (xy)^2 + (yx)^2 + xy^2x + yx^2y \\ &= 2\left[(xy)^2 + x^2y^2\right], \text{so } xy + yx = (xy + yx)^3 \\ &= 2\left[(xy)^2 + x^2y^2\right][xy + yx] = \left[(xy)^2 + x^2y^2\right][2(xy + yx)] = 0,\end{aligned}$$

as claimed.

We note that if elements a and b satisfy $a^{n(a)} = a$ and $b^{n(b)} = b$, for $n(a) > 1$ and $n(b) > 1$, then $a^t = a$ and $b^t = b$, where

$$t = (n(a) - 1)(n(b) - 1) + 1.$$

Moreover, if $n(a)$ and $n(b)$ are both even, then this common exponent t is also even.

In preparation for the general result, we must prove the claim for rings with unity.

LEMMA 3. *Let R be a ring with unity* 1 *such that for each $x, y \in R$ there exists an even integer $n(x, y) > 0$, with $(xy + yx)^{n(x,y)} = xy + yx$. Then R is anticommutative.*

Proof. Given $x, y \in R$, there exists an even integer $t \geqslant 2$ such that

$$(xy + yx)^t = xy + yx \text{ and } (-xy - yx)^t = -xy - yx.$$

Putting $y = 1$, we get $2x = (2x)^t = (-2x)^t = -2x$, so $4x = 0$.

Thus $2x = (2x)^t = 2^t x^t = 0$ since $t \geqslant 2$, and so R has characteristic 2. As mentioned above, a ring of characteristic 2 is anticommutative if and only if it is commutative, and in such a ring the current hypothesis reduces to that of Theorem 1. The result follows from Theorem 1.

We are now ready to prove the main theorem. Our proof parallels Herstein's.

Proof of the Main Theorem. First, let us consider a semisimple ring R satisfying the hypothesis. Such a ring is isomorphic to a subdirect sum of primitive rings R_i, each of which, as a homomorphic image of R, satisfies the hypothesis placed on R. Thus, to show that R is anticommutative, it is sufficient to show that each R_i is anticommutative, so we may assume that R is primitive.

If R is primitive, the first possibility is that $R \simeq D$, where D is a division ring. In this case the theorem is true by Lemma 3. Otherwise, for some $k > 1$, D_k, the complete matrix ring over a division ring D, would be a homomorphic image of a subring of R. Thus D_k would inherit the property

$$(xy + yx)^{n(x,y)} = xy + yx.$$

The following elements x and y in D_k show that this is impossible:

$$x = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

These elements satisfy $xy + yx = y \neq 0$ and $y^2 = 0$. Thus $y^{n(x,y)} = 0 \neq xy + yx$. We conclude that the theorem holds for all semisimple rings.

Now let R be any ring satisfying the hypothesis. If $J(R)$ is the Jacobson radical of R, then $R/J(R)$, as a homomorphic image of R, is a semisimple ring which inherits the hypothesis on R, so by our previous result $R/J(R)$ is anticommutative. Thus for all $x, y \in R$, $xy + yx \in J(R)$. Now $J(R)$ has the property that if $ab = a$,

with $b \in J(R)$, then $a = 0$. Hence

$$(xy + yx)(xy + yx)^{n(x,y)-1} = xy + yx$$

implies that $xy + yx = 0$ for all $x, y \in R$, so R is anticommutative.

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A Reordering of the Sylow Theorems

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Introduction. This paper grew partly out of a sense of dissatisfaction with the usual treatment of the Sylow theorems in textbooks on group theory, and partly out of a desire to find a new approach in the spirit of McKay's delightful proof of Cauchy's theorem given in [1]. We concentrate our attention on the first anzahl theorem of Frobenius, which states that if p is a prime such that p^n divides the order of a finite group G, then the number of subgroups of order p^n of G is congruent to $1 \bmod p$. In Section 2, we give a new proof of this theorem, taking in Sylow's first and third theorems in the process. Cauchy's theorem and Sylow's second theorem are the main tools, and proofs of these are presented in Section 1. In Section 3 we prove a related result about supergroups.

First we summarize the elementary facts relating to the Frobenius decomposition of a group relative to two subgroups. If A and B are subgroups, not necessarily distinct, of a finite group G, then G can be decomposed as the disjoint union of double cosets relative to A and B,

$$G = Ag_0B \cup \cdots \cup Ag_kB.$$

The index of B in G is given by

$$(G:B) = \sum_{i=0}^{k} \left(|A|/|g_i^{-1}Ag_i \cap B|\right),$$

each term in the summation being an integer. If $A \subseteq B$, then one of the double cosets is B itself, and the corresponding term in the summation is 1.

Next we recall that if G is a finite group of order $p^n r$ where p is a prime, n is a nonnegative integer, and p does not divide r, then any subgroup of G of order p^n is called a p-Sylow subgroup of G.

1. Preliminary Results. Throughout this section, G will denote a finite group and p a prime which divides the order h of G.

1.1 CAUCHY'S THEOREM. *The number of solutions in G of the equation $x^p = 1$ is a nonzero multiple of p.*

Proof (McKay). Let

$$S = \{(a_1, a_2, \dots, a_p),\ a_i \in G, 1 \le i \le p,\ |a_1 a_2 \cdots a_p = 1\}.$$

The number of elements of S is clearly h^{p-1}.

Define an equivalence relation $\sim$ on S by $x \sim y$ if x is a cyclic permutation of y. If $x = (a_1, a_2, \dots, a_p) \in S$, then the equivalence class of x consists of exactly one element if all the a_i are equal and exactly p elements otherwise, since p is prime. Let r be the number of classes with just one element and t the number of classes with p elements. Then $r + tp = h^{p-1}$, which implies that p divides r. Also, r is nonzero since $(1, 1, \dots, 1) \in S$. Since r is the number of solutions of $x^p = 1$ in G, the theorem is proved.

1.2 COROLLARY. *The number of subgroups of order p in G is congruent to* 1 mod p.

Proof. By 1.1, G has elements of order p, and each such element generates a subgroup of order p. Let $\{P_1, P_2, \dots, P_k\}$ be the complete set of subgroups of order p in G. By Lagrange's theorem, if $i \neq j$ then $P_i \cap P_j = \{1\}$, so the number of elements of $P_1 \cup P_2 \cup \cdots \cup P_k$ is $k(p-1) + 1$. Since these are precisely the solutions of $x^p = 1$ in G, it follows from 1.1 that $k \equiv 1 \bmod p$.

1.3 SYLOW'S SECOND THEOREM. *If G has p-Sylow subgroups, then they form a single conjugacy class of subgroups of G.*

Proof. Suppose P is a p-Sylow subgroup of G; then any subgroup of G conjugate to P has the same order as P so is also a p-Sylow subgroup of G. On the other hand, if Q is also a p-Sylow subgroup of G, let

$$G = Pg_0Q \cup \cdots \cup Pg_kQ$$

be a Frobenius decomposition of G. Since p is prime to $(G : Q)$, it is also prime to $|P|/|g_i^{-1}Pg_i \cap Q|$ for some i. Thus $g_i^{-1}Pg_i = Q$, and the result is established.

1.4 LEMMA. *If p is the smallest prime divisor of the order of G and S is a subgroup of G whose index in G is p, then S is normal in G.*

Proof. Let $G = S \cup Sg_1S \cup \cdots \cup Sg_kS$ be a Frobenius decomposition of G. Then

$$p = (G : S) = 1 + \sum_{i=1}^{k} \left(|S|/|g_i^{-1}Sg_i \cap S|\right).$$

Since each term in the summation divides the order of G, each term is 1 by the hypothesis on p. It follows easily that S is normal in G.

2. Main Result.

2.1 THEOREM (Frobenius). *Let p be a prime. If p^n divides the order of a finite group G, then G contains at least one subgroup of order p^n. The number of such subgroups is congruent to* 1 mod p.

Proof. The theorem is trivially true when $n = 0$, and was proved for $n = 1$ in 1.2. Suppose inductively that the result holds for $n = m \geq 1$. For subgroups P of G of order p^m, there are two possibilities to be considered.

(i) For some P, p is prime to the index of P in its normalizer N in G. Then P is the unique p-Sylow subgroup of N, by 1.3. Let

$$G = N \cup Pb_1N \cup \cdots \cup Pb_kN$$

be a Frobenius decomposition of G. Then, for all i, $b_i \notin N$, so that no $b_i^{-1}Pb_i$ is a subgroup of N, for otherwise it would be a p-Sylow subgroup of N distinct from P. Hence, for each i, p divides $|P|/|b_i^{-1}Pb_i \cap N|$. Thus $(G:N)$ is congruent to 1 mod p, and p^{m+1} does not divide the order of G.

(ii) For each P, p divides the index of P in its normalizer N in G. Then any subgroup Q of G of order p^{m+1} containing P is a subgroup of N by 1.4. The natural homomorphism from N to N/P establishes a one-to-one correspondence between such subgroups Q and the subgroups of N/P of order p. It follows by 1.2 that such Q exist and that their number is congruent to 1 mod p. Since the total number of P is congruent to 1 mod p by hypothesis, this implies that the number of ordered pairs (P, Q), where P is a subgroup of G of order p^m and Q is a subgroup of G of order p^{m+1} containing P, is congruent to 1 mod p. Since the number of P contained in each Q is congruent to 1 mod p by hypothesis, it follows that the total number of Q is congruent to 1 mod p.

Since each subgroup of G of order p^{m+1} contains some P by hypothesis, the result follows by induction.

If we do not require Frobenius' result but only the weaker first and third theorems of Sylow, then the proof of 2.1 can be adapted as follows: If p divides the order of a finite group G, 1.2 ensures that G has a p-subgroup P which is contained in no larger p-subgroup of G. Let N be the normalizer of P in G. We consider the natural homomorphism from N to N/P and invoke 1.2 to deduce that p is prime to the index of P in N. The reasoning at (i) in 2.1 shows that P is a p-Sylow subgroup of G and that the index of N in G is congruent to 1 mod p. It follows in the usual way from Sylow's second theorem that the number of p-Sylow subgroups of G is congruent to 1 mod p.

On the other hand, after Frobenius' result has been established, it seems natural to ask if the methods we have employed can be easily adapted to give a more comprehensive picture of the p-subgroups of a finite group. We believe they can, and indicate this by proving a complementary anzahl theorem in Section 3. We shall

need the following definition:

DEFINITION. If G is a group of order $p^n r$, where p is a prime, $n \geq 0$, and p does not divide r, then a p-series in G is any ordered $(n+1)$-tuple $(P_0, \ldots, P_n)$ of subgroups of G such that the order of P_i is p^i $(0 \leq i \leq n)$ and $P_i \subset P_{i+1}$ $(0 \leq i < n)$.

3. A Complementary Result. Throughout this section, G will denote a finite group of order $p^n r$ where p is prime, $n \geq 0$, and p does not divide r.

3.1 LEMMA. *If P is a subgroup of G of order p^k where $k < n$, then the number of subgroups of G of order p^{k+1} containing P is congruent to* $1 \bmod p$.

Proof. This is included in the proof of 2.1.

3.2 LEMMA. *The number of p-series in G is congruent to* $1 \bmod p$.

Proof. $P_0 = \{1\}$, and if $n \neq 0$ and $P_0, \ldots, P_r$ have been chosen $(0 \leq r < n)$, then the number of choices for P_{r+1} is congruent to $1 \bmod p$ by 3.1. The result follows by multiplication.

3.3 LEMMA. *If P is a subgroup of G of order p^k, where $0 \leq k \leq n$, then the number of p-series $(P_0, \ldots, P_n)$ in G in which $P_k = P$ is congruent to* $1 \bmod p$.

Proof. The number of p-series in P is congruent to $1 \bmod p$ by 3.2. If $k = n$ the result follows immediately. If $k < n$ and if $P_0, \ldots, P_r$ have been chosen $(k \leq r < n)$, then the number of choices for P_{r+1} is congruent to $1 \bmod p$ by 3.1, and the result follows by multiplication.

3.4 THEOREM. *If $k \leq m \leq n$ and P is a subgroup of G of order p^k, then the number of subgroups of G of order p^m which contain P is congruent to* $1 \bmod p$.

Proof. Let $\{Q_1, \ldots, Q_r\}$ be the complete set of subgroups of G of order p^m which contain P; such subgroups certainly exist by repeated application of 3.1. For each i, $1 \leq i \leq r$, the number of p-series $(P_0, \ldots, P_m)$ in Q_i in which $P_k = P$ is congruent to $1 \bmod p$, by applying 3.3 to Q_i. The number of p-series $(P_0, \ldots, P_n)$ in G with $P_k = P$ and $P_m = Q_i$ is therefore congruent to $1 \bmod p$ by the argument used in 3.3. Adding the results for $i = 1, \ldots, r$ we deduce that the number of p-series $(P_0, \ldots, P_n)$ in G with $P_k = P$ is congruent to $r \bmod p$. Then, by 3.3, r is congruent to $1 \bmod p$, and the proof is complete.

I hope that the new outlook of this paper on the Sylow theorems will prove useful to teachers of group theory as an alternative to that of standard texts.

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$a(\mathrm{mod}\ p) \leqslant b(\mathrm{mod}\ p)$ for All Primes p Implies $a = b$

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The reader might wonder about the abundance of authors for such a short note. The assertion of the title was conjectured by P. P. Pálfy, and P. Erdős pointed out that it easily follows from the Sylvester-Schur theorem. Then it was set as a problem in the 1984 Hungarian annual M. Schweitzer memorial mathematics contest for college students. The most elegant solution was given by M. Szegedy, and that is what we present here.

THEOREM. *Let a and b be positive integers. If, divided by any prime number, the residue of a is less than or equal to the residue of b, then a and b are equal.*

Proof. Let us use the notation x_p for the residue of x modulo p, i.e., $x \equiv x_p (\mathrm{mod}\ p)$ and $0 \leqslant x_p < p$. If we choose $p > \max(a, b)$, then $a = a_p \leqslant b_p = b$. We will suppose $a < b$ and prove the theorem by contradiction.

We have $1 \leqslant b - a < b$ and $(b-a)_p = b_p - a_p \leqslant b_p$, so $b - a$ and b satisfy the hypothesis of the theorem if and only if a and b do. So, without loss of generality, we may choose the smaller of a and $b - a$, call it a and have $1 \leqslant a \leqslant b/2$.

We can reach a contradiction immediately by making use of the Sylvester-Schur theorem [**1**]. It yields a prime $p > a$ such that p divides $\binom{b}{a}$, i.e., $p \mid (b - a + 1) \cdots (b-1)b$. Exactly one of the factors will be divisible by p, say $b - k$, $0 \leqslant k < a$. But then $b_p = k < a = a_p$, contrary to the assumption.

However, we can give a more elementary, self-contained proof. Let

$$A = 1 \cdot 2 \cdots (a-1)a \quad \text{and} \quad B = (b-a+1) \cdots (b-1)b,$$

so that $B/A = \binom{b}{a}$. Let $\alpha(p^k)$ and $\beta(p^k)$ be the number of factors in A and B, respectively, which are divisible by p^k. Thus

$$A = \prod_p p^{\sum_{k=1}^{\infty} \alpha(p^k)}, \quad B = \prod_p p^{\sum_{k=1}^{\infty} \beta(p^k)}.$$

Since both A and B are products of a consecutive integers and multiples of p^k appear p^k integers apart in the sequence of integers, we have

$$|\alpha(p^k) - \beta(p^k)| \leqslant 1.$$

By hypothesis, $a_p \leqslant b_p$. That is, the first multiple of p in the sequence $a, a-1, \dots, 1$ will occur not later than the first multiple of p in the sequence $b, b-1, \dots, b-a+1$. Thus $\alpha(p) \geqslant \beta(p)$. But if $p > a$, then $\alpha(p) = 0$. So, $\beta(p) = 0$ also, and neither A nor B is divisible by p.

We have

$$\binom{b}{a} = \frac{B}{A} = \prod_{p \leqslant a} p^{\sum_{k=1}^{\infty} \beta(p^k) - \sum_{k=1}^{\infty} \alpha(p^k)}.$$

Denoting by $\kappa(p)$ the exponent of the highest power of p for which $\beta(p^k) > 0$ we get

$$\sum_{k=1}^{\infty} (\beta(p^k) - \alpha(p^k)) = \beta(p) - \alpha(p) + \sum_{k=2}^{\kappa(p)} (\beta(p^k) - \alpha(p^k))$$
$$- \sum_{k=\kappa(p)+1}^{\infty} \alpha(p^k) \leqslant \sum_{k=2}^{\kappa(p)} 1 = \kappa(p) - 1.$$

Therefore

$$\frac{B}{A} \Bigg| \prod_{p \leqslant a} p^{\kappa(p)-1},$$

or put in another way

$$\frac{(b-a+1)\cdots(b-1)b}{\prod_{p \leqslant a} p^{\kappa(p)}} \Bigg| \frac{1 \cdot 2 \cdots (a-1)a}{\prod_{p \leqslant a} p}.$$

Here, after factoring, there remain in the right-hand side exactly $a - \pi(a)$ factors each at most a, and in the left-hand side at least $a - \pi(a)$ such factors which are $\geqslant b - a + 1$. Since $b \geqslant 2a$, $b - a + 1 \geqslant a + 1 > a$, so we have a contradiction.

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THE TEACHING OF MATHEMATICS

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The Classification of 1-Manifolds: A Take-Home Exam

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1. Introduction. An effective and much used method for introducing students to a new mathematical topic (e.g., modern algebra) is to pick some important subtopic (say, groups) and then present a discussion of the simplest or most familiar special

case (for example, the integers). Applying this doctrine to topology, it is clear that the study of manifolds is a very central subtopic and the simplest special case is surely that of 1-dimensional manifolds. The "classification theorem" of our title says in effect that 1-manifolds are not only simple but they are also familiar, being in fact nothing more than circles or intervals. The theorem itself is probably no great surprise. It is however important and useful in at least one approach to topology in obtaining some of the deeper results connected with fixed point theory. Further, the proof of the theorem is both instructive and nontrivial. It seemed worthwhile, therefore, for pedagogical reasons to present the detailed treatment which follows.

This exposition was motivated by recent experience in trying to teach courses in algebraic and differential topology at the advanced undergraduate level. These subjects, it seems to me, present some special difficulties not present in other courses. Consider, for example, courses in modern algebra or integration theory or point-set topology. In these areas the subject matter has become quite standardized, there are numerous texts that treat the material, and it is possible to get down to business quite rapidly and start presenting some of the important results.

In the topology courses, on the other hand, it seems that no matter what approach one takes it is necessary to do a fair amount of hard work before one can get to the meat of the subject, and I found many of my students were not prepared for this from their experiences in other courses. The problem becomes particularly acute if one demands the same degree of rigor and precision as in, say, a point-set topology course. On the other hand, if in order to "get somewhere" one takes a more relaxed, informal approach, many students become unsure as to when a careful argument is needed and when a wave of the hands is enough.

I certainly have no ready solution for this dilemma which may well be inherent in the nature of the subject itself, but I do want to propose a way of at least getting off to a good start. My thesis is that the problem of classifying all 1-dimensional manifolds provides an excellent bridge between the pure point-set ideas which the students are presumably already familiar with and the new combinatorial material to which they are being introduced. Further, the result can be derived completely rigorously without taking an inordinate amount of time.

Here are some other reasons for working through the 1-manifold theorem:

1. A number of undergraduate texts present the classifications of 2-manifolds at an early stage. It seems rather natural to do the easier 1-manifold theorem first as a sort of warm-up.

2. The classification theorem represents a typical example of a theorem which adduces a global conclusion from local hypotheses, i.e., knowing what a space looks like in the neighborhood of each of its points enables one to conclude exactly what it is "in the large." Such theorems, of course, are central in many branches of geometry and analysis.

3. The 1-manifold theorem is perhaps not so important in the development of algebraic topology, but it plays an absolutely pivotal role in differential topology as

presented in the famous exposition of Milnor [3] based on the work of Hirsch [2] on the Brouwer fixed-1 point theorem.

4. One might say that 1-manifolds themselves are not very exciting. There are only four connected ones (manifolds in this paper will always include manifolds with boundary) and they are the obvious ones. I claim, however, that the proof of this fact is interesting because it requires the use of the Hausdorff separation axiom and it is precisely at the point where this axiom is used that the combinatorial aspect of the problem becomes apparent. I am referring to Proposition 1 of the presentation to follow, and its corollary.

I should remark that proofs of the 1-manifold theorem in the smooth case are given both by Milnor [3] and by Guillemin and Pollack [1] but both of these proofs make use of differentiability. The proof for the topological case which is presented here is not to my knowledge presented in any text. In fact, I have not been able to find anyone who was able to tell when the theorem was first proved or by whom, and I would be most interested in any information on this matter. In any case I don't imagine any proof of the result can be very different from the one presented here.

I have organized the material in the form of a take-home exam because the topic seems ideally suited for this. In fact, I have tried to arrange things so that a person teaching an undergraduate topology course could use this exposition directly as it stands. I even think the material would be suitable for use following the well-known R. L. Moore method in which all proofs are presented in class by the students. The instructor may wish to conduct a somewhat shorter exercise by considering only the case of manifolds without boundary. I have organized the proofs into a series of lemmas and propositions and have provided hints with the purpose of bringing the work to what I consider the appropriate level for upper division undergraduate math majors. I have not included proofs but would be glad, upon request, to send my own set of answers to anyone interested.

2. The Theorem. It will be assumed in what follows that the reader is familiar with standard point set topology and the elementary topological properties of the real numbers, specifically, that connected subsets are intervals, that homeomorphisms between intervals are monotonic, and that open subsets of the reals are unions of disjoint open intervals.

DEFINITION. A 1-manifold is a second countable Hausdorff topological space X such that

(M) X can be covered by open sets each of which is homeomorphic either to the open interval $\langle 0,1\rangle$ or the half-open interval $[0,1\rangle$. Sets of the first type will be called ***O-sets***, of the second type ***H-sets***, of either type ***I-sets***, and the corresponding homeomorphisms to these intervals will be called ***O-charts***, ***H-charts***, and ***I-charts***, respectively. If X can be covered by O-sets it is a ***manifold without boundary***, otherwise it is a ***manifold with boundary***.

CLASSIFICATION THEOREM. *There are exactly four connected* 1*-manifolds* (*up to homeomorphism*) *and they are given by the following table*:

	Without boundary	With boundary
Compact	a circle	a closed interval
Non-compact	an open interval	a half-open interval

PROPOSITION 0. *Each of the four spaces of the table above is a* 1*-manifold.*

We next want to show the necessity of the Hausdorff Axiom.

EXAMPLE 1. Let $X = \langle 0,1\rangle \cup \{p\}$ where p is a singleton. A basis for the open sets of X are all open sets of $\langle 0,1\rangle$ plus all sets of the form $(U - \{1/2\}) \cup \{p\}$ where U is open in $\langle 0,1\rangle$ and $1/2 \in U$. Prove that X is a T_1-space which satisfies condition (M) but is not Hausdorff.

From here on U and V will stand for I-sets in a 1-manifold and ϕ and ψ will be associated I-charts.

LEMMA. *Suppose $U \cap V$ and $U - V$ are nonempty and let (x_n) be a sequence in $U \cap V$ converging to x in $U - V$. Then the sequence $\psi(x_n)$ has no limit point in $\psi(V)$.*

Hint: Use the Hausdorff property.

We say that U and V ***overlap*** if $U \cap V$, $U - V$ and $V - U$ are nonempty.

DEFINITION. An open subinterval of $\langle 0,1\rangle$ is ***lower*** if it is of the form $\langle 0, b\rangle$ and ***upper*** if it is of the form $\langle a, 1\rangle$. A subinterval which is either upper or lower is called ***outer***. It is easy to see that an open interval in $\langle 0,1\rangle$ is outer if and only if it contains a sequence with no limit point in $\langle 0,1\rangle$. Similarly, in $[0,1\rangle$, a subinterval is called ***upper*** and ***outer*** if it is of the form $\langle a,1\rangle$. (There are, by definition, no lower open subintervals of $[0,1\rangle$.) An open subinterval of $[0,1\rangle$ is outer if and only if it contains a sequence with no limit point in $[0,1\rangle$.

PROPOSITION 1. *If U and V overlap and W is a component of $U \cap V$, then $\phi(W)$ and $\psi(W)$ are outer intervals.*

Hint: Note that $\phi(W)$ is a proper subinterval of $\phi(U)$. Using the lemma show that $\phi(W)$ is an open interval. Then construct an appropriate sequence in $\phi(W)$ and use the lemma again.

COROLLARY. *If U and V are I-sets, then $U \cap V$ has at most two components. If either U or V is an H-set, then $U \cap V$ is connected.*

PROPOSITION 2. *If X is connected and $U \cap V$ has two components, then X is a circle.*

Hints: (a) Let the components be Z and W and choose O-mappings ϕ and ψ so that $\phi(W)$ and $\psi(W)$ are lower and $\phi(Z)$ and $\psi(Z)$ are upper.

(b) Let $a = \sup \phi(W)$, $a' = \inf \phi(Z)$, and $b = \sup \psi(W)$, $b' = \inf \psi(Z)$. Let f map $[0,1]$ to the unit square by a piecewise linear mapping with

$$f(0) = (0,0), \quad f(a) = (1,0), \quad f(a') = (1,1), \quad f(1) = (0,1).$$

Let g map $[b, b']$ linearly with $g(b) = (0,0)$, $g(b') = (0,1)$. Define η on $U \cup V$ by $\eta(x) = f \circ \phi(x)$ for $x \in U$ and $\eta(x) = g \circ \psi(x)$ for $x \in V - U$.

(c) Prove that η is a homeomorphism of X onto the unit square using compactness of $U \cup V$ and connectedness of X.

PROPOSITION 3. Hypotheses: *U and V overlap and $U \cap V$ is connected*. Conclusion: (i) *If U and V are O-sets, so is $U \cup V$.*

(ii) *If U is an H-set and V an O-set, then $U \cup V$ is an H-set.*

(iii) *If U and V are H-sets, then $U \cup V = X$ and X is a closed interval.*

Hint: Letting $W = U \cap V$ choose ϕ and ψ so that $\phi(W)$ and $\psi(W)$ are upper and let $b = \inf \psi(W)$; define η on $U \cup V$ by $\eta(x) = \phi(x)$ for $x \in U$ and $\eta(x) = 1 + b - \psi(x)$ for $x \in V - U$.

All right; now give the proof of the Classification Theorem for the compact case. (*Hint*: Use induction on the number of sets in a finite open covering.)

For the noncompact case we must use separability of the space C, for there exist nonseparable Hausdorff 1-manifolds, the so-called Long Lines, which will be discussed in the Appendix.

Assuming second countability, prove the Classification Theorem for the noncompact case. *Hint*: Consider first the case without boundary. Let (U_i), $i = 1, 2, \ldots$ be a countable covering of X by O-sets and define a nested sequence (V_i) of O-sets inductively as follows. $V_1 = U_1$ and $V_{n+1} = V_n \cup U_k$ where k is the smallest subscript such that U_k meets V_n. Prove that $V = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} U_n = X$ (this is the crucial step). Finally, define mappings ψ_n from V_n to R inductively as follows: ψ_1 is an O-mapping of V_1. Now suppose $\psi_n(V_n) = \langle a, b \rangle$. Let $\tilde{\psi}_n$ be an O-mapping on V_{n+1} such that $\tilde{\psi}_n \circ \psi_n^{-1} \langle a, b \rangle = \langle \alpha, \beta \rangle$ and use this to define an extension of ψ_n to a mapping ψ_{n+1} of V_{n+1}.

For the case with boundary a slight modification of the above construction and argument is needed.

Appendix: The Long Line. In order to show that second countability is necessary for the classification theorem we present here an example of a Hausdorff space which satisfies property (M) but is not second countable.

For this section some familiarity with transfinite ordinals is required. Consider the set L of all pairs (α, x) where α is a countable ordinal and $x \in [0, 1\rangle$ and order

these pairs lexicographically, that is, $(\alpha, x) > (\beta, y)$ if $\alpha > \beta$ or if $\alpha = \beta$ and $x > y$. It is easy to see that L is a Hausdorff space in the order topology.

For each countable ordinal α define $H_\alpha = \{(\beta, x) \in L | \beta < \alpha\}$.

PROPOSITION 4. *H_α is an H-set.*

Granting Proposition 4, it follows at once that (M) is satisfied by the sets H_α, so L is a manifold with boundary. However, it is not second countable because the $\{H_\alpha\}$ cannot be reduced to a countable covering. Namely, if $\{H_{\alpha_i}\}$, $i = 1, 2, \ldots$, were such a covering, then any countable ordinal would be in some H_{α_i}, but there are only a countable number of ordinals in each H_{α_i}. Hence $\bigcup_{i=1}^{\infty} H_\alpha$ is a countable set, contradicting the fact that there are uncountably many countable ordinals.

To prove Proposition 4 one must show that (i) H_α has the least upper bound property, and (ii) H_α has a countable dense subset. Both are easily proved. Finally one uses the fact that any set with properties (i) and (ii) is order-isomorphic to a real interval, in this case a half-open interval. This is not hard either. First map the rational points of $[0, 1\rangle$ order-isomorphically onto the countable dense subset and then extend $[0, 1\rangle$ using the least upper bound property.

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The Error for Quadrature Methods: A Complex Variables Approach

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By a quadrature method for integration over an interval $[a, b]$, we mean a set of distinct points $x_0 < x_1 < \cdots < x_n$, a set of constants $\alpha_0, \ldots, \alpha_n$, and a formula

$$Q_n(f) := \sum_{j=0}^{n} \alpha_j f(x_j) \tag{1}$$

that serves as an estimate of the integral

$$\int_a^b f(x)\, dx.$$

*The research of this author was supported, in part, by a National Science Foundation research grant.

In this note, we present a new proof of a recent result [3] due to Claus Schneider that provides a simple and elegant method for determining the error and degree of precision for general quadrature formulae. The essential feature of our proof is the use of complex variable techniques, which simplify the algebraic manipulation of divided differences, and provide a straightforward derivation of a formula for the error

$$E_n(f) := \int_a^b f(x)\,dx - Q_n(f).$$

We shall prove

THEOREM 1 [3: Prop. 2.1]. *Let* $x_0 < x_1 < \cdots < x_u = a < \cdots < x_v = b < \cdots < x_n$, *and let* $Q_n(f)$ *as given in* (1) *be any quadrature rule approximating* $\int_a^b f(x)\,dx$, *where* f *is continuous on* $[x_0, x_n]$. *Suppose that* $F' = f$ *and define the polynomials*

$$\Omega(x) := \prod_{j=0}^{n} (x - x_j), \tag{2}$$

$$q_n(x) := \Omega^2(x)\left\{\frac{b-a}{(x-a)(x-b)} - \sum_{j=0}^{n} \frac{\alpha_j}{(x-x_j)^2}\right\}. \tag{3}$$

Then

$$E_n(f) := \int_a^b f(x)\,dx - Q_n(f) = (Fq_n)[x_0, x_0, \ldots, x_n, x_n]. \tag{4}$$

Here, $(Fq_n)[x_0, x_0, \ldots, x_n, x_n]$ is the $(2n+1)$st divided difference of the product Fq_n at the points $x_0, x_0, \ldots, x_n, x_n$. Notice that there are no restrictions on the signs of the weights α_j. Moreover, since some of the weights may be zero, there is no loss of generality in our assumption that the endpoints a, b are nodes of the quadrature formula.

Before embarking on the proof of Theorem 1, we briefly discuss divided differences.

DEFINITION 1. Let $P_m(x) = \sum_{k=0}^{m} a_k x^k$ be the unique polynomial of degree at most m that interpolates (agrees with) the function g at the points $t_0, t_1, \ldots, t_m$. Then the mth divided difference of g in these points is given by

$$g[t_0, \ldots, t_m] := a_m. \tag{5}$$

If $t_0, \ldots, t_m$ are distinct, this definition is unambiguous. In the case that some of the t_j are repeated, we are to understand "interpolating polynomial" in the *Hermite sense*. That is, if t_j is repeated i times, then we mean that the polynomial P_m and its first $i-1$ derivatives agree with g and its first $i-1$ derivatives at t_j.

In the case of distinct t_j, a simple induction yields the familiar recursive definition:

$$g[t_j] = g(t_j), \quad j = 0, \dots, n,$$
$$g[t_0, \dots, t_k] = \frac{g[t_0, \dots, t_{k-1}] - g[t_1, \dots, t_k]}{t_0 - t_k},$$

where k ranges from 1 to n. More important for us is the following known representation theorem for the divided difference of an analytic function (cf. [**2**, §3.6]).

THEOREM 2. *Let $t_0, \dots, t_m$ be $m + 1$ (not necessarily distinct) points, and let C be a simple closed rectifiable curve in the z-plane surrounding $t_0, \dots, t_m$. If g is analytic inside and on C, then*

$$g[t_0, \dots, t_m] = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z - t_0) \cdots (z - t_m)}\, dz. \tag{6}$$

Proof. The representation (6) can be derived from the *Hermite error formula*

$$g(z) - P_m(z) = \frac{1}{2\pi i} \int_C \frac{\omega(z) g(t)}{\omega(t)(t - z)}\, dt, \tag{7}$$
$$\omega(z) := \prod_{j=0}^{m} (z - t_j),$$

for z inside C, where P_m is the interpolating polynomial of Definition 1. Using the Cauchy integral representation for $g(z)$, formula (7) is equivalent to

$$P_m(z) = \frac{1}{2\pi i} \int_C \left[\frac{\omega(z) - \omega(t)}{z - t} \right] \frac{g(t)}{\omega(t)}\, dt. \tag{8}$$

Notice that the right-hand side of (7) vanishes at the points t_j (the zeros of $\omega(z)$) in the Hermite sense, and that the integral in (8) is a polynomial of degree at most m in z. (These two observations show that the right-hand side of (8) is indeed the interpolating polynomial P_m.) Now since

$$\frac{\omega(z) - \omega(t)}{z - t} = z^m + \cdots,$$

we see from (8) that the coefficient a_m of z^m in P_m is just the integral

$$a_m = \frac{1}{2\pi i} \int_C \frac{g(t)}{\omega(t)}\, dt,$$

which verifies formula (6). □

Theorem 2 removes much of the mystery surrounding divided differences with coincident points, and permits a simple proof of Theorem 1. Schneider's proof of

Theorem 1, which makes use of Leibniz's rule for divided differences and other identities, is algebraic in nature. Our proof shows that the result follows in a natural way from the Cauchy Integral Theorem.

The Proof of Theorem 1. First assume that f is analytic on the interval $[x_0, x_n]$; then we can assume f has been analytically continued to an open domain containing $[x_0, x_n]$. Since $\int_a^b f(x)\,dx = F(b) - F(a)$, we have

$$\begin{aligned} E_n(f) &= F(b) - F(a) - \sum_{j=0}^{n} \alpha_j F'(x_j) \\ &= \frac{1}{2\pi i}\int_C \frac{F(z)}{z-b}\,dz - \frac{1}{2\pi i}\int_C \frac{F(z)}{z-a}\,dz - \sum_{j=0}^{n} \alpha_j \frac{1}{2\pi i}\int_C \frac{F(z)}{(z-x_j)^2}\,dz \\ &= \frac{1}{2\pi i}\int_C F(z)\left\{\frac{b-a}{(z-b)(z-a)} - \sum_{j=0}^{n}\frac{\alpha_j}{(z-x_j)^2}\right\}dz, \end{aligned}$$

where C is a suitably chosen contour. Now observe from (2) and (3) that the portion of the integrand in brackets is just $q_n(z)/\Omega^2(z)$. Hence

$$E_n(f) = \frac{1}{2\pi i}\int_C \frac{F(z)q_n(z)}{\Omega^2(z)}\,dz. \tag{9}$$

Since q_n is a polynomial, $F(z)q_n(z)$ is analytic inside and on C and so, by Theorem 2, $E_n(f)$ is the divided difference given in (4). This proves the result for analytic f.

In the general case where f is continuous on $[x_0, x_n]$, we construct the interpolating polynomial p that satisfies

$$p(x_i) = F(x_i), \quad p'(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.$$

Since

$$\int_a^b p'(x)\,dx = p(b) - p(a) = F(b) - F(a) = \int_a^b f(x)\,dx,$$

we see that $E_n(f) = E_n(p')$. Moreover we have

$$(pq_n)[x_0, x_0, \ldots, x_n, x_n] = (Fq_n)[x_0, x_0, \ldots, x_n, x_n]$$

because, in general, $g[x_0, x_0, \ldots, x_n, x_n]$ is a linear combination of the numbers $g(x_0), g'(x_0), \ldots, g(x_n), g'(x_n)$ (see [**1**, p. 12]). Thus, applying the first part of the proof to p', we obtain (4) in the general case. □

We now apply Theorem 1 to the study of precision in quadrature methods.

DEFINITION 2. The quadrature method (1) has *precision m* if it integrates exactly every polynomial of degree m or less, but does not integrate exactly some polynomial of degree $m + 1$. That is, for every polynomial p with degree $\leqslant m$, $Q_n(p) = \int_a^b p(x)\,dx$, but $Q_n(x^{m+1}) \neq \int_a^b x^{m+1}\,dx$.

In Theorem 1, the polynomial q_n is of degree at most $2n$. As the next corollary shows, the smaller the precise degree of q_n, the higher is the precision of the quadrature scheme.

COROLLARY 1. *Let* $d := \deg q_n$. *If* $d \leqslant 2n - 1$, *then the degree of precision of the quadrature scheme* (1) *is* $2n - d - 1$.

Proof. Let P_k be a polynomial of degree k and P_{k+1} be an antiderivative of P_k. Then the product $P_{k+1}q_n$ has degree $k + 1 + d$. Since

$$E_n(P_k) = (P_{k+1}q_n)[x_0, x_0, \ldots, x_n, x_n],$$

and the jth order divided difference of a polynomial of degree at most $j - 1$ is zero*, then $E_n(P_k) = 0$ for $k + 1 + d \leqslant (2n + 1) - 1$; that is, for $k \leqslant 2n - d - 1$. Hence the method integrates exactly every polynomial of degree $2n - d - 1$ or less.

It remains to show that $E_n(x^{2n-d}) \neq 0$. With $F(z) := z^{2n-d+1}$, formula (9) asserts that

$$E_n(x^{2n-d}) = \frac{1}{2\pi i}\int_C \frac{F(z)q_n(z)}{\Omega^2(z)}\,dz, \tag{10}$$

where C can be taken as any circle centered at the origin having sufficiently large radius. Since the integrand in (10) is a rational function with numerator degree $2n + 1$ and denominator degree $2n + 2$, then (after cancelling common factors) we have, in a neighborhood of infinity,

$$\frac{F(z)q_n(z)}{\Omega^2(z)} = \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots,$$

where $A_1 \neq 0$. Hence, from (10),

$$E_n(x^{2n-d}) = \frac{1}{2\pi i}\int_C\left[\frac{A_1}{z} + \frac{A_2}{z^2} + \cdots\right]dz = A_1 \neq 0. \square$$

We remark that if we say that the quadrature scheme (1) has degree of precision -1 if it will not even integrate constants exactly, then Corollary 1 also holds for $d = 2n$.

The reader is invited to apply Corollary 1 to familiar quadrature formulae such as the trapezoid rule, Simpson's rule, and Gaussian quadrature. For example, Simpson's rule is a special case of (1) where $n = 2$, $\alpha_0 = \alpha_2 = (b - a)/6$, $\alpha_1 = 2(b - a)/3$, $x_0 = a$, $x_1 = (b + a)/2$, and $x_2 = b$. A routine calculation shows that $q_2(x)$ reduces to a constant polynomial ($d = 0$) and so (as is well known) Simpson's rule has precision $2n - d - 1 = 4 - 1 = 3$.

Using the fact (proved by Rolle's theorem) that for smooth functions g and real points t_j,

$$g[t_0, \ldots, t_m] = \frac{g^{(m)}(\mu)}{m!}, \quad \text{for some } \mu \in [\min t_i, \max t_i], \tag{11}$$

we can derive the following representation for the error $E_n(f)$ in (4).

*This is an immediate consequence of Definition 1.

COROLLARY 2. *If* $f \in C^{(2n)}[x_0, x_n]$, *then for some* $\mu \in [x_0, x_n]$,

$$E_n(f) = \frac{1}{(2n+1)!} \sum_{j=0}^{d} \binom{2n+1}{j} q_n^{(j)}(\mu) f^{(2n-j)}(\mu), \tag{12}$$

where $d = \deg q_n$.

Proof. From (4), (11), and Leibniz's rule for differentiation we have

$$\begin{aligned} E_n(f) = (Fq_n)[x_0, x_0, \dots, x_n, x_n] &= \frac{(Fq_n)^{(2n+1)}(\mu)}{(2n+1)!} \\ &= \frac{1}{(2n+1)!} \sum_{j=0}^{2n+1} \binom{2n+1}{j} q_n^{(j)}(\mu) F^{(2n+1-j)}(\mu) \\ &= \frac{1}{(2n+1)!} \sum_{j=0}^{d} \binom{2n+1}{j} q_n^{(j)}(\mu) f^{(2n-j)}(\mu). \square \end{aligned}$$

We remark that formula (12) can be used to give an alternative proof of Corollary 1, which we previously established without appealing to Rolle's theorem.

REFERENCES

1. C. de Boor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.
2. P. J. Davis, Interpolation and Approximation, Dover, New York, 1974.
3. C. Schneider, Error Analysis for Numerical Integration—an Algorithmic Approach, Habilitationsschrift, Mainz, 1983.

PROBLEMS AND SOLUTIONS

EDITED BY PAUL T. BATEMAN, HAROLD G. DIAMOND, KENNETH B. STOLARSKY (ADVANCED PROBLEMS), AND DOUGLAS B. WEST (ELEMENTARY PROBLEMS)

DEDICATED TO THE MEMORY OF IRVING REINER

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ELEMENTARY PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed by June 30, 1987, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3189. *Proposed by M. Lutzky, Silver Spring, MD.*

Prove that the product of two skew-symmetric matrices of order $2N$ has no simple eigenvalues.

E 3190. *Proposed by Vasanth B. Solomon, Drake University, Des Moines, IA.*

Show that

$$\sum_{r=0}^{j} \frac{(-1)^r (N-2r)\binom{j}{r}}{(N-r)\cdots(N-r-j)} = 0 \quad \text{for } j>0 \quad \text{and} \quad N>2j.$$

E 3191. *Proposed by Gheorghe Răuţu, Centre of Mathematical Statistics, Bucharest, Romania.*

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ which have the Darboux property and which satisfy the equality

$$f(x+y) = f(x+f(y))$$

for $x, y \in \mathbb{R}$.

E 3192. *Proposed by D. Fisher, Harvey Mudd College, and M. Martelli, Bryn Mawr College.*

Find the largest k such that the following is true:
If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in a normed space with $\mathbf{u}+\mathbf{v}+\mathbf{w}=0$, then

$$\|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{u}\| \geqslant k(\|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\|).$$

E 3193. *Proposed by Andrew Lenard, Indiana University.*

Let θ be an (undirected) acute angle. Show that if a one-to-one mapping T of the Euclidean plane E onto itself has the property that whenever points P and Q subtend the angle θ at the point R then also the points $T(P)$ and $T(Q)$ subtend the angle θ at the point $T(R)$, the T is a similarity transformation of E.

E 3194. *Proposed by Ira Gessel, Brandeis University.*

Let S be the smallest set of rational functions containing x and y, and closed under subtraction and reciprocals. Show that $1 \notin S$.

Editorial remarks. Several solvers cited a more general result from pp. 92–93 of Louis Comtet, *Advanced Combinatorics*, Reidel, Boston, 1974. Namely, the $r \times r$ matrix, A, with integer entries $a_{ij} = \binom{m+i-1}{k+j-1}$, has the determinant,

$$\det A = \prod_{s=1}^{r} \frac{\binom{m+s-1}{k}}{\binom{k+s-1}{k}}.$$

This determinant reduces to the given quotient with $r = k + 1$ and $m = n + k$. Ira Gessel points out that the result follows from P. A. MacMahon, *Combinatorial Analysis*, vol. 2, section 495 where it is shown that

$$\prod_{s=1}^{m} \frac{\binom{n+k+s-1}{k}}{\binom{k+s-1}{k}}$$

is the number of plane partitions with at most k rows, at most m columns, and largest part at most n.

Also solved by I. Gessel, D. Jeffords, O. G. Ruehr, M. Vowe (Switzerland), P. Y. Wu (Taiwan), and the proposers.

The Word x^2y^2xy is Universal in A_n

E 3048 [1984, 437]. *Proposed by J. L. Brenner, Palo Alto, California, and D. M. Silberger, SUNY at New Paltz, New York.*

Prove that every element a in the alternating group $A_n (n > 4)$ can be written in the form $a = xxyyxy (x, y \in A_n)$.

Solution by the proposers. We exhibit a permutation $y \in A_n$ such that $y^2 = 1$ and ay has period prime to 3. Then, solving $3t \equiv 1 \bmod k$, where k is the period of ay, we set $x = (ay)^t$, whence we have that $a = xxxy = xxyyxy$ since $yy = 1$. To show how to construct y we consider five cases where the period of a is not prime to 3.

CASE 1. None of the cyclic components of a is a 3-cycle. Then, for each cyclic component c of a whose length $L(c)$ is a multiple of 3, we have that $L(c) > 5$; we multiply c by two disjoint transpositions of symbols adjacent in c, in order to

shorten c to a cycle of length $L(c)-2$. Paradigm Example:

$$(1\,2\,3\,4\,5\,6)\cdot(1\,2)(3\,4)=(1\,3\,5\,6).$$

CASE 2. a has exactly $2i$ cyclic components of length 3, for i a positive integer. Partition these 3-cycles into pairs. For each pair, fuse the two members of the pair with one transposition, and shorten the result of this fusion to a 5-cycle, using a transposition disjoint from the first transposition. Paradigm Example:

$$(1\,2\,3)(4\,5\,6)\cdot(1\,2)(3\,4)=(1\,3\,5\,6\,4).$$

The j-cyclic components of a, for $j\neq 3$, are handled as in Case 1.

CASE 3. a has exactly $2i+1$ cyclic components of length 3, for i a positive integer. Paradigm Example:

$$(1\,2\,3)(4\,5\,6)(7\,8\,9)\cdot(1\,2)(3\,4)(6\,7)(8\,9)=(1\,3\,5\,6\,8\,7\,4).$$

In the foregoing example, four disjoint transpositions are employed to fuse three component 3-cycles of a, and to shorten the resultant 9-cycle to a cycle whose length 7 is prime to 3. The remaining cyclic components of a are handled as in Cases 1 and 2.

CASE 4. a has at least two cyclic components, but exactly one component that is a 3-cycle. In this case we fuse the troublesome 3-cycle together with one or two of the other cyclic components of a, and when necessary shorten the result of this fusion to a cycle whose length is prime to 3. We arrange that the number of transpositions used in this process is even; the details are as follows:

Subcase. Besides its 3-cyclic component, a has some cyclic component r of odd length $L(r)>3$. If $L(r)-1$ is not a multiple of 3, then we fuse the 3-cycle together with r, and shorten the result to a cycle of length $L(r)+2$; this process uses two disjoint transpositions. Paradigm Example:

$$(1\,2\,3)(4\,5\,6\,7\,8)\cdot(1\,2)(3\,4)=(1\,3\,5\,6\,7\,8\,4).$$

On the other hand, if $L(r)-1$ is a multiple of 3, then $L(r)>6$, and we use four disjoint transpositions in a process which yields a cycle of length $L(r)$. Paradigm Example:

$$(1\,2\,3)(4\,5\,6\,7\,8\,9\,10)\cdot(1\,2)(3\,4)(5\,6)(7\,8)=(1\,3\,5\,7\,9\,10\,4).$$

Subcase. a has exactly one cyclic component of odd length. Then a has at least two cyclic components r and s whose lengths are even. If $L(r)+L(s)$ is not a multiple of 3, then with two disjoint transpositions we merely fuse the 3-cycle together with r and s; e.g.,

$$(1\,2\,3)(4\,5)(6\,7)\cdot(3\,4)(5\,6)=(1\,2\,3\,5\,7\,6\,4).$$

On the other hand, if $L(r)+L(s)$ is a multiple of 3, then we employ four transpositions, thus both fusing and shortening; e.g.,

$$(1\,2\,3)(4\,5)(6\,7\,8\,9)\cdot(1\,2)(3\,4)(5\,6)(7\,8)=(1\,3\,5\,7\,9\,6\,4).$$

CASE 5. a has exactly one cyclic component, and that component is a 3-cycle. Then, since $n > 4$, there are at least two points fixed by a. Paradigm Example:

$$(1\,2\,3)\cdot(3\,4)(1\,5) = (1\,5\,2\,3\,4).$$

Terminological note. In current parlance, the above argument proves that the word x^2y^2xy is *universal* in every alternating group A_n for which $n > 4$.

Also solved by the University of South Alabama Problem Group and G. L. Walls.

A Sum Involving Stirling Numbers

E 3052 [1984, 438]. *Proposed by Peter Ungar, New York University*.

Show that

$$\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\binom{n}{k}(n-k)^n = n^n\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, The Netherlands. Since

$$(n-k)^n = \sum_{m=0}^{n}(-1)^m\binom{n}{m}k^m n^{n-m},$$

by binomial expansion, the given sum can be written as

$$\sum_{m=0}^{n}(-1)^{m-1}\binom{n}{m}n^{n-m}S(m-1,n),$$

where

$$S(p,n) = \sum_{k=1}^{n}(-1)^k\binom{n}{k}k^p \qquad (n>0).$$

Now

$$\begin{aligned}
S(0,n) &= (1-1)^n - 1 = -1,\\
S(-1,n) &= \sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1} = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\int_0^1 x^{k-1}\,dx\\
&= \int_0^1\sum_{k=1}^{n}(-1)^k\binom{n}{k}x^{k-1}\,dx = \int_0^1\frac{(1-x)^n-1}{x}dx\\
&= -\int_0^1(y^{n-1}+y^{n-2}+\cdots+1)\,dy, \qquad y = 1-x\\
&= -\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right),\\
S(p,n) &= 0
\end{aligned}$$

for $0 < p < n$, as is easily seen by noting that for $p > 0$, $(-1)^n S(p, n)$ is just the Stirling number of the second kind, that is, the number of ways in which a p element set can be partitioned into n nonempty subsets. It follows that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k}(n-k)^n = n^n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 1\right).$$

Also solved by 36 other readers and the proposer.

Curves Containing Rectangles

E 3056 [1984, 515]. *Proposed by Bruce Reznick, University of Illinois at Urbana-Champaign.*

For which integers $n > 1$ and real numbers r does the curve $y = x^n + rx$ contain the vertices of a rectangle?

Solution by The University of South Alabama Problem Group. We will show that n must be odd, and r may be any number less than $-2\sqrt{n}/(n-1)$. It is easy to show that it is necessary that n is odd and r is negative, and, furthermore, if (x_i, y_i), $i = 0, 1, 2, 3$, are consecutive vertices (in a counterclockwise direction) of a parallelogram which are on the curve $y = x^n + rx$ with $x_1 > x_0 > x_2 > x_3$, then $x_1 = -x_3$ and $x_0 = -x_2$. Using

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2$$

and simplifying, we find that the parallelogram is a rectangle if and only if

$$1 + \left(\frac{x_1^n - x_0^n}{x_1 - x_0} + r\right)\left(\frac{x_1^n + x_0^n}{x_1 + x_0} + r\right) = 0.$$

Let $A = A(x_1, x_0)$ be the continuation of $(x_1^n - x_0^n)/(x_1 - x_0)$ to the domain $0 \le x_0 \le x_1$ and $B = B(x_1, x_0)$ be the continuation of $(x_1^n + x_0^n)/(x_1 + x_0)$ to the same domain. The above orthogonality requirement is equivalent to

$$r = -\left((A + B) \pm \left((A - B)^2 - 4\right)^{1/2}\right)/2,$$

so we see that $A - B \ge 2$. To maximize r, we set

$$r = -\left((A + B) - \left((A - B)^2 - 4\right)^{1/2}\right)/2.$$

Let c be a positive constant. The function $A(x, y) + B(x, y)$, when subjected to the constraints $x + y = c$, $0 \le y \le x$, is minimum when $x = y = c/2$, and the function $A(x, y) - B(x, y)$, when subjected to the same constraints, is maximum

when $x = y = c/2$. Hence if $w > 0$, then the curves

$$A - B = w \quad\text{and}\quad A + B = \left(\frac{n+1}{n-1}\right)w \quad \text{for } 0 \le y \le x$$

intersect only at the point

$$x = y = \left(\frac{2w}{n-1}\right)^{1/(n+1)},$$

so if

$$u > \left(\frac{n+1}{n-1}\right)w \quad\text{and}\quad 0 \le y \le x,$$

the curves $A + B = u$ and $A - B = w$ have an intersection point (x, y) with $x > y > 0$ (and no intersection point if $u < (n+1)w/(n-1)$).

The function

$$r(w) = -\left(\left(\frac{n+1}{n-1}\right)w - \left(w^2 - 4\right)^{1/2}\right)/2, \quad w \ge 2,$$

takes on its maximum value $-2\sqrt{n}/(n-1)$ when $w = (n+1)/\sqrt{n}$, so we see that if we set $A - B = (n+1)/\sqrt{n}$ with $n > 1$ odd, and let $A + B$ take on all values greater than $(n+1)^2/(\sqrt{n}(n-1))$, then r takes on all values less than $-2\sqrt{n}/(n-1)$, and so for precisely such r, $y = x^n + rx$ contains the vertices of a rectangle.

Also solved by O. P. Lossers (The Netherlands), M. D. Meyerson, M. Pelant (Czechoslovakia), and the proposer. Partially solved by W. Janous (Austria).

Stirling Derangements

E 3057 [1984, 515]. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Prove that for all $n \geqslant 3$,

$$\sum_{k=1}^{n} k^3\binom{n}{k}D_{n-k} = 5n!,$$

where $D_m = m!\sum_{r=0}^{m}(-1)^r 1/r!$ denotes the derangement number (of $1, 2, \ldots, m$).

Solution I by David H. Leuck, Farmington, Connecticut. To be more general, one can show that for any nonnegative integer r

$$\sum_{k=0}^{n} k^r\binom{n}{k}D_{n-k} = n!\sum_{m=0}^{\min\{r,n\}} S(r, m), \tag{$*$}$$

where $S(r, m)$ is a Stirling number of the second kind.

If $r \geq 0$,

$$\begin{aligned}
\sum_{k=0}^{n} k^r \binom{n}{k} D_{n-k} &= n! \sum_{k=0}^{n} \frac{k^r}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \\
&= n! \sum_{k+j \leq n} \frac{(-1)^j}{j!k!} k^r \\
&= n! \sum_{m=0}^{n} \frac{1}{m!} \sum_{k+j=m} \frac{(-1)^j m!}{j!k!} k^r \\
&= n! \sum_{m=0}^{n} \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^r \\
&= n! \sum_{m=0}^{n} S(r, m).
\end{aligned}$$

Assertion $(*)$ follows, then, since $S(r, m) = 0$ for $m = r + 1, r + 2, \ldots$.

For the case $n \geq 3 = r$ we have,

$$\begin{aligned}
\sum_{k=1}^{n} k^3 \binom{n}{k} D_{n-k} &= \sum_{k=0}^{n} k^3 \binom{n}{k} D_{n-k} \\
&= n! \sum_{m=0}^{3} S(3, m) \\
&= n!(0 + 1 + 3 + 1) = 5n!.
\end{aligned}$$

Solution II by J. M. Freeman, S. C. Locke, and H. Niederhausen, Florida Atlantic University. We consider the following problem.

How many pairs of functions (f, g) are there such that

$$f: \{1, 2, \ldots, r\} \to \{1, 2, 3, \ldots, n\}$$

and g is a permutation of $\{1, 2, 3, \ldots, n\}$ which fixes the range of f?

If we let k denote the number of fixed points of a suitable function g, then there are $\binom{n}{k}$ ways to choose which k points are fixed, D_{n-k} ways to arrange the remaining points and k^r choices for the function f. Summing over all possible k, we obtain for the number of pairs (f, g)

$$\sum_{k=1}^{n} \binom{n}{k} k^r D_{n-k}.$$

On the other hand, we can consider that the cardinality of the range of f is i, for some i. For each unordered partition of $\{1, 2, \ldots, r\}$ into i nonempty sets $S_1, S_2, \ldots, S_i$ there are $n!/(n - i)!$ ways to choose the images of $S_1, S_2, \ldots, S_i$ and

$(n-i)!$ ways to permute the remaining elements of $\{1,2,\dots,n\}$, for a total of $n!$ ways to form g. Summing over all partitions, we get $m \cdot n!$, where m is the number of unordered partitions of r into at most n nonempty parts. If $n \geq r$, this is the rth Bell number, B_r.

Thus,

$$\sum_{k=1}^{n} \binom{n}{k} k^r D_{n-k} = B_r n!, \quad \text{for} \quad n \geq r.$$

In particular, there are 5 partitions of $\{1,2,3\}$; thus $B_3 = 5$ and, hence,

$$\sum_{k=1}^{n} \binom{n}{k} k^3 D_{n-k} = 5n!$$

Also solved by 38 other readers and the proposer.

Empty Cells

E 3061 [1984, 580]. *Proposed by T. Ferguson and C. Melolidakis, University of California, Los Angeles.*

For each of n independent trials, a ball is placed in cell i with probability $2^{-(i+1)}$ for $i = 0,1,2,\dots$. Let K denote the number of empty cells less than the largest occupied cell. Show that for all $n \geqslant 1$, $P_n(K=k) = 2^{-(k+1)}$ for $k = 0,1,2,\dots$.

Solution by Donald E. Knuth, Stanford University.

Equivalently, place a ball in cell 0 with probability 1/2; then if it is still unplaced, repeat the same process with cell numbers increased by 1. We can place n balls by first seeing how many of them go into cell 0, then (if any are left) we can place the remaining $j > 0$ balls in higher-numbered cells with the same process. It follows that

$$P_n(K=k) = 2^{-n}\delta_{k0} + 2^{-n}\sum_{j=1}^{n-1}\binom{n}{j}P_j(K=k) + 2^{-n}P_n(K=k-1),$$

and the stated result follows by induction on k and n.

Also solved by D. Callan, Cal Poly Pomona Problems Group, D. A. Darling, V. Hernandéz and P. Vélez (Spain), E. Hertz, O. P. Lossers (The Netherlands), R. S. Pinkham, W. D. Rehder, J. Vukmirović (Yugoslavia), R. Zahn, and the proposer.

Factorial Products as a Factorial

E 3063 [1984, 581]. *Proposed by Ion Cucurezeanu, Constanta, Romania.*

Find all natural numbers $n > 1$ and $m > 1$ such that

$$1!3!5! \cdots (2n-1)! = m!.$$

Solution by D. Enkers, Rand Afrikaans University, Johannesburg, South Africa.

The only pairs (n, m) satisfying the above conditions are $(2,3)$, $(3,6)$, and $(4,10)$. In order to prove this, let $n \geq 5$. Note that for $n = 5$, $1!3! \cdots 9! > 10!$. Hence, if $1!2! \cdots 9! = m!$ for some m, then $m \geq 11$; hence $11 | m!$ while $11 \nmid 1!3! \cdots 9!$. A similar argument can be used if $n = 6$: $1!3! \cdots 11! > 12!$ with 13 as prime.

For $n \geq 7$, an easy inductive proof shows that $1!3! \cdots (2n-1)! > (4n)!$ and for each n Bertrand's postulate guarantees the existence of a prime p with $2n < p < 4n$ which can be used to the same effect.

Also solved by 42 other readers and the proposer.

Cubes as Sums of Consecutive Squares

E 3064 [1984, 649]. *Proposed by Ion Cucurezeanu, Constanta, Romania.*

Find all integers m, $m > 1$, such that m^3 is a sum of m squares of consecutive integers.

Solution by Gertrude Ehrlich, University of Maryland. We seek all positive integers $m > 1$ such that, for some integer k,

$$m^3 = (k+1)^2 + \cdots + (k+m)^2. \tag{1}$$

Squaring, collecting terms and summing, we have

$$m^3 = mk^2 + 2 \cdot \frac{m(m+1)}{2}k + \frac{m(m+1)(2m+1)}{6},$$

which reduces to the quadratic

$$k^2 + (m+1)k + \frac{1}{6}(m+1)(2m+1) - m^2 = 0,$$

with solutions

$$k = -\frac{1}{2}(m+1) \pm \frac{1}{6}t, \quad \text{where } t = \sqrt{33m^2+3}. \tag{2}$$

From (2) it follows that, for m an integer, k is an integer if and only if t is an integer. For if k is an integer, then t is rational and the square root of an integer, hence t is an integer. Conversely, if t is an integer, then 3 divides t^2, hence 3 divides t, and $m + 1$ and $t/3$ have the same parity. But then $(m + 1)/2$ and $t/6$ are either both integers, or both halves of odd integers, and so k is an integer.

Thus, to find the solutions of (1), it suffices to solve the Diophantine equation $33m^2 + 3 = t^2$ or, equivalently,

$$t^2 - 33m^2 = 3. \tag{3}$$

The obvious solution (6, 1) of (3) is ruled out by the conditions of the problem, but it is essential for finding the set of all solutions.

Consider the Pell equation

$$x^2 - 33y^2 = 1. \tag{4}$$

The fundamental solution (i.e., the positive solution (x, y) with $x + \sqrt{33y}$ as small as possible) is $(x, y) = (23, 4)$, hence the set of all solutions of (4) is given by $(\pm 1, 0)$ and

$$(\pm x_n, \pm y_n), \quad \text{where } x_n + y_n\sqrt{33} = (23 + 4\sqrt{33})^n, \tag{5}$$

for n any positive integer [**1**].

From every solution (x, y) of (4), using the solution (6, 1) of (3), we obtain a solution (t, m) of (3) by forming

$$(6 + \sqrt{33})(x + y\sqrt{33}) = t + m\sqrt{33},$$

where

$$t = 6x + 33y, \quad m = x + 6y. \tag{6}$$

Then $t - m\sqrt{33} = (6 - \sqrt{33})(x - y\sqrt{33})$, whence

$$\begin{aligned} t^2 - 33m^2 &= (t - m\sqrt{33})(t + m\sqrt{33}) \\ &= (6 - \sqrt{33})(6 + \sqrt{33})(x - y\sqrt{33})(x + y\sqrt{33}) \\ &= (6^2 - 1^2 \cdot 33)(x^2 - 33y^2) = 3 \cdot 1 = 3, \end{aligned}$$

and so (t, m) is a solution of (3).

In fact, every solution of (3) may be obtained in this way. For let (t, m) be any solution of (3), and let $x = 2t - 11m$, $y = (6m - t)/3$. Since $t^2 - 33m^2 = 3$, 3 divides t^2, hence 3 divides t, and so x and y are both integers. It is easily verified that (x, y) is a solution of (4), and that $t = 6x + 33y$, $m = x + 6y$. From (5) and (6), it is apparent that the solutions of (3) with $m > 1$ are obtained from the solutions $(+x_n, +y_n)$ in (5), and that all values of m occurring in solutions are odd. The trivial solution (1, 0) of (4) yields the "trivial" solution (6, 1) of (3).

We conclude that the values of m called for in the problem are given by $m_n = x_n + 6y_n$, where $x_n + y_n\sqrt{33} = (23 + 4\sqrt{33})^n$ for n any positive integer.

For computational purposes, one easily obtains the following recursive characterizations of the solutions (x_n, y_n) of (4) and the corresponding solutions

(t_n, m_n) of (3):

$$(x_1, y_1) = (23,4); \quad (x_{n+1}, y_{n+1}) = (23x_n + 132y_n, 23y_n + 4x_n), \tag{7}$$

$$(t_0, m_0) = (6,1); \quad (t_{n+1}, m_{n+1}) = (23t_n + 132m_n, 23m_n + 4t_n). \tag{8}$$

Using (8), we obtain the first six values of $m > 1$:

$$m_1 = 47, \quad m_2 = 2161, \quad m_3 = 99359, \quad m_4 = 4568353,$$
$$m_5 = 210044879, \quad m_6 = 9657496081.$$

We note that the two choices of sign allowed in (2) lead to equivalent decompositions of m^3. For if $k = -\frac{1}{2}(m+1) + t/6$ and $k' = -\frac{1}{2}(m+1) - t/6$, then $k' + 1 = -(k+m)$.

EXAMPLES. For $n = 1$, we have $m_1 = 47$, $k_1 = 21$, $k_1' = -69$, yielding the equivalent decompositions

$$47^3 = 22^2 + \cdots + 68^2 = (-68)^2 + \cdots + (-22)^2.$$

For $n = 2$, we have $m_2 = 2161$, $k_2 = 988$, $k_2' = -3150$, yielding the equivalent decompositions

$$2161^3 = 989^2 + \cdots + 3149^2 = (-3149)^2 + \cdots + (-989)^2.$$

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Also solved by A. Bager (Denmark), J. C. Binz (Switzerland), T. Elliott, L. L. Foster, C. Georghiou (Greece), C. Groenewoud, J. W. Grossman and D. G. Malm, C. Hurd, W. Janous (Austria), R. Knoebbel (West Germany), M. Leitz (West Germany), C. Levesque (Canada), O. P. Lossers (The Netherlands), K. D. McLenithan, J. Metzger, W. A. Newcomb, M. D. Nutt (Canada), C. C. Oursler, V. Schindler (West Germany), A. J. Schwenk, J. S. Sumner, M. Vowe (Switzerland), and the proposer. Partially solved by S. Asadulla (Canada), B. S. Garbow and J. J. More, D. G. Hernandez, H. V. Krishna (India), and J. T. Ward.

Asymptotics and Derivatives

E 3067 [1984, 649]. *Proposed by Colin Bennett, University of South Carolina.*

In this MONTHLY, 88(7) (1981), 526–527, D. J. Newman presented the following result: Let $f(x)$ and $g(x)$ be nonnegative C^1-functions, both of which increase without bound as $x \to +\infty$. Write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to +\infty$. If f is convex and $1/g$ is convex, then

$$f \sim g \Rightarrow f' \sim g'. \tag{1}$$

By considering the equivalent conditions on the reciprocal functions, one might expect to have the following result: If $f(x)$ is concave and $xg(x)$ is convex, then (1) holds.

Show by example that this fails but that the result is true under the additional mild growth condition on g:

$$g(x) \leqslant cg(2x) \quad (x \geqslant 0), \tag{2}$$

for some constant $c < 1$.

Solution by the proposer. The functions

$$f(x) = 4\log(x+1) \quad \text{and} \quad g(x) = 4\log(x+1) + \sin(\log(x+1))$$

provide the required counterexample.

The hypotheses imply that both $f(x)$ and $g(x)$ increase without bound as $x \to \infty$, and that $g(0) = 0$. We may also assume without loss of generality that $f(0) = 0$. From (2) and the convexity of $xg(x)$ we obtain

$$\left(1 - \frac{c}{2}\right)xg(x) \le xg(x) - \frac{x}{2}g\left(\frac{x}{2}\right) \le \frac{x}{2}(xg(x))' = \frac{x^2}{2}g'(x) + \frac{x}{2}g(x),$$

so that

$$(1-c)g(x) \le xg'(x). \tag{3}$$

The concavity of $f(x)$ implies that $1/(xf(x))$ is convex, so we may apply Theorem 1 to the functions $xg(x)$ and $xf(x)$ to deduce that

$$xg'(x) + g(x) \sim xf'(x) + f(x).$$

Hence, if we set

$$p(x) = \frac{f(x)}{g(x)}, \quad q(x) = \frac{xf'(x) + f(x)}{xg'(x) + g(x)}, \tag{4}$$

then

$$p(x) \to 1, \quad q(x) \to 1, \quad \text{as } x \to \infty. \tag{5}$$

Now it follows from (4) that

$$x(f' - g') = (q-1)xg' + (q-p)g,$$

so, since both g and g' are nonnegative, we have

$$x|f' - g'| \le |q-1|xg' + |q-p|g.$$

Using this in conjunction with (3) we find that $|f' - g'| \le \varepsilon g'$, where

$$\varepsilon = \varepsilon(x) = |q-1| + |q-p|/(1-c).$$

But (5) shows that $\varepsilon \to 0$ as $x \to \infty$. Hence $f' \sim g'$ and the proof is complete.

Maximum Intersection of Circle and Triangle

E 3068 [1984, 650]. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $T = \Delta ABC$ be a triangle with inradius r and circumradius R. We consider a circular disc C with radius d, $r \leqslant d \leqslant R$, in a position such that Area $(T \cap C)$ is a maximum. Prove that as d varies continuously in the closed interval $[r, R]$, the

center of the disc C (in the maximum position) moves on a conic τ passing through the incenter, circumcenter and the Lemoine point of ΔABC. Also ΔABC is self-polar with respect to the conic τ.

Composite solution. The statement of the problem requires correction: it is to be assumed that T is an acute-angled triangle.

Under this assumption, let P be the center of a circular disk C of radius d, $r \leqslant d \leqslant R$, positioned so that the area of $T \cap C$ is a maximum. This is also the position minimizing the sum of the areas of the three segments of C outside T. As shown in the solution to E 2842 (by Jordi Dou), [1981, 766] when this sum is minimized the chords of C lying on the sides of T are in proportion to the sides of T. That is, if a,b,c are the lengths of the sides of T then $2ka$, $2kb$, $2kc$ are the lengths of the respective chords, for some k. Let x, y, z be the distances from P to the sides of T. An application of the Pythagorean theorem gives

$$d^2 = x^2 + k^2a^2 = y^2 + k^2b^2 = z^2 + k^2c^2. \tag{1}$$

From this it follows easily that

$$(b^2 - c^2)x^2 + (c^2 - a^2)y^2 + (a^2 - b^2)z^2 = 0. \tag{2}$$

Equation (2) can also be derived by a direct application of the calculus.

It is known that (2) describes a conic τ in trilinear coordinates (x, y, z) for P. The incenter, circumcenter, and Lemoine point of T, with coordinates proportional to $(1,1,1)$, $(\cos A, \cos B, \cos C)$, and (a, b, c) respectively, all satisfy (2) and, therefore, lie on τ. The arc of τ between the incenter and the circumcenter is the required locus of P as d varies between r and R.

The polar of a point (x_0, y_0, z_0) with respect to τ has equation

$$(b^2 - c^2)x_0x + (c^2 - a^2)y_0y + (a^2 - b^2)z_0z = 0,$$

from which it follows that each side of T is the polar of the opposite vertex; hence T is self-polar with respect to τ.

J. Dou and P. Yff noted that τ is in fact a rectangular hyperbola with center on the circumcircle of T and passing through the three excenters of T. Yff pointed out that the Euler line of T is tangent to τ at the circumcenter. Dou provided an analysis of the situation when T is not assumed to be acute-angled.

Solved by J. Dou (Spain), L. Kuipers (Switzerland), P. Yff (Lebanon), and the proposer.

ADVANCED PROBLEMS

6536. *Proposed by Pei Yuan Wu, National Chiao Tung University, Hsinchu, Taiwan.*

Let T be a bounded linear operator on a complex Hilbert space H such that $\dim \ker (T) < \infty$. Define

$$\alpha(T) = \inf_S \{ \|T - S\| : \dim \ker (S) > \dim \ker (T) \}$$

and

$$\gamma(T) = \inf_x \{ \|Tx\| \colon x \perp \ker(T) \text{ and } \|x\| = 1 \}.$$

It is proved on p. 365 of J. B. Conway, *A Course in Functional Analysis* (Springer-Verlag, New York, 1985) that $\gamma(T) \leqslant \alpha(T)$. Show that

$$\gamma(T) = \alpha(T).$$

6537. *Proposed by F. S. Cater, Portland State University, Oregon.*

Let (u_n) be a sequence of positive numbers. For each positive integer n and each $x \in (0,1)$, let $(2^n x) = 2^n x - [2^n x]$, where $[2^n x]$ denotes the greatest integer in $2^n x$. Let

$$A = \left\{ x \colon 0 < x < 1 \quad \text{and} \quad \liminf_{n \to \infty} u_n^{-1}(2^n x) = 0 \right\}.$$

Prove that the measure of A is 0 or 1 according as Σu_n converges or diverges.

6538. *Proposed by Joseph Rotman, University of Illinois, Champaign-Urbana.*

Let $\phi \in \operatorname{Aut}(S_6)$ have the following values on transpositions:

$$\phi(12) = (12)(36)(45)$$
$$\phi(13) = (16)(24)(35)$$
$$\phi(14) = (13)(25)(46)$$
$$\phi(15) = (15)(26)(34).$$

Find $\phi(16)$ explicitly.

SOLUTIONS OF ADVANCED PROBLEMS

Codimensions in Banach Algebras

6494 [1985, 290]. *Proposed by A. Wilansky, Lehigh University.*

Let X be a commutative Banach algebra with 1 and M, N maximal ideals. Let S be the linear span of $\{mn \colon m \in M, n \in N\}$. Show that the codimension of S must be finite if $M \neq N$. (This is not necessarily true if $M = N$.)

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, The Netherlands. Since M and N are ideals, $S \subset M \cap N$. Let $M \neq N$. Then $M + N$ is an ideal that properly contains the maximal ideal M. Hence, $X = M + N$. In particular, $1 = m + n$ for some $m \in M$ and $n \in N$, so

$$p = p1 = pm + pn \in S$$

whenever $p \in M \cap N$. Hence, $S = M \cap N$, a subspace of codimension 2 if the

groundfield is $\mathbb{C}$ (and of codimension at most 4 if it is $\mathbb{R}$). To establish the parenthetical comment, let $\beta = (\beta_i)_{i=1}^{\infty}$ denote a bounded sequence. Let B be the Banach algebra of all linear operators

$$T_{\alpha,\beta}\colon l^2 \to l^2,$$

where

$$T_{\alpha,\beta}(x_1, x_2, x_3, \dots) = \alpha(x_1, x_2, x_3, \dots) + (\beta_1 x_2, 0, \beta_2 x_4, 0, \dots).$$

Then B is commutative with

$$1 = T_{1,0}$$

and

$$M = \{T_{\alpha,\beta} \in B : \alpha = 0\}$$

is the unique maximal ideal of B. Clearly, $S = \{0\}$ has infinite codimension.

Ulrich Everling (Federal Republic of Germany) discussed the possible values of the codimension in detail, and supplied a proof of the

THEOREM. *If a real Banach algebra A is a field, then its dimension over the reals is one or two.*

(Everling believes this is somewhere in the literature, but was unable to locate a reference).

Also solved by O. P. Lossers (The Netherlands), John C. Tripp, and the proposer.

Comparative Asymptotics of Certain Slowly Diverging Series

6495 [1985, 290]. *Proposed by Robert E. Shafer, Berkeley, California.*

For $x > 1$, define

$$\psi(x) = \lim_{N\to\infty}\left\{\log(x+N) - \sum_{n=0}^{N}\frac{1}{x+n}\right\},$$

$$P(x) = \lim_{N\to\infty}\left\{\log\log(x+N) - \sum_{n=0}^{N}\frac{1}{(x+n)\log(x+n)}\right\}.$$

Prove that $P(x) > \log\psi(x)$ for x sufficiently large.

Solution by William A. Newcomb, Lawrence Livermore National Laboratory, Livermore, California. Let $f(x)$ be a function such that for $x \geqslant c$ it and all its derivatives approach zero monotonically as $x \to \infty$. Then, by the Euler-Maclaurin summation formula,

$$\lim_{N\to\infty}\left(\int_x^{N+x} f(t)\,dt - \sum_{n=0}^{N} f(n+x)\right)$$
$$= -\frac{1}{2}f(x) + \frac{1}{12}f'(x) + R,$$

where

$$|R| \leqslant \frac{1}{720}|f'''(x)|.$$

In particular, for $f(x) = x^{-1}$, we have

$$f^{(n)}(x) = (-1)^n n! x^{-n-1},$$

so

$$\psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^4}\right)$$

as $x \to \infty$. From the Taylor expansion

$$\log(a+t) = \log a + \frac{t}{a} - \frac{t^2}{2a^2} + O\left(\frac{t^3}{a^3}\right) \quad \left(\frac{t}{a} \to 0\right),$$

we deduce

$$\log \psi(x) = \log\log x - \frac{1}{2x\log x} - \frac{1}{12x^2\log x} - \frac{1}{8x^2(\log x)^2} + O\left(\frac{1}{x^3(\log x)^2}\right).$$

For $f(x) = (x\log x)^{-1}$ we have

$$f^{(n)}(x) = \frac{(-1)^n}{x^{n+1}\log x} Q_n\left(\frac{1}{\log x}\right),$$

where Q_n is a polynomial of degree n determined by the recurrence relation

$$Q_{n+1}(t) = (n+1+t)Q_n(t) + t^2 Q_n'(t)$$

and the initial condition $Q_0 = 1$. Clearly, Q_n has only positive coefficients, so the monotonicity condition is satisfied for all n. We have, then, again by use of the Euler-Maclaurin formula,

$$P(x) = \log\log x - \frac{1}{2x\log x} - \frac{1}{12x^2\log x}\left(1 + \frac{1}{\log x}\right) + O\left(\frac{1}{x^4\log x}\right),$$

and therefore

$$P(x) - \log\psi(x) = \frac{1}{24x^2(\log x)^2} + O\left(\frac{1}{x^3(\log x)^2}\right),$$

which is clearly positive for sufficiently large x.

Also solved by I. E. Leonard, O. P. Lossers (The Netherlands), and the proposer.

Spectral Radii as Possibly Unattained Infima

6496 [1985, 362]. *Proposed by Ryszard Szwarc, University of Wrocław, Poland.*

Let T be a bounded operator with spectral radius 1 on a given Hilbert space H, and let $c > 1$. Prove that there is an invertible operator A on H such that $\|ATA^{-1}\| \leqslant c$.

Combined solution. Since the spectral radius of T is

$$\lim_{n\to\infty} \|T^n\|^{1/n} = 1,$$

we may define a positive Hermitian operator A by

$$A^2 = \sum_{n=0}^{\infty} c^{-2n}(T^*)^n T^n = I + c^{-2}T^*T + \cdots .$$

This A is invertible since $I \leqslant A$. Define a norm equivalent to $\|\cdot\|$ by

$$\|x\|_c = \|Ax\|.$$

By the definition of A,

$$\|Tx\|_c^2 = c^2\|x\|_c^2 - c^2\|x\|^2 \leqslant c^2\|x\|_c^2,$$

so for any y in H we have

$$\|ATA^{-1}y\|^2 = \|TA^{-1}y\|_c^2 \leqslant c^2\|A^{-1}y\|_c^2 = \|y\|^2$$

and the result follows.

If we replace T by rT where $r > 0$, the result may be stated in a formally more general way: if r is the spectral radius of T and $c > r$, then

$$\|ATA^{-1}\| \leqslant c$$

for some invertible A. In other words,

$$\inf\{\|ATA^{-1}\| : A \text{ is invertible}\} = \text{spectral radius}(T).$$

In this form it was proved by G. -C. Rota, On models for linear operators, Comm. Pure Appl. Math., 13(1960), 469–472. Related questions and generalizations are considered by B. Sz.-Nagy, Completely continuous operators with uniformly bounded iterates, Publ. Math. Inst. Hung. Acad. Sci., 4 (1959), 89–92; F. Gilfeather, Norm conditions on resolvents . . . , Proc. Amer. Math. Soc., 68(1978), 44–48; and P. Halmos, *A Hilbert Space Problem Book*, Cor. 4 to Problem 153.

The infimum in the last formula is attained for spaces of dimension at most one, but need not be attained in any higher dimension. In fact, the matrix operator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is already a counterexample in dimension 2. Here $(T - I)^2 = 0$, so T has spectral radius 1. Also,

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

so if the 2-vector v satisfies

$$A^{-1}v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then

$$AT^nA^{-1}v = A\begin{pmatrix} n \\ 1 \end{pmatrix},$$

and

$$|(ATA^{-1})v| = \left|A\binom{n}{1}\right| \to \infty.$$

But

$$\|(ATA^{-1})^n\| \leqslant \|ATA^{-1}\|^n,$$

so the infimum of $\|ATA^{-1}\|$ is not attained. (The editor thanks H. Lotz, M. J. Pelling, and especially H. Porta for helpful comments and analysis.)

Solutions were received from K. N. Boyadzhiev (Bulgaria), F. Gilfeather, B. Sz.-Nagy (Hungary), Pei Yuan Wu (Taiwan), and the proposer.

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q-Analogues of a Gamma Function Identity

6497 [1985, 362]. *Proposed by Richard Askey, University of Wisconsin.*

Let $0 < q < 1$, $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$. Show that

$$\int_0^\infty \frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\frac{d_qt}{t} = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} \tag{1}$$

and

$$\int_0^\infty \frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\frac{dt}{t} = \frac{-\log q}{1-q}\frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \tag{2}$$

where

$$(x;q)_\infty := \prod_{n=0}^\infty (1-xq^n),$$

$$\Gamma_q(x) := (q;q)_\infty(1-q)^{1-x}/(q^x;q)_\infty,$$

and

$$\int_0^\infty f(t)\,d_qt := (1-q)\sum_{-\infty}^\infty f(q^n)q^n.$$

These extend the gamma function identity

$$\int_0^\infty \frac{dt}{t(1+t)^b(1+t^{-1})^a} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

to q-gamma functions (for properties see R. Askey, Ramanujan's extensions of the gamma and beta functions, this MONTHLY, 87 (1980) 346–359).

Solution by George Gasper, Northwestern University, Evanston, Illinois. We shall evaluate these beta-type integrals by an elementary method that is also applicable to some more general integrals. For definiteness, let us first consider the integral in (2). Denote this integral by $I(a, b)$, temporarily assume that $a > 0$ and $b > 0$, and set $t = sq^a$ to obtain

$$\begin{aligned} I(a,b) &= \int_0^\infty \frac{(-sq^{a+b};q)_\infty(-q/s;q)_\infty}{(-sq^a;q)_\infty(-q^{1-a}/s;q)_\infty}\frac{ds}{s} \\ &= \int_0^\infty \frac{(-sq^{a+b+1};q)_\infty(-q/s;q)_\infty}{(-sq^a;q)_\infty(-q^{1-a}/s;q)_\infty}(1+sq^{a+b})\frac{ds}{s} \\ &= I(a,b+1) + q^b I(a+1,b). \end{aligned}$$

Similarly, by setting $t = sq^{1-b}$ we obtain

$$I(a,b) = I(a+1,b) + q^a I(a,b+1).$$

By solving for $I(a, b+1)$ and $I(a+1, b)$ in terms of $I(a, b)$, we find that

$$I(a,b+1) = \frac{1-q^b}{1-q^{a+b}}I(a,b), \qquad I(a+1,b) = \frac{1-q^a}{1-q^{a+b}}I(a,b),$$

and hence, by induction,

$$I(a+j,b+k) = \frac{(q^a;q)_j(q^b;q)_k}{(q^{a+b};q)_{j+k}}I(a,b) \tag{3}$$

for $j, k = 0, 1, \ldots,$ where

$$(a;q)_j = (a;q)_\infty/(aq^j;q)_\infty.$$

Next, observe that (by partial fractions)

$$I(1,1) = \int_0^\infty \frac{1}{(1+t)(q+t)}\,dt = -\frac{\log q}{1-q}.$$

Upon letting $j, k \to \infty$ in (3), we find on the one hand that (with $a = b = 1$)

$$I(\infty,\infty) = (1-q)(q;q)_\infty I(1,1),$$

while on the other hand

$$I(\infty,\infty) = (1-q)(q;q)_\infty \frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} I(a,b).$$

Thus, it follows from (3) that (2) holds for positive integer values of a and b. Hence both sides of (2), which are analytic functions of $z = q^a$ and $w = q^b$ for $|z| < 1$

and $|w| < 1$, are equal for infinitely many values of z and of w having 0 as a limit point. This shows that (2) holds when $0 < q < 1$, $\operatorname{Re} a > 0$, and $\operatorname{Re} b > 0$.

To derive (1) by this method one need but consider the special case when a and b are positive integers, and replace the substitutions $t = sq^a$ and $t = sq^{1-b}$ by the shifts $n = m + a$ and $n = m + 1 - b$ in the index of summation of the infinite series defining the integral in (1). One finds that this integral also satisfies the above recurrence relations. Then, observe that

$$\int_0^\infty \frac{1}{(1+t)(q+t)}\,d_qt = \sum_{n=-\infty}^{\infty}\left(\frac{1}{1+q^{1-n}} - \frac{1}{1+q^{-n}}\right) = 1.$$

The above method can also be used to extend (2) to

$$\int_0^\infty t^{c-1}\frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\,dt = \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a+c)\Gamma_q(b-c)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+b)} \tag{4}$$

when $0 < q < 1$, $\operatorname{Re}(a+c) > 0$, $\operatorname{Re}(b-c) > 0$, and a limit is taken when c is an integer. For, upon denoting the integral in (4) by $I(a, b, c)$, this method gives

$$I(a+j, b+k, c) = \frac{(q^{a+c};q)_j(q^{b-c};q)_k}{(q^{a+b};q)_{j+k}}I(a,b,c)$$

for $j, k = 0, 1, \ldots$. This reduces the proof to the evaluation of $I(a, b, c)$ for a particular value of a and of b such as, e.g., the well-known special case

$$I(0,1,c) = \int_0^\infty \frac{t^{c-1}}{1+t}\,dt = \Gamma(c)\Gamma(1-c), \qquad 0 < \operatorname{Re} c < 1.$$

Also see Askey and Roy, More q-beta Integrals, Rocky Mountain Math. J., to appear.

It can also be shown that (1) extends to

$$\int_0^\infty t^{c-1}\frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\,d_qt$$
$$= \frac{(-q^c;q)_\infty(-q^{1-c};q)_\infty}{(-1;q)_\infty(-q;q)_\infty}\frac{\Gamma_q(a+c)\Gamma_q(b-c)}{\Gamma_q(a+b)}, \tag{5}$$

when $0 < q < 1$, $\operatorname{Re}(a+c) > 0$, and $\operatorname{Re}(b-c) > 0$. This formula is equivalent to Ramanujan's summation formula for a ${}_1\psi_1$ series (see Askey's paper mentioned in the statement of the proposed problem).

Also solved by the proposer.

EDITED BY J. H. EWING AND ALLAN EDMONDS

Random Walks and Electric Networks. By Peter G. Doyle and J. Laurie Snell. The Carus Mathematical Monographs, Number 22, Mathematical Association of America, 1984. xiii + 159 pp.

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The greatest revolution in the physical sciences during this century has been the formulation (or reformulation) of the basic laws of nature in probabilistic terms. Mathematics has also followed this path, and the modern mathematician sees differential equations, complex variables, and even number theory and combinatorics (thanks in part to Erdős and Kac for the last two) in a much different light than did his nineteenth century counterpart. An excellent way for the typical mathematically trained person (whose college analysis is often mostly 19th century) to get a feeling for what has happened, is to read Mark Kac's Carus monograph, *Statistical Independence in Probability, Analysis, and Number Theory*. But the real core of classical applicable mathematics is the Laplace equation $\nabla^2\phi = 0$, and its relatives. Here the Kac monograph needs to be supplemented.

Doyle and Snell begin with a symmetric random walk on $\{0, 1, \ldots, N\}$ and let $p(x)$ denote the probability of reaching N, starting at x. If E is an event and F and G are disjoint events, one of which must occur, then

$$P(E) = P(F)P(E \text{ given } F) + P(G)P(E \text{ given } G).$$

From this we quickly deduce

$$p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x+1)$$

so $p(x)$ is a discrete harmonic function, i.e., a solution to the one-dimensional (discrete) Laplace equation. Now they sketch all of one-dimensional potential theory with the probabilistic interpretation fresh in the reader's mind (a bit of martingale terminology is also included). Next, they widen the plot by pointing out that if n and $n-1$ are connected by a one ohm resistor ($n = 1, \ldots, N$) and the points 0 and N are kept at a one volt potential difference (with 0 grounded) then $p(x) = v(x)$, the voltage at x. Thus in one dimension we have three ways of looking at potential theory: it is the study of Laplace's equation, the study of electrical circuits, or the study of random walks. (Among the well-chosen exercises in Chapter 1, the authors could have included the minimization of

$$\sum_{n=0}^{N-1} [f(n+1) - f(n)]^2$$

over all real functions f, such that $f(0) = 0$ and $f(N) = 1$, as a precursor to the Dirichlet principle of exercise 1, p. 64).

The objective is to show that this triple point of view yields important insights into higher dimensional problems. One of the first exciting observations is that the two-dimensional Laplace equation can be "solved" by tossing dice (the "Monte Carlo method"). Is this the way to teach partial differential equations to beginners? Erich Zauderer's recent introductory but comprehensive treatise on partial differential equations takes exactly this point of view. I heartily recommend his book, and its strategy of starting with random walks (a move that earned him criticism in some quarters). Doyle and Snell next present some nice numerical examples of walks in two dimensions, again for finite graphs, and show how the exact values are calculated via matrix inversion. This leads nicely into a miniature theory of Markov chains. The discussion then turns to the electric circuit point of view, and we learn Kirchhoff's laws, the probabilistic interpretation of current and voltage, and the general concept of a flow. "When a unit voltage is applied between a and b, making $v_a = 1$ and $v_b = 0$, the voltage v_x at any point x represents the probability that a walker starting from x will return to a before reaching b."

The triple point of view now has a big payoff. Physicists and engineers generally first try to solve their problems not by means of differential equations, but by conservation of energy. With a moderate high school background in electricity, we know that the energy dissipation in a resistor is I^2R. It's now a small step to a correct formulation of the Dirichlet principle and its dual, the Thomson principle. All of this is done in the context of finite networks, but with faith in limiting processes we can now sense the "big picture" of potential theory. (The authors might have given a reference to some earlier descriptions of discrete potential theory; e.g., H. A. Heibronn, *On Discrete Harmonic Functions*, Proc. Camb. Phil. Soc., 45 (1949), 194–206.)

What is not brought out in this book (and indeed could not reasonably be expected in a book of only 159 small pages) is the extent to which random walks have a life of their own, independent of potential theory. An excellent and easily readable survey of their importance in physics, chemistry, and biology is given by George H. Weiss in *Random Walks and Their Applications*, American Scientist, 71 (1983), 65–70. (For a similar, but considerably more lengthy and technical discussion, see G. H. Weiss and R. J. Rubin, *Random Walks: Theory and Selected Applications*, Adv. Chem. Phys., 52 (1982), 363–505.) Weiss also mentions that a recent bibliography on random walks of over 800 references is far from complete. Another good overview can be obtained by perusing M. N. Barber and B. W. Ninham, *Random and Restricted Walks: Theory and Applications* (Gordon and Breach, New York, 1970). One specialty of particular interest to chemists is the self-avoiding walk. An excellent and mathematically sound introduction is given by Stuart G. Whittington, *Statistical Mechanics of Polymer Solutions and Polymer Absorption*, Advances in Chemical Physics, 51 (1982), 1–48. Apparently some major progress has been made here recently by Nijenhuis. Yet more on random walks can

be found in various issues of the Journal of Statistical Physics (especially vol. 30 (1983)).

The second half of Doyle and Snell is concerned with infinite networks, and especially the famous Pólya recurrence theorem. This asserts that walks on lattices are recurrent (return to their starting point with probability one) or transient (not recurrent) depending upon whether or not the lattice has dimension at most 2. The authors want us to grasp this in the most direct, physical, and intuitive way possible and present a surprising number and variety of proofs. Thus Pólya's theorem now joins the ranks of quadratic reciprocity, the inequality of the arithmetic and geometric means, the prime number theorem, and the fundamental theorem of algebra as a theorem with a significant number of different proofs. Perhaps the most striking are those that rely on embedding certain electrical circuits into the lattice, and upon clever use of certain variational principles developed earlier for electrical circuits (monotonicity theorems for cutting and shorting). Of special value (in connection with one of their proofs) is the careful discussion of, and distinction between, n-dimensional random walk and n independent random walks (Chapter 7).

The scholarship and breadth of view displayed here is impressive, but there is a significant oversight. The authors do not seem to be aware of the electrical circuit approach to the discrete Laplace equation given by Balth. van der Pol and H. Bremmer on pp. 365–372 of their book *Operational Calculus* (Cambridge, 1955). This conceptual use of electrical circuits can also be found in Appendix IV of M. Kac, *Probability and Related Topics in Physical Sciences*, Interscience, London, 1959; the Appendix is by Balth. van der Pol. These would certainly belong in the final list of references of any second edition. I also think F. Spitzer's *Principles of Random Walk* belongs there as well, if only as another source for the McCrea-Whipple calculations (not every library has the 1940 Proc. Roy. Soc. Edinburgh).

As one who became acquainted with random walks through Watson's treatise on Bessel functions, I am struck by the fact that this book almost entirely avoids Fourier analysis, and Bessel or any other nonelementary special functions. This may help the book—it has a simple and friendly appearance with hardly any intricate formulas (except on page 126) to frighten the novice. Though some of us (especially those trained in analytic number theory) enjoy formulas, I'll agree that (1) they are available elsewhere (for example, in the van der Pol references given above) and (2) current guidelines call upon authors to give conceptual rather than calculational proofs.

The extent to which this book is self-contained, and its small number of references (37) also keep it nonintimidating. (On p. 96 the authors should have included some page or at least chapter numbers in their references to the method of shorting and cutting in the treatises of Rayleigh, Maxwell, Jeans, and Pólya and Szegő. In Exercise 1 on p. 143 this is done properly.) It *is* wonderful to have a nontrivial mathematical work on random walks that the mathematical public can read cover to cover.

Doyle and Snell do connect their material with some additional interesting scientific topics—invariance under time reversal (pp. 42–43) and fractional dimension (p. 116). In connection with the latter it is interesting to examine *Polymer Statistics and Universality* by Fereydoon Family, AIP Conf. Proc. Number 109, *Random Walks and Their Applications in the Physical and Biological Sciences*, Amer. Inst. Physics (1984), 33–72; this studies random walks, fractional dimension, and the now fashionable technique of renormalization. The final chapter of Doyle and Snell introduces some simple but perhaps original material on the type of an infinite graph—in particular, they show that a random walk on the Penrose tiling is recurrent.

There seem to be very few misprints (on p. 61 in item (c), i should be j; on p. 95 "d 1" should be "$d - 1$"; on p. 135, 4 lines from bottom, G should be $\bar{G}$; on p. 137 line 3 should read "draw the edges...").

The reader of "Random Walks and Electric Networks" will learn how to see classical mathematics from a probabilistic point of view, and will also see how physics and mathematics can be integrated without any loss of mathematical rigor. This book is a beautiful introduction to random walks. The thought-provoking exercises provided throughout the book should enable it to be used as the basis of a challenging honors course at good universities. I am happy to place it on my shelf next to the Carus monograph of Kac.

Curves and Singularities: A Geometrical Introduction to Singularity Theory. By J. W. Bruce and P. J. Giblin. Cambridge University Press, 1984. xii + 209 pp.

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The singularities of a geometric object are its most distinctive features, and they often provide the key to its structure. Singularities occur in many settings. In optics, they are the points where light rays focus or wave fronts pile up; in differential geometry, they include points of stationary curvature—e.g., the vertices of a plane curve or the rib lines and umbilics of a surface; in dynamics, they are the zeros of a vector field; in vision, they are the points which form the outline of the retinal image. The classical work on singularities, in these fields and others, has been unified in the past two decades following the pioneering work on understanding the local structure of smooth maps $f: R^n \to R^p$ that Whitney carried out in the 1950s. Singularity theory, as this field is now called, is an active research area based on the interplay between commutative algebra and differential topology. The mathematician or scientist wanting a clear and detailed introduction to the subject can get it in a number of expository articles, texts, and monographs. What Bruce and Giblin bring to the existing stock is a conviction that beginning graduate students and

well-prepared undergraduates will find the ideas and methods of singularity theory both accessible and interesting. In *Curves and Singularities* they successfully realize that conviction with fresh, innovative presentations, a clear sense of what is important, and uncompromising attention to mathematical detail.

We can get a glimpse of some of the principal features of singularity theory by considering the simple case of smooth functions $f\colon R \to R$. According to the inverse function theorem, each *non*-critical point of f has a neighborhood U on which f is invertible. If we treat the inverse as a smooth coordinate change, then, in terms of the new coordinate on U, f is just the identity map. From this point of view, all non-critical points of all smooth functions look alike. They are called *regular points*. Besides being monotonously alike, the regular points of a function are the most abundant: they form an open set and, typically, a dense one as well. The remaining points are both rare and more interesting. On this account they are called *singular*.

For a start, singular points are far from alike. For example, no smooth coordinate change $x = h(u)$ that is defined on an interval containing the origin will convert $y = x^k$ into $y = u^m$ if $k \neq m$, or even into $y = -u^k$ if k is even. Thus, $+x^2$, $-x^2$, x^3, $+x^4$, $-x^4$, etc., represent distinct singularities, though they hardly exhaust the possibilities. There are also infinitely many distinct "flat" singularities, so-called because their derivatives of all orders vanish; $\exp(-1/x^2)$ is a familiar example, and $y = 0$ also falls into this category. Thus the first item of business in singularity theory is classification: we must sort singularities into distinct equivalence classes using an equivalence relation that identifies two functions if coordinate changes will make their graphs coincide.

We find, moreover, that the classification has a rich hierarchical structure; that is, some singularities are more singular than others. For example, the degenerate critical point of the function $f(x) = x^3$ can be eliminated by perturbing f to a function (see Fig. 1) which has either no critical points or else a pair of ordinary extrema. By contrast, every small perturbation of $y = x^2$ will have an ordinary minimum near the origin. In other words, x^3 can "decay" into $+x^2$ and $-x^2$, but not the reverse. (We sometimes stumble across the fragility of the x^3-singularity when we ask our calculus students to graph a function which has a horizontal inflection; their sketches are likely to have either two critical points or none.) Thom, using instead the imagery of a budding flower, gave the name *unfolding* to the process of embedding a singularity in a family which exhibits other singularities. The number of parameters needed for a minimal complete unfolding is called the *codimension* of that singularity. The codimension helps us know where a singularity belongs in the hierarchy: x^3 has codimension 1 while x^2 has codimension 0, so x^3 is a higher singularity than x^2; x^k has codimension $k - 2$, and all the flat singularities, like $\exp(-1/x^2)$, have infinite codimension. Matters naturally get much more complicated when two or more variables are involved, but unfolding codimension remains the most important landmark in determining the position of a singularity in the hierarchy.

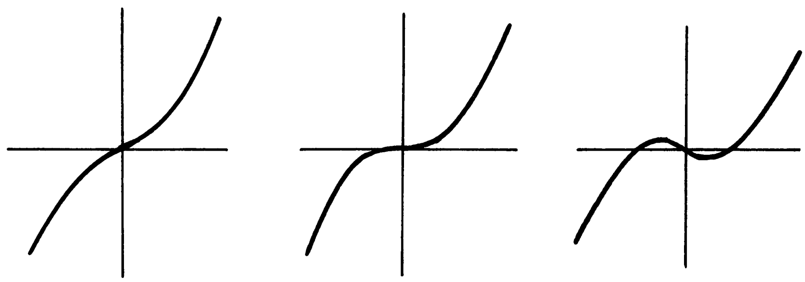


FIG. 1. Perturbations of $f(x) = x^3$.

We can realize singularities as submanifolds of some appropriate space (and this will explain why the "weight" of a singularity is described as a codimension). This realization is built around a map called the k-jet extension, which gives a partial description of the singularity structure of a function. It can be considered a modification of the kth degree Taylor polynomial, and in the case of a function $f\colon R \to R$ it is defined as

$$j^k f\colon R \to R^k$$

$$j^k f(t) = (a_1, a_2, a_3, \ldots, a_k) \quad a_j = f^{(j)}(t)/j!.$$

The k-tuple is called *the k-jet of f at t* (notice that we ignore the constant term, which is geometrically irrelevant). The target R^k is, in this context, called *the space of k-jets* and is denoted J^k. For any $k \geqslant 1$ the k-jet extension tells us when t is a critical point of f: the condition is, obviously, $a_1 = 0$. Let $C^k \subset J^k$ denote this subspace of critical jets. If $k \geqslant 2$, the k-jet extension carries out, in effect, the "second derivative test": t is an ordinary minimum if $a_1 = 0$, $a_2 > 0$; an ordinary maximum if $a_1 = 0$, $a_2 < 0$; of indeterminate type if $a_1 = a_2 = 0$. Fig. 2 shows how the jet spaces grow and reveal greater detail as k increases; the images of the different k-jet extensions of $g(x) = x^3 - 3x$ are also illustrated. The singularity x^3 is represented by the $(k-2)$-dimensional submanifold of C^k given by $a_2 = 0$, $a_3 \neq 0$. However, its codimension—that is, the number of dimensions of its complement—in J^k is two, independently of k, while its codimension in C^k is one, in agreement with the value already defined in terms of unfoldings. As k increases, other submanifolds of higher codimension appear, corresponding to singularities x^m with $m > 3$. In fact C^k is the disjoint union of these submanifolds of different dimension. Such a union is called a *stratified set*; the submanifolds are its *strata*. The hierarchical structures encountered in singularity theory are stratified sets; as Fig. 2 shows, they are a kind of generalization of a simplicial complex.

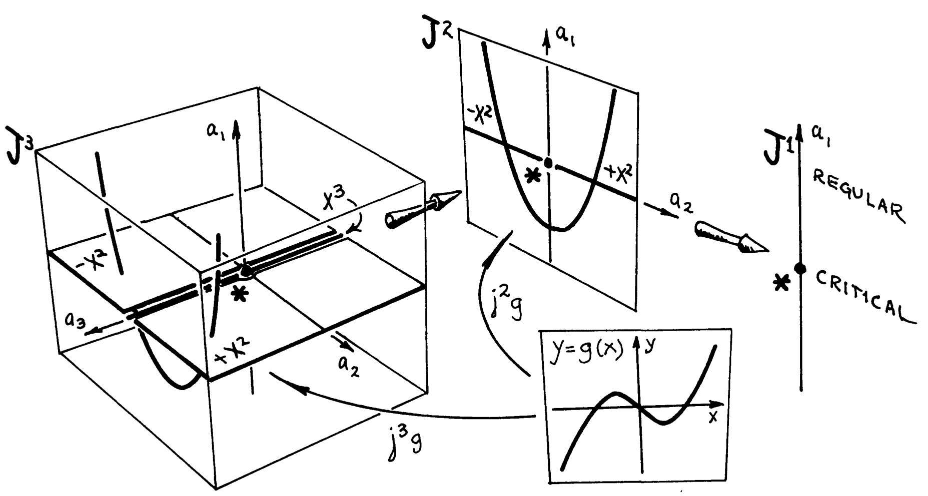


FIG. 2. The jet spaces J^k and the k-jet extensions of $g(x) = x^3 - 3x$. The asterisk denotes a stratum of indeterminate type.

Suppose now that we have, in a given geometric context, a good idea what the singularities are and how they are related. Can we draw any inferences about which ones might actually appear in a specific problem? After all, singularity types are infinitely numerous, yet in practice we seem to encounter only a small number which crop up again and again. Just consider the ubiquitous cusp of coffee-cup caustics and the other physical and behavioral models of catastrophe theory. There is indeed a criterion which permits us to turn our attention away from nearly the entire list of singularities and concentrate on a few; it says that the most important singularities are the ones of lowest codimension. But to see exactly why this is true, and just how low the codimension must be, we need to invoke the notions of stability, transversality, and genericity which are at the heart of singularity theory.

Let us look back at $f(x) = x^3$ and $g(x) = x^3 - 3x$. As Fig. 3 shows, the image of the k-jet of f meets the x^3-stratum of J^k. When f is perturbed to $F(x) = x^3 + ux$, for instance, the image of j^kF misses that stratum if $u \neq 0$, and F, now lacking an x^3-singularity, is qualitatively different from f. Nothing like this can happen to g. In fact, since the image of any perturbation G must stay close to g's image, and since j^kg cuts straight through the x^2-strata, j^kG must cut through these strata as well. Hence all perturbations of g look like g: they have one maximum and one minimum. We say g is *stable* and f is *unstable*.

It is intuitively clear from the figures why g is stable and f is not: the image of any k-jet extension of the sort we have been considering is a one-dimensional curve, so there is always room in the ambient space to push it away from any stratum of

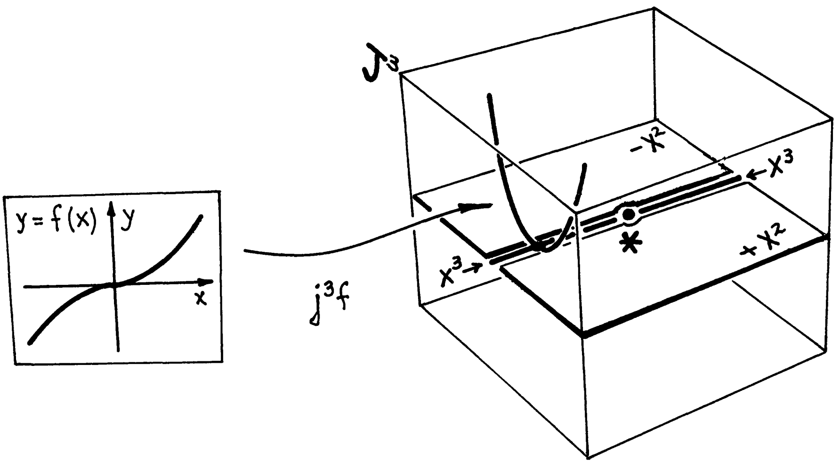
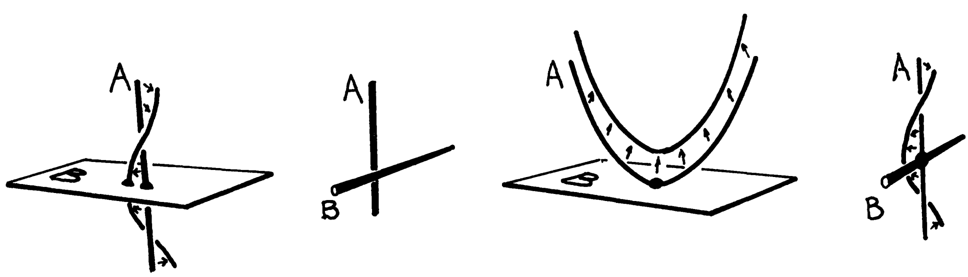


FIG. 3. The 3-jet extension of $f(x) = x^3$.

codimension two (or greater) but not from a stratum of codimension one, *at least when it cuts cleanly* (*i.e., non-tangentially*) *through that stratum*. This is the crucial distinction. Some intersections can be altered and others cannot. See Fig. 4, which shows the essentially different possibilities. When no perturbation of A will change the way it meets (or avoids!) B, A is said to be *transverse* to B. The k-jet extension of a stable function is transverse to all the strata of J^k, and consequently it can actually meet only those strata whose codimension does not exceed the dimension of its domain. These strata are essentially the ones we are after, the ones which correspond to the singularities we encounter repeatedly. We say that a singularity X is *generic* if the functions whose k-jet extensions are transverse to the X-stratum in J^k form an open dense set. (The point is that a generic singularity should be



unalterable intersections
A is transverse to B.

alterable intersections
A is not transverse to B.

FIG. 4

associated with a "big" set of functions; in the absence of a measure, we take "big" to mean "open and dense.") Stability is an open condition but density is problematic. In the context we have been pursuing, though, the stable functions are dense, so there are precisely two generic singularities—namely, the non-degenerate critical points with normal forms $y = +x^2$ and $y = -x^2$. In the context of maps from the plane to the plane, there are exactly two generic singularities, called the fold (normal form $u = x, v = y^2$) and the cusp (normal form $u = x, v = y^3 + xy$). This is the seminal result which Whitney published in 1955.

The foregoing sketch has scarcely hinted at the formidable technicalities which overlay singularity theory and, at times, overwhelm geometric intuition. *Curves and Singularities* traces a path through the subject that meets technical rigor head-on while limiting attention to low-dimensional cases—just curves, where possible. The setting Bruce and Giblin choose is the classical differential geometry of curves, and this is an ideal choice because the material is familiar and intuitive, but still complex enough to display the new ideas vividly. They look at curves from every angle. For example, they reinvigorate a classical backwater like the theory of envelopes by viewing it as an aspect of the theory of singularities of a family of curves. No essentials are ignored: questions of classification, unfolding, transversality, and genericity are covered using curves as models.

By succeeding so well in reaching its intended audience, the book in fact serves as an excellent introduction to singularity theory for any reader. But that reader might do well to remember that Great Britain and the United States remain two countries separated by a common language: *touch* must be translated as *tangent* exclusively—when a British carpenter saws a board in two, the sawblade does not touch the board!

Statistical Inference. By Vijay K. Rohatgi. John Wiley and Sons, New York, 1984. xiv + 940 pp., $44.95.

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Illiteracy. It is rare these days to go for very long without reading of a national problem of illiteracy: mathematical, geographical, historical, cultural, not to mention the kind modified by "functional." Next to that last "ordinary" kind, I believe that statistical illiteracy has the most potential for harm.

Statistical illiteracy comes in two basic varieties; the first is due to lack of exposure to the subject, and the second to the act of exposure to it. The first kind results in vulnerability to spurious arguments put forward by advertisers, govern-

ments and charlatans of various ilks or, obversely, results in cynical rejection of any statistical evidence, valid as well as invalid. The second kind frequently results in the same evils as the first, but offers the potential for more besides: the ability to personally advance fallacious arguments draped in statistical trappings, innocently or otherwise.

The unfortunate and perhaps surprising fact is that both types are often well-represented, even after graduation, among students who enroll in the traditional two-semester, calculus-based sequence in probability and statistics. Skeptics are encouraged to gauge students' ability to handle a random sample of (1) review exercises from Freedman, Pisani, and Purves' *Statistics* [**13**], or (2) examples from Chapter 15 of Campbell's *Flaws and Fallacies in Statistical Thinking* [**11**].

When statistical illiteracy is due to exposure to statistics, it is because the exposure is either inadequate, improper, or both. Examination of the texts and syllabi for many of the proliferation of one-semester "cookbook" courses, often taught in different academic departments for their own students, provides ample evidence of both inadequacy and impropriety. But even in the traditional core sequence, the treatment of statistics is often incomplete and out of date. As stated in the report of the statistics subpanel of the CUPM:

> "The traditional undergraduate course in statistical theory has little contact with statistics as it is practiced and is not a suitable introduction to the subject. Such a course gives little attention to data collection, to analysis of data by simple graphical techniques, and to checking assumptions such as normality. ... It is now inexcusable to present the two-sample t-test for means and the F-test for variances as equally legitimate when a large literature demonstrates that the latter is so sensitive to nonnormality as to be of little practical value..." [**12**, p. 95]

Most scholars would be justly pleased when the tools of their trade are found applicable to diverse disciplines; and statistics certainly has found manifold applications in unexpected areas. But when the result is work that is worthless or worse, the pleasure should cease.

Worse than worthless? A little knowledge may be a dangerous thing, but a little statistical knowledge can be hazardous to the general public and not only to the possessor. Among others, Douglas Altman has written at length [**1–10**] about the ethical and practical consequences of poorly understood statistics in medical journals. About half of the papers sampled in each of several studies contain errors detectable even without examination of raw data [**8**], [**9**]. Among the untoward consequences: subjects are put at risk or inconvenienced for no benefit; other resources are diverted from more worthwhile uses; other patients may later receive inferior treatment as a result; the researcher(s) involved may use the same substandard statistical methods again, and others may copy them [**9**].

The impact of statistical abuse in economics, education, and social sciences may be less immediate, but may easily be as harmful in the long run.

Two can be good. The traditional sequence has probability in the first semester, followed by statistics in the second semester. This is the logical pattern, since probability provides the theoretical underpinnings for most statistical techniques. It is also preferred by the statistics subpanel [**12**, p. 96], but with modifications: "... the statistics course should be revised to incorporate the topics and flavor of... data analysis... ." Among specific topics recommended for inclusion: practical experience with simple random sampling and the variability inherent in repeated samples; the need for experiments and experimental design; basic ideas of control and randomization (stratified sampling, matching, blocking) to reduce variability; histograms, line and bar graphs and their abuses; box plots; outliers in data; bivariate frequency tables and the misleading effects of too much aggregation.

One is not enough. I am a probabilist, not a statistician, and I find it exhilarating to plow through the traditional first-semester's probability, ending with the big bang of the Central Limit Theorem. That is a great place to pause. But try as I might, I can't convince myself that it is a good place to stop. But most students do stop there, and thus see no statistics at all.

Why such attrition? For some, to be sure, the first semester was hard enough, and there is no way they are going to subject themselves to more of the same or worse. But lack of ability or motivation is not the only cause. At my current institution, secondary education majors are required to take the first semester; since this usually occurs in the senior year, they are unable to continue the sequence due to student teaching. Also, students who are not strictly math majors are easy casualties; it is difficult to persuade such students to "stick around—the things you can apply are just around the corner."

Education majors constitute the most troubling lot for me. Hyperbole aside, if progress is ever to be made in combating statistical ignorance among the general population, it will have to begin in the schools. And prospective high school teachers are unlikely to be able to impart an intelligent familiarity with statistics to their students, when they are unlikely to acquire it themselves.

When one is all there is. The CUPM report describes in some detail a proposed unified course, which includes a non-trivial amount of statistics in the first semester (perhaps still not fully "adequate" but much better than nothing, and at least "proper"). The outline comprises about four weeks of working with, organizing, and describing data (including the topics mentioned in "Two can be good," above); about four weeks of probability (including random variables, the standard distributions, law of large numbers and central limit theorem [without proof], but excluding combinatorics, generating functions, continuous joint distributions), and about six weeks of statistical inference (sampling distributions, tests of significance, point and interval estimation, simple linear regression). Questions of checking and justifying assumptions (such as normality) are properly identified as an integral part of the process.

But the CUPM report laments the absence of a suitable text at a post-calculus level, a situation which seems not to have changed since 1981. Finding myself in

agreement with their recommendations, and eager to "do it right," I was dreading the time and trouble that doing it right would entail: a partially suitable text, several additional "recommended, not required" texts (copies on reserve in the library), juggled reading assignments, reams of dittoed handouts.... In the back of my mind was the even more dreadful and probably ridiculous idea of writing my own text. Then Rohatgi's book arrived.

Although CUPM is nowhere mentioned, Rohatgi seems to be speaking to the recommendations. From the preface [p. vii]:

> "My approach to the subject separates this text from the rest in several respects. First, probability is treated here from a modeling viewpoint with strong emphasis on applications. Second, statistics is not relegated to the second half of the course—indeed, statistical thinking is encouraged and emphasized from the beginning."

Further, addressing the ill effects of attrition after the first semester (or quarter) [p. viii]:

> "I believe that the design of this text alleviates these problems and enables the student to acquire an outlook approaching that of a modern mathematical statistician and an ability to apply statistical methods in a variety of situations."

I was elated! Perhaps "cautiously optimistic" is better. The late Jim Williams once began a functional analysis class by describing his vision of the ideal text, which he had intentions of authoring. After a pause, he said "Fortunately or unfortunately, I won't ever write it—Walter Rudin already has." We students didn't really believe that Rudin's book was perfect any more than Jim did, but we did tend to approach it with a positive attitude. It was with such a positive attitude that I approached *Statistical Inference*.

Unfortunately, I am disappointed. The first quote above is accurate; there is a general flavor of concern with inferential thinking throughout. Further, the book is excellent in many respects. But I do not think that it lives up to the promise implied by the second quote. One semester's worth of work in it is not a successful antidote to statistical illiteracy, nor does it adequately fulfill the CUPM recommendations.

What there is and what there isn't. The inferential flavor shows early on. Aside from explicit verbal references, it appears in the examples and exercises, sometimes simply through judicious wording ("A lot of 100 fuses is known to contain 20 defective fuses. If a sample of 15 fuses contains only one defective fuse, is there reason to doubt the randomness of the selection?" [p. 42]) But the main innovation consists of Chapter 4, in which a brief introduction to estimation and hypothesis testing appears (both of which are pursued in detail in Chapters 9–11).

Again without reference to it, Rohatgi seems to have applied with a vengeance another point from the statistics subpanel: "A firm grasp of statistical reasoning is more important than coverage of a few additional specific procedures" [**12**, p. 98]. I

say "with a vengeance," since Chapter 4 includes none of the following: method of moments, maximum likelihood, t-tests, z-tests, χ^2-tests, F-tests. In fact, the first semester's work does not even include the names of the standard distributions: binomial, Poisson, uniform, exponential, normal, geometric, hypergeometric. Many of these distributions do appear, but they are not identified until Chapters 6 and 7, with which the second semester is supposed to begin.

What Chapter 4 does contain is a worthy treatment of the *ideas* behind estimators, confidence intervals, and hypothesis testing, including analysis of error estimates, the importance of P-values, and the proper interpretation of confidence coefficients. The presentation is marred by the seemingly constant reminders that the present examples and exercises (mostly samples of size one or two) are limited in scope, and that "real" problems will be tackled later (in the second semester).

The central limit theorem does not appear until Chapter 9. The material on graphical representation of data is skimpy, and does not address abuses.

There are over 1400 exercises and 400 examples, and they include some unaccustomed delights: the Warner model of randomized response to sensitive survey questions [p. 666]; some former MONTHLY problems [e.g., p. 578 #18]; the Collins case (or how not to use probability in a courtroom [p. 65]); the elections of 1936, 1948, and 1980 [p. 75]; Bertrand's paradox [p. 395]; and even one hypothesis testing exercise based on the attempted rescue of hostages in Iran [p. 243]. The level of difficulty of the exercises ranges from routine to very challenging (a few of each, with most in the moderate range).

There are quite a few misprints. Most are verbal rather than mathematical, and thus more irritating than harmful. Some misprints are serious, however. Examples: the sum of a geometric series with kth term r^k is given as $1/r$ [p. 20]; on p. 114, the probability assigned to a certain planar region is correct, but the derivation is entirely inappropriate; "$|x| < r$" is given as "$|r| < x$" on p. 129. The many errors in the answers to exercises will likely upset students.

The long and the short of it. For students completing an entire year's coursework, *Statistical Inference* includes solid coverage of all of the standard topics, with the one notable exception of regression. The brief early exposure to estimation and hypothesis testing may make the later full treatment easier to digest. Students who quit after one term will have seen some interesting and worthwhile things, but not enough to make them, in my mind, statistically literate.

Reading this book requires quite a lot of the student. The style of writing ranges from chatty in places to rather pedantic in others. Several calculations (integrations, etc.) are carried out in excruciating detail (not always bad), but with little or no step-by-step explanation (almost always bad). The heavy reliance on an occasionally inconsistent symbolism will tend to disorient students, especially in the beginning section on hypothesis testing.

I may continue to use this book; I'm not sure yet. But regardless, I will await the appearance of an introductory, calculus-based probability and statistics text which possesses mathematical honesty, a reasonable level of exposition and rigor, and a

good mix of exercises (most of which *Statistical Inference* has most of the time); but which also incorporates most of the CUPM recommendations. As for the last requirement, Rohatgi cannot be faulted for failing to live up to standards which he made no explicit claim of adopting.

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TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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General, P*, L*. *Dictionary of Computing, Second Edition.* Valerie Illingworth, Edward L. Glaser, I.C. Pyle. Oxford U Pr, 1986, 416 pp, $29.95. [ISBN: 0-19-853913-4] In dictionary format, over 4000 terms used in computing, electronics, mathematics and logic, including "hashing," "hacker," "homomorphisms," "HP-IB," etc. Concise definitions at a mixture of levels, benefiting a wide audience. Authoritative. Ideal reference for persons retraining in computer science, or teachers of computer literacy; valuable asset for novice, working professional alike. RB

Precalculus, T(13: 1). *Essentials of Precalculus Mathematics, Third Edition.* Dennis T. Christy. Harper & Row, 1986, xiii + 602 pp. [ISBN: 0-06-041308-5] For students needing a concrete approach. Modest algebra and geometry background is assumed. Author "believes college students who need precalculus mathematics learn best by 'doing'." Readable. Plenty of examples and exercises. Usual tables. Accommodates both students with calculators and those without. (*First Edition*, TR, March 1977; *Second Edition*, TR, February 1982.) JK

Precalculus, T(13: 1, 2), S. *Precalculus.* John R. Durbin. Wiley, 1986, xii + 689 pp, $28.95. [ISBN: 0-471-88603-3] A thorough introduction to the standard prerequisite topics for college calculus. Brief treatments of complex numbers, systems of linear equations, mathematical induction, and analytic geometry of conic sections are also given. Thousands of exercises, including some for calculators, many phrased verbally. Each chapter ends with summary, review exercises, and "cumulative" (since beginning of book) exercises. PZ

Precalculus, T(13: 2). *Precalculus with Trigonometry: Functions and Applications.* Paul A. Foerster. Addison-Wesley, 1987, viii + 615 pp, $26.95. [ISBN: 0-201-21507-1] A one-semester course in trigonometry is followed by chapters on functions, linear regression and curve fitting; probability, statistics, vectors, matrices, conic sections and quadric surfaces, polar coordinates and parametric equations, complex numbers and infinite series, and limits and derivatives of polynomial functions; optional computer graphics exercises. JNC

Education, T(15-17), S, P, L. *The Child's Understanding of Number.* Rochel Gelman, C.R. Gallistel. Harvard U Pr, 1986, xv + 260 pp, $8.95 (P). [ISBN: 0-674-11637-2] A study of the preschool child's conception of number and how that conception develops. The authors explain how preschoolers have greater cognitive capacities than previously thought, and consider representation and reasoning about number, counting, formal and informal mathematical learning. Suitable for courses in cognitive development, psychology of mathematics, early math education. RB

Education, P, L. *Computers in the Classroom, Second Edition.* Ed: Henry S. Kepner, Jr. National Education Association, 1986, 175 pp, $10.95 (P). [ISBN: 0-8106-1829-X] A book concerning instructional computing at primary and secondary institutions, almost completely revised to reflect fast, fundamental changes since 1982. Current and future

status of instructional computing (teacher's viewpoint); computer literacy, exceptional students; update on computer use in business education, language arts, fine arts, social studies and at all grade levels; computer science curriculum. RB

History, L. *Girolamo Saccheri's Euclides Vindicatus.* Ed: George Bruce Halsted. Chelsea, 1986, xxx + 255 pp, $16.95. [ISBN: 0-8284-0289-2] Second English edition of Saccheri's 1733 book, with original Latin on facing pages. Saccheri was first to state the axiom systems of non-Euclidean geometry, and first to prove basic theorems of hyperbolic geometry, preceding Gauss, Bolyai and Lobachevski by 100 years; his lost work was dramatically rediscovered in 1889. RB

History, S*(16-17), P*, L*. *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System.* Michael J. Crowe. Dover, 1985, xv + 270 pp, $7 (P). [ISBN: 0-486-64955-5] A most welcome republication (TR, May 1968 and January 1976; Extended Review, November 1969). A second preface, prepared for this edition brings to readers "some of the relevant studies of specific areas which have appeared since the book's first publication in 1967." JK

Combinatorics, P. *Lecture Notes in Mathematics-1202: Möbius Functions, Incidence Algebras and Power Series Representations.* Arne Dür. Springer-Verlag, 1986, xi + 134 pp, $12.10 (P). [ISBN: 0-387-16771-4] Written for the specialist in combinatorics with an interest in Möbius inversion, incidence algebras are constructed from a combinatorial category and their structure is studied, paying particular attention to realizations as formal power series. Index, references. JS

Discrete Mathematics, T*(14: 1, 2), S, L. *Discrete Mathematics.* Norman L. Biggs. Clarendon Pr, 1985, xiv + 480 pp, $29.95. [ISBN: 0-19-853252-0] A comprehensive and well-written introduction to the subject. Divided into three sections. Numbers and Counting, which includes counting techniques, designs, permutations and modular arithmetic; Graphs and Algorithms, which includes efficiency of algorithms, sorting, trees, and network flow; and Algebraic Methods, which includes introductory material on group, rings, fields, codes, generating functions and symmetry. Lots of examples and exercises, and selected answers. CEC

Discrete Mathematics. *Discrete Mathematics for Engineers, English Edition, Revised and Enlarged.* O.P. Kuznetsov, G.M. Adel'son-Vel'skii. Transl: V.I. Kisin. Topics in Comp. Math., V. 2. Gordon & Breach, 1985, xiv + 410 pp, $195. [ISBN: 2-88124-201-4] Theoretical treatment of mathematics used in computer science. Some algebra, logic, graph theory; main emphasis on algorithms, Turing machines, decidability, formal systems and metatheory, automata, computational complexity. Nice, somewhat advanced treatment, but note the price. RM

Number Theory, S, P, L*. *Disquisitiones Arithmeticae, English Edition.* Carl Friedrich Gauss. Transl: Arthur A. Clarke. Springer-Verlag, 1986, xx + 472 pp, $58. [ISBN: 0-387-96254-9] A slightly revised reprint of the 1966 translation of this monumental work which was originally published in 1801. William C. Waterhouse has introduced small changes in a number of places where a more precise rendering of the original text seemed desirable. CEC

Number Theory, S(16-18), P, L. *Primality and Cryptography.* Evangelos Kranakis. Ser. in Comp. Sci. Wiley, 1986, xv + 235 pp, $41.95. [ISBN: 0-471-90934-3] A cram course in probability and number theory, aimed at a survey of primality tests, pseudorandom generators, and "public-key" cryptosystems. Factoring algorithms are intentionally neglected. (An efficient—i.e., probabilistic polynomial time—factoring algorithm would put a huge dent in the crypto-business.) BC

Linear Algebra, T*(14: 1). *Elementary Linear Algebra with Applications.* W. Keith Nicholson. Prindle, Weber & Schmidt, 1986, xiii + 557 pp. [ISBN: 0-87150-902-4] A compact and thorough presentation "for students with only a good knowledge of high school algebra;" many proofs are optional; applications at the end of each chapter. Hopefully, the poor binding on the review copy is an exception. JNC

Algebra, P. *Lecture Notes in Computer Science-228: Applied Algebra, Algorithmics and Error-Correcting Codes.* Ed: Alain Poli. Springer-Verlag, 1986, vi + 265 pp, $21.40 (P). [ISBN: 0-387-16767-6] Proceedings of the 2nd International Conference which was held in Toulouse, France in October 1984. CEC

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Algebra, P. *Localization in Noetherian Rings.* A.V. Jategaonkar. London Math. Soc. Lect. Note Ser., V. 98. Cambridge U Pr, 1986, xii + 324 pp, $34.50 (P). [ISBN: 0-521-31713-4] An exposition of the au-

thor's approach to the idea of localization in Noetherian rings—through the ideas of the second layer of undecomposable injectives and the links between prime ideals. SG

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Algebra, T(13: 1), S, P, L. *Algebraic Structures.* C.F. Gardiner. Math. & Its Applic. Halsted Pr, 1986, 280 pp, $69.95. [ISBN: 0-470-20306] Intended as a sequel to author's *Modern Algebra* (TR, November 1982), the text moves briskly through group theory, rings and fields (including Galois theory), linear groups and representations, to computational group theory and Todd-Coxeter algorithm for finite presentations of groups. Exercises, some solutions, index, references. JS

Algebra. *Basic Abstract Algebra.* P.B. Bhattacharya, S.K. Jain, S.R. Nagpaul. Cambridge U Pr, 1986, xvii + 454 pp, $69.50; $27.50 (P). [ISBN: 0-521-30990-5] A senior-level or beginning-graduate-level introduction to algebra; covers groups including normal series and some structure theorems; rings including unique factorization and rings of fractions; Galois theory; modules over PIDs. Relatively short on problems. The book falls somewhere between Herstein and Lang. SG

Algebra, P. *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory.* Spencer J. Bloch, et al. Contemp. Math., V. 55. AMS, 1986. *Part I*, xv + 406 pp; *Part II*, xvii + 411 pp, $62 set (P). [ISBN: 0-8218-5054-7] A collection of 40 papers covering a wide range of application of K-theory to arithmetic and geometry. SG

Calculus, T*(14: 2), S, L. *Multivariable Calculus, Linear Algebra, and Differential Equations, Second Edition.* Stanley I. Grossman. Academic Pr, 1986, xiv + 977 pp, $34. [ISBN: 0-12-304380-8] The book consists of five parts. The first is standard multivariable calculus; next an introduction to linear algebra; the next section combines these sections and includes Taylor's theorem in n-variables; fourth an introduction to ordinary differential equations; and finally a section on power series solutions to differential equations. The author seems to have succeeded in including anything which appears in second-year calculus. Many examples, exercises, and illustrations. CEC

Calculus, T(13: 2). *Calculus and Analytic Geometry for Engineering Technology.* Bernard J. Rice, Jerry D. Strange. Breton (Distr: Prindle, Weber & Schmidt), 1986, xii + 643 pp. [ISBN: 0-534-06378-0] Standard introduction to calculus intended for non-mathematical science majors. Exercises contain extensive applications to engineering, physics. Interesting use of BASIC programs to illustrate ideas such as limits, derivatives, integrals. Emphasis placed strongly on methods rather than understanding. MR

Calculus, T(13: 2). *Applied Mathematical Analysis.* Stanley I. Grossman. Wadsworth, 1986, x + 773 pp, $25.50. [ISBN: 0-534-05766-7] This is essentially a new edition of *Applied Calculus* with additional chapters on calculus of trigonometric functions, differential equations, sequences and series, and matrices and systems of linear equations. JNC

Calculus, T(13: 2). *Calculus with Analytic Geometry, Fifth Edition.* Edwin J. Purcell, Dale Varberg. Prentice-Hall, 1987, xiii + 904 pp. [ISBN: 0-13-111105-1] Changes in this edition include 600 new problems (mostly applied and challenge), renaming chapter review problem sections as chapter review, and moving mathematical induction into the appendix—changes not worth making a good edition obsolete. (*Third Edition*, TR, October 1978; *Fourth Edition*, TR, January 1985.) JNC

Calculus, T(13: 1, 2), S, L. *Calculus and the Computer.* Sheldon P. Gordon. Prindle, Weber & Schmidt, 1986, xiv + 237 pp, (P). [ISBN: 0-87150-950-4] Applications of simple BASIC programming (appropriate for microcomputing) to topics in elementary calculus. A typical short section includes brief exposition of topic (e.g., Monte Carlo methods), a BASIC program, and exercises. Not linked to a particular calculus text. Symbolic computing is not treated. PZ

Calculus, T(13: 2). *Technical Calculus with Analytic Geometry, Third Edition.* Allyn J. Washington. Benjamin/Cummings, 1986, xv + 620 pp, $31.95. [ISBN: 0-8053-9512-1] Primarily for students pursuing technical programs. New edition features computer programs in BASIC keyed to text material, several text-incorporated student aids, more figures, examples, and exercises (*Second Edition*, TR, November 1981). Unsophisticated light treatment of analytic geometry through differential equations and Fourier series. JK

Calculus, T*(13-14: 1-3). *The Calculus with Analytic Geometry, Fifth Edition.* Louis Leithold. Harper & Row, 1986, xv + 1369 pp. [ISBN: 0-06-043926-2] Many alterations have been made in the structure of this text. Fifteen sections have been relegated to chapter ends to serve as supplementary material (examples are Newton's method and inverse hyperbolic functions). Still traditional. Still readable. Still a solid calculus text. Still large. Over 7000 exercises, revised and graded. (*First Edition*,

TR, June-July 1969; *Second Edition*, TR, June-July 1972 and January 1973; *Third Edition*, TR, June-July 1976; *Fourth Edition*, TR, August-September 1981.) JK

Real Analysis, T(15-16), S, P, L. *The Theory of Fourier Series and Integrals.* P.L. Walker. Wiley, 1986, viii + 192 pp, $29.95. [ISBN: 0-471-90112-1] A development of the topic, intended both for students seeking applications of elementary real analysis, and for scientists and engineers skeptical about the relevance and difficulty of theory. Prerequisites: continuity, uniform convergence of sequences of functions, Riemann integration. Applications of theory, particularly to heat conduction. 69 exercises; appendix reviewing prerequisites. RB

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Differential Equations, T(14-15: 1, 2). *Introduction to Differential Equations with Applications.* Fred Brauer, John A. Nohel. Harper & Row, 1986, xii + 628 pp. [ISBN: 0-06-040942-8] Suitable for students in engineering, physical, mathematical, biological and social sciences. Flexible as regards degree of emphasis and course length. Readable. Plenty of worked-out examples and exercises, routine and otherwise. Honest treatment of the usual applications. Nice chapter on qualitative study of differential equations. JK

Differential Equations, T*(14-15: 1, 2), S*. *Ordinary Differential Equations: An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences.* Morris Tenenbaum, Harry Pollard. Dover, 1985, xvii + 808 pp, $14.95 (P). [ISBN: 0-486-64940-7] Unabridged republication of the work originally published in 1963. Well-written. Very well organized. Exceptionally detailed and thorough. An abundance of exercises and interesting problems. Excellent discussions of applied problems. Answers to almost every exercise and problem. None of the usual topics in ordinary differential equations have been overlooked. Great for review or refreshing. JK

Differential Equations, T(14: 1). *A First Course in Differential Equations with Applications, Third Edition.* Dennis G. Zill. Prindle, Weber & Schmidt, 1986, xiii + 562 pp [ISBN: 0-87150-928-8] ; *Differential Equations with Boundary-Value Problems.* xiii + 640 pp. [ISBN: 0-87150-933-4] Now contains material on Bessel functions and Legendre polynomials, Gauss-Jordan elimination, the eigenvalue problem for matrices, complex numbers, determinants and Cramer's rule. New problems, examples, figures, applications. Emphasis on problem-solving and applications. The bigger book replaces the smaller one's chapter on partial differential equations with three, on orthogonal functions and Fourier series; boundary-value problems; the integral transform method. (*First Edition*, TR, August-September 1980; *Second Edition*, TR, March 1983.) DFA

Differential Equations, S(15-17), P, L. *The Predator-Prey Model: Do We Live in a Volterra World?* Manfred Peschel, Werner Mende. Springer-Verlag, 1986, xi + 251 pp, $22. [ISBN: 0-387-81848-0] An attempt to unify many models of growth under the hyperlogistic model $x' = Kx^k(B - x^w)^n$ and its extension to multivariable settings. Includes numerous examples from ecology, economics, and chemistry, as well as simulation programs in BASIC. Assumes only elementary mathematics, mostly linear algebra and calculus. LAS

Partial Differential Equations, P. *Mixed Elliptic-hyperbolic Partial Differential Operators: A Case-study in Fourier Integral Operators.* R.J.P. Groothuizen. CWI Tract, V. 16. Math Centrum, 1985, iii + 147 pp, Dfl. 21,40 (P). [ISBN: 90-6196-287-0] Considers partial differential equations involving the Tricomi operator $\partial_t^2 + t\Delta_x$ and the operator $\partial_t^2 + t\Delta_x + \alpha(\partial_t)$, called the pseudo Tricomi operator. These operators are significant because they provide examples of differential equations of mixed elliptic/hyperbolic type. Uses Fourier integral operators to investigate effects of the transition of the operators from elliptic to hyperbolic. AM

Partial Differential Equations, P. *Finite Element Methods and Navier-Stokes Equations.* C. Cuvelier, A. Segal, A.A. van Steenhoven. Math. & Its Applic. D Reidel, 1986, xvi + 483 pp, $64. [ISBN: 90-277-2148-3] Part I provides an introduction to finite element techniques for elliptic partial differential equations. Part II applies these techniques to the study of Navier-Stokes equations for incompressible fluid flows. Part III contains error analyses and convergence proofs. Part IV presents an overview of current research. AO

Partial Differential Equations, T(17-18: 1), P. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms.* Vivette Girault, Pierre-Arnaud Raviart. Ser. in Comput. Math., V. 5. Springer-Verlag, 1986, x + 374 pp, $90. [ISBN: 0-387-15796-4] Covers recent developments for stationary problems with incompressible fluids. External problems are not considered. Focuses on the theory underlying the methods presented. AO

Numerical Analysis, P. *Approximation Theory.* Ed: Carl de Boor. Proc. of Symp. in Appl. Math.,

V. 36. AMS, 1986, xi + 131 pp, \$26. [ISBN: 0-8218-0098-1] Six lectures from AMS short course at New Orleans meeting, January 1986. First paper describes basic problems of approximation theory; remaining are introductions to some areas of current research. LC

Functional Analysis, P. *Nonlinear Functional Analysis and Its Applications.* Ed: S.P. Singh. NATO ASI Ser. C, V. 173. D Reidel, 1986, xi + 418 pp, \$69.95. [ISBN: 90-277-2211-0] Proceedings of a NATO Institute on functional analysis during April-May, 1985. Topics include degree theory, differential equations, dynamical systems, and boundary value problems. JS

Functional Analysis, P. *Fractional Calculus.* Ed: A.C. McBride, G.F. Roach. Res. Notes in Math., V. 138. Pitman, 1985, 214 pp, \$38.95 (P). [ISBN: 0-273-08753-3] 16 papers from a workshop on fractional calculus held at Ross Priory, University of Strathclyde in August 1984. Includes open problems. LC

Functional Analysis, P. *Lecture Notes in Mathematics-1184: One-parameter Semigroups of Positive Operators.* W. Arendt, et al. Springer-Verlag, 1986, x + 460 pp, \$36.30 (P). [ISBN: 0-387-16454-5] Other authors are A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, U. Schlotterbeck. Results on characterization, spectral theory, and asymptotic behavior in four different types of underlying spaces: Banach spaces; $C_0(X)$, where X is locally compact; Banach lattices; non-commutative operator algebras. DFA

Analysis, P. *Constants in Some Inequalities of Analysis.* Solomon G. Mikhlin. Transl: Reinhard Lehmann. Wiley, 1986, 108 pp, \$27. [ISBN: 0-471-90559-3] Mainly covers author's results on constants in inequalities that arise in Sobolev space theory, numerical analysis, etc. LC

Analysis, P. *Fourier Techniques and Applications.* Ed: John F. Price. Plenum Pr, 1985, viii + 231 pp, \$35. [ISBN: 0-306-42100-3] 13 papers from a week of applied Fourier analysis held at the University of New South Wales, Australia, August 29-September 2, 1983. LC

Algebraic Geometry, P. *Cohomology of Sheaves.* Birger Iversen. Universitext. Springer-Verlag, 1986, xi + 464 pp, \$36 (P). [ISBN: 0-387-16389-1] Comprehensive account of subject with applications in analysis (if the sheave consists of continuous, smooth, or analytic functions) and algebraic topology. Main focus is on topological spaces with compact support with duality theorems (Poincare, Verdier, Alexander) playing prominent role. Terse and formal but with applications throughout. (1984 Aarhus U paperback edition, TR, August-September 1986.) MR

Algebraic Geometry, P. *Arithmetic Geometry.* Ed: Gary Cornell, Joseph H. Silverman. Springer-Verlag, 1986, xv + 353 pp, \$34. [ISBN: 0-387-96311-1] A valuable publication whose goal is to provide an introduction to the mathematics behind Gerd Falting's celebrated proof of Mordell's conjecture. Expository papers by a variety of mathematicians cover such topics as group schemes, Abelian and Jacobian varieties, heights, minimal models, and intersection theory. Included are some historical comments by Faltings and a translation of Falting's original paper. SG

Differential Geometry, T(18), P. *Lecture Notes in Mathematics-1195: Minimal Surfaces in R^3.* J. Lucas, M. Barbosa, A. Gervasio Colares. Springer-Verlag, 1986, x + 124 pp, \$11.60 (P). [ISBN: 0-387-16491-X] Examples of complete minimal surfaces, classical and recent, presented with minimal theory; notes for a Brazilian short course, August 1984. Systematic, self-contained development of necessary theory for construction of examples, with some technical proofs omitted. Intended for a graduate level introduction, and as a catalog of surfaces. RB

Differential Geometry, P. *Low Dimensional Topology.* Ed: Roger Fenn. London Math. Soc. Lect. Note Ser., V. 95. Cambridge U Pr, 1985, 258 pp, \$27.95 (P). [ISBN: 0-521-26982-2] Proceedings of the third topology seminar of the University of Sussex, England held August 2-6, 1982. JAS

Geometry, T(15-16: 2). *Combinatorics of Finite Geometries.* Lynn Margaret Batten. Cambridge U Pr, 1986, x + 173 pp, \$39.50; 16.95 (P). [ISBN: 0-521-26764-1; 0-521-31857-2] Introductory text with chapters on linear, projective, affine, polar spaces, generalized quadrangles and partial geometries. Extensive exercises and references. LC

Geometry, T*(15: 1), S*, L. *Euclidean and Non-Euclidean Geometry: An Analytical Approach.* Patrick J. Ryan. Cambridge U Pr, 1986, xvii + 215 pp, \$42.50; \$14.95 (P). [ISBN: 0-521-25654-2; 0-521-27635-7] Relies on the group concept and analytic techniques of linear algebra to construct and study models of plane geometry and non-Euclidean geometry; a much needed study of the results of classical geometry providing excellent preparation for study in group theory, differential geometry, topology and mathematical physics. Prerequiste: linear algebra. JNC

Algebraic Topology, P*. *User's Guide to Spectral Sequences.* John McCleary. Publish or Perish, 1985, xiii + 423 pp, \$40. [ISBN: 0-914098-21-7] A very nice presentation aimed at anyone with a basic course in algebraic topology. Contains the necessary "what is a spectral sequence," topological background, development of important spectral sequences, and guid-

ance for using this powerful computational tool. This user's manual includes enough information to trouble shoot when a spectral sequence fails to work. The book is set beautifully by TEXbut the organization of the bibliography is awkward: to find the reference GWW the reader needs to recall George W. Whitehead's initials and look under W not G. JAS

Differential Topology, P. *A Survey of Minimal Surfaces.* Robert Osserman. Dover, 1986, 207 pp, $8 (P). [ISBN: 0-486-64998-9] The well-known 1969 edition (TR, November 1970), which surveys major developments during the 20 prior years in two-dimensional minimal surfaces in a Euclidean space of arbitrary dimension, is supplemented by a 24-page appendix listing progress in the topic from 1970 through 1985, and a sizable list of additional recent references. RB

Topology, T(15-16), S. *Einführung in die Topologie.* Horst Herrlich. Berliner Studienreihe zur Mathematik 1. Heldermann Verlag, 1986, viii + 214 pp, 36 DM (P). [ISBN: 3-88538-101-X] An introduction to the topology of metric spaces intended primarily for independent study, with numerous examples, exercises. Summaries precede each chapter. Foundations; maps, continuity; complete metric spaces; totally bounded metrics, compactness; connectedness; function spaces; topological spaces and uniform spaces; historical remarks. Collaborative work with H. Bargenda, C. Trompelt. RB

Topology, P. *Categorical Topology.* Ed: H.L. Bentley, et al. Sigma Ser. in Pure Math., V. 5. Heldermann Verlag, l984, xv + 635 pp, $88 (P). [ISBN: 3-88538-005-6] Proceedings of the international conference held at the University of Toledo, Toledo, Ohio on August 1-5, 1983. JAS

Dynamical Systems, P. *Chaos.* Ed: Arun V. Holden. Princeton U Pr, 1986, vii + 324 pp, $50; $19.95 (P). [ISBN: 0-691-08423-8; 0-691-08424-6] Fifteen technical papers by world-wide authors concerning the mathematical concept of chaos and its applications to fields such as biology. Applications overview, graphical representations precede expository papers on 1-, 2-dimensional iterated maps; chaos in feedback systems, lasers, ecology, cellular metabolism, cardiac rhythms, giant squid axons; measurement of chaos. RB

Dynamical Systems, P. *Dimensions and Entropies in Chaotic Systems: Quantification of Complex Behavior.* Ed: G. Mayer-Kress. Ser. in Synergetics, V. 32. Springer-Verlag, 1986, ix + 257 pp, $41. [ISBN: 0-387-16254-2] Twenty-nine papers from a workshop at the Center for Nonlinear Studies, Los Alamos, focused on theoretical, algorithmic, and experimental approaches to computing fractal dimensions, entropy, and Lyapunov exponents for chaotic systems. Includes three papers on chaos in the brain. BC

Dynamical Systems, S(16-18), P, L. *Nonlinear Dynamics and Chaos: Geometrical Methods for Engineers and Scientists.* J.M.T. Thompson, H.B. Stewart. Wiley, 1986, xvi + 376 pp, $42.95. [ISBN: 0-471-90960-2] A richly illustrated introduction to geometric methods in nonlinear dynamical systems, written for engineers, scientists, analysts, experimentalists in all disciplines concerned with time evolution of real systems. Prerequisites: elementary differential equations. Basic concepts of nonlinear dynamics; iterated maps; topological structures of geometric dynamics; further applications. Emphasis: illustrate theory with examples. RB

Statistics, S(16-18), P, L. *Lecture Notes in Statistics-29: Statistics in Ornithology.* Ed: B.J.T. Morgan, P.M. North. Springer-Verlag, 1985, xxv + 418 pp, $29 (P). [ISBN: 0-387-96189-5] A collection of twenty-six papers dealing with statistics in ornithology, including an introductory one by the editors which serves as a setting for those which follow. KK

Computer Literacy. *Personal Computers & Data Communications.* Dimitris N. Chorafas. Computer Science Pr, 1985, xi + 340 pp, $19.95 (P). [ISBN: 0-88175-052-2] A business-oriented survey of personal computers and simple machines that the author calls workstations. Covers a lot of ideas readably, thereby providing a wide background for the topic. Beware: some of the material contains uninterpreted claims from manufacturers (the Intel 8086 is claimed to have 14 general registers), and comparisons between common systems (MS-DOS) and obscure systems (MOD-400) which are not very informative. Also fails to cover many important ideas at a more advanced level of microcomputer. JAS

Computer Literacy, L? *Real World UNIX: Managing a Business with the UNIX Operating System.* John D. Halamka. Sybex, 1984, xxi + 209 pp, $9.95 (P). [ISBN: 0-89588-093-8] A smooth-reading introduction to UNIX with lots of extra material about tools for the business environment. Almost strictly microcomputer oriented. At least some bugs: the index shows vi mentioned on page 55 where no mention occurs within several pages, and where some of what is said is contradicted by the existence of vi. JAS

Computer Programming, P. *65816/65802 Assembly Language Programming.* Michael Fischer. Osborne McGraw-Hill, 1986, xii + 691 pp, $19.95 (P). [ISBN: 0-07-881235-6] Reference (no exercises) for the recently introduced 65816 and 65802 microprocessors, produced by Western Design Center and available as coprocessors/upgrades for Apple II series computers. Brief introductions to computer organi-

zation and assemblers; architecture and instructions for the processors; programming introduced through example programs and routines; complete specifications of chips. RB

Computer Programming, T?(13-14). *Advanced Programming: Design and Structure Using Pascal.* Lawrence H. Miller. Addison-Wesley, 1986, xv + 574 pp, $27.95. [ISBN: 0-201-05531-7] Designed for ACM '85 CS2. Brief review of Pascal, followed by program design and development, usual data structures (linked lists, applications to searching, sorting). Shallow coverage of recursion (but nice discussion of divide-and-conquer paradigm), complexity, specification and verification. Case study of function plotter carried throughout. RM

Computer Programming, S. *Apple Pascal: A Self-Study Guide for the Apple II Plus, IIe, and IIc.* Lowell A. Carmony, et al. Computer Science Pr, 1985, vii + 233 pp, $18.95 (P). [ISBN: 0-88175-076-X] Describes Apple operating system, editor, as well as Pascal. Suitable for self study by high school students. RM

Computer Programming, T(13: 1, 2). *Structured Programming with True Basic.* Larry Joel Goldstein, C. Edward Moore, Peter J. Welcher. Prentice-Hall, 1986, xi + 531 pp, $26.95 (P). [ISBN: 0-13-855008-5-01] Complete introduction to True BASIC; assumes no prior computer experience. Suitable for a high-school course, too. Many sample programs in complete detail, numerous exercises at various levels. Stresses modular programming. Chapters on graphics and sound, random numbers and simulation, games. Very readable. DFA

Computer Programming, T(13: 1), S. *Apple Machine/Assembly Language Programming.* John S. Hinkel. Gorsuch Scarisbrick, 1986, x + 221 pp, $17 (P). [ISBN: 0-89787-413-7] Self-contained introduction to assembly language programming on the Apple II family of microcomputers. Assumes familiarity with a high-level language. No reference manuals needed. Only software required is the editor/assembler in the DOS Tool Kit or in the ProDOS Assembler Tools from Apple. Many programming examples and exercises. DFA

Computer Programming, S(13-16), L. *68000 Microprocessor Handbook, Second Edition.* William Cramer, Gerry Kane. Osborne McGraw-Hill, 1986, viii + 142 pp, $14.95 (P). [ISBN: 0-07-881205-4] This *Second Edition* presents the 68008, 68010, 68012, and 68020 in addition to the 68000. Not a programming book, although it does include a description of the instruction sets; it deals mostly with architecture, addressing, and signal descriptions. JAS

Software Systems, T(16-18), P, L. *Relational Database Technology.* Suad Alagić. Texts & Mono. in Comp. Sci. Springer-Verlag, 1986, xi + 259 pp, $33. [ISBN: 0-387-96276-X] A textbook presenting an integrated view of the field of relational databases, including formal properties of the relational model, design of relational database systems, implementation. Introduction to the relational model, normal forms, logical design issues; structural design issues; data integrity and concurrency; distributed technology. Prerequisite: maturity. Examples and exercises. RB

Computer Science, P. *Parallel Array Processing.* P.G. Ducksbury. Ser. in Electrical & Electronic Eng. Halsted Pr, 1986, 123 pp, $37.95. [ISBN: 0-470-20330-7] A report on University of London research: solution of two-dimensional partial differential equations using an ICL Distributed Array Processor computer system. The author used variations on the finite element optimization approach to solve linear, non-linear steady state partial differential equations. Comparisons with conventinal sequential computation. A practical application of a parallel computer system. RB

Computer Science, T(17-18), S, P. *Structural Methods in Pattern Recognition.* Laurent Miclet. Transl: J. Howlett. Springer-Verlag, 1986, xviii + 160 pp, $38. [ISBN: 0-387-91277-0] Pattern recognition aims to simulate responsive behavior through representation and decision. Speech recognition, visual image processing, artificial intelligence now urge new interest in this field. Miclet develops mathematical formalism and consequent algorithms of structural approach to pattern recognition, compares with statistical approach, illustrates with 12 real case studies. Without exercises. RB

Computer Science, P. *On Knowledge Base Management Systems: Integrating Artificial Intelligence and Database Technologies.* Ed: Michael L. Brodie, John Mylopoulos. Topics in Inform. Syst. Springer-Verlag, 1986, xxi + 660 pp, $38. [ISBN: 0-387-96382-0] Forty papers on the integration of concepts and techniques in artificial intelligence and databases in so-called Knowledge-Based Management Systems; proceedings of workshop at Islamorada, Florida, February 1985. Knowledge bases vs. databases; extensions of each; retrieval/interface/reasoning; hardware design. Fourth in publisher's series on integration of artificial intelligence, database, programming language technologies. RB

Computer Science, P. *Lecture Notes in Computer Science-223: Structure in Complexity Theory.* Ed: Alan L. Selman. Springer-Verlag, 1986, vi + 401 pp, $25 (P). [ISBN: 0-387-16486-3] Proceedings of a June 1986 conference in Berkeley, California that followed (and had some overlap with) ACM STOC '86. Technical papers with emphasis on structural

aspects of complexity classes and global aspects of complexity theory. RM

Computer Science, T(15-17: 1), S, P, L. *Algorithms: The Construction, Proof, and Analysis of Programs.* Pierre Berlioux, Philippe Bizard. Transl: Annwyl Williams. Wiley, 1986, ix + 145 pp, $24.95 (P). [ISBN: 0-471-90844-4] Covers techniques for formally proving the correctness of programs and the application of these techniques to program design. Invariants are used to simultaneously construct a program and a proof of its correctness. AO

Computer Science, T, S, P, L*. *Programming Languages: A Grand Tour, Third Edition.* Ed: Ellis Horowitz. Comp. Software Eng. Ser. Computer Science Pr, 1987, ix + 512 pp, $39.95 (P). [ISBN: 0-88175-142-1] Thirty source articles (many classics) on programming language design issues. Revised and enlarged chapters: Algol family; abstract data types; concurrency. New chapter on "old languages with new faces" (e.g., True BASIC). Prolog, more Modula-2 added. Ada, C reference manuals removed (overview articles retained). The result is a broader, updated, less bulky anthology. RB

Computer Science, P. *Lecture Notes in Computer Science-220: RIMS Symposia on Software Science and Engineering II.* Ed: Elichi Goto, Keijiro Araki, Taiichi Yuasa. Springer-Verlag, 1986, xi + 323 pp, $20.50 (P). [ISBN: 0-387-16470-7] Fifteen selected papers by Japanese authors from the 1983 and 1984 symposia, in series sponsored by Research Institute for Mathematical Sciences, Kyoto University. Implementation of functional languages, including Prolog; parallel architectures design, practical experience, including data-flow machines; concurrency and distributed data bases; algebraic language specification; circuit simulation using computer algebra. RB

Computer Science, P. *Advanced Computer Graphics.* Ed: Tosiyasu L. Kunii. Springer-Verlag, 1986, xi + 504 pp, $89. [ISBN: 0-387-70011-0] Proceedings of the fourth international Tokyo conference in computer graphics, incorporating technical papers on research results, computer art technology, business/technical trends. Computational geometry research; rendering of textures; visual interfaces and languages; databases of images; computer animation; graphics hardware/software architecture; CAD/CAM applications in mechanical and VLSI design; trends. RB

Computer Science, T(16-17: 1), P, L. *The Theory of Database Concurrency Control.* Christos Papadimitriou. Princ. of Comp. Sci Ser. Computer Science Pr, 1986, xi + 239 pp, $39.95. [ISBN: 0-88175-027-1] A unified exposition of the primary techniques for controlling concurrency in shared storage systems. Algorithms are explained, analyzed, and compared. The mathematical tools used include graph theory, geometry, logic, and complexity theory. AO

Computer Science, P, L. *The Connection Machine.* W. Daniel Hillis. Ser. in Art. Intell. MIT Pr, 1985, xiii + 190 pp, $22.50. [ISBN: 0-262-08157-1] Describes a fundamentally new computer architecture especially designed for artificial intelligence applications. Covers both the hardware and software aspects of the machine's design. AO

Computer Science, P. *Two Issues in Public-Key Cryptography: RSA Bit Security and a New Knapsack Type System.* Ben-Zion Chor. ACM Dist. Diss. MIT Pr, 1986, 78 pp, $20. [ISBN: 0-262-03121-3] A doctoral dissertation (MIT) published in connection with the ACM annual contest for the best computer science/engineering dissertation. The author is first to demonstrate high security of the well-known RSA public-key cryptography system; he then presents a new "knapsack" type public-key system, perhaps one of the first plausible alternatives to RSA. RB

Computer Science, P. *Canonical Forms in Finitely Presented Algebras.* Philippe Le Chenadec. Res. Notes in Theor. Comp. Sci. Pitman, 1986, 201 pp, $22.95 (P). [ISBN: 0-273-08721-5] Theory of "rewriting systems" (syntactic replacement or simplification rules to give reduction to normal forms) to solve problems in algebras (e.g., monoids, groups, rings) defined by identities. Exploits Knuth-Bendix completion procedure (compilation process to translate presentations into complete rewrite rule sets). RM

Computer Science, T(13-14: 1), S. *Turbo Pascal: A Problem-Solving Approach.* Elliot B. Koffman. Addison-Wesley, 1986, xvii + 610 pp, $25.95 (P). [ISBN: 0-201-11743-6] An established author's revised text for first course in computer science, conforming to ACM CS1 and ETS Advanced Placement guidelines. Stress on problem solving through stepwise algorithm development. Procedures introduced early; expanded case studies; interactive programming; substantial chapter on recursion; "spiral approach" to key topics. Brief appendix on differences with Standard Pascal. RB

Computer Science, T(18: 1), P. *Local Area & Multiple Access Networks.* Ed: Raymond L. Pickholtz. Adv. in Telecomm. Networks Ser. Computer Science Pr, 1986, xi + 291 pp, $50. [ISBN: 0-88175-143-X] Twelve original, self-contained technical papers on Local Area Networks (LAN's) focusing on the widely used IEEE 802 protocol standards. Tutorial of IEEE 802; topics in CSMA/CD ("Ethernet-like") and token ring networks; Metropolitan Area Networks; LAN vs. PABX (or PBX). Snapshot of the

present, glimpse of new developments for the technician or researcher. RB

Computer Science, P. *Lecture Notes in Computer Science-227: VLSI Algorithms and Architectures.* Ed: F. Makedon, et al. Springer-Verlag, 1986, viii + 328 pp, $20.50 (P). [ISBN: 0-387-16766-8] Proceedings of a July 1986 workshop in Loutraki, Greece. Technical papers (systolic arrays, pipelined and parallel architectures, signal processing, polynomial algorithms, etc.). RM

Applications, T?(14-15), S. *Classification Algorithms.* Mike James. Wiley, 1985, xii + 211 pp, $34.95. [ISBN: 0-471-84799-2] Readable introduction to pattern recognition/discrimination analysis; emphasis on practical methods for determining optimal classification rule for assigning an object to a group, based on observations of the object. Statistical methods, normal Bayes rule, error rate estimation, feature selection, non-parametric methods. Includes BASIC programs, but no exercises. RM

Applications (Artificial Intelligence), P, L. *Computers and Communications: A Vision of C&C.* Koji Kobayashi. MIT Pr, 1986, xvi + 190 pp, $16.95. [ISBN: 0-262-11111-X] Kobayashi, an engineer turned 21-year Chief Executive Officer of NEC Corporation (Japan), began proposing "C&C" (Computers and Communication) in 1976. The ultimate goal: real-time machine translation of language, both written and spoken. The author presents his vision and the goals of the famous "fifth generation computer" project, for a general readership. RB

Applications (Artificial Intelligence), P. *The Architecture of Cognition.* John R. Anderson. Cognitive Sci. Ser., V. 5. Harvard U Pr, 1983, xiii + 345 pp, $9.95 (P). [ISBN: 0-674-04426-6] An exposition of ACT*, the author's unified theory of higher-level cognition, encompassing categorical free recall, learning theory, psycholinguistics and cognitive psychology. ACT* is based on productions (cf. many expert systems in artificial intelligence), and was developed with the help of computer simulation. A cognitive science work of interest in applied artificial intelligence. RB

Applications (Biology), P. *Some Mathematical Questions in Biology Muscle Physiology.* Ed: Robert M. Miura. Lect. on Math. in the Life Sci., V. 16. AMS, 1986, x + 234 pp, $35 (P). [ISBN: 0-8218-1166-5] How do muscles contract? The current mechanical model points to cross-bridges between sliding actin and myosin filaments, but the biochemical details are still being worked out. Six symposium papers address the question. (Teachers of mathematical modelling or control theory might get some ideas here.) BC

Applications (Biology), P. *Disordered Systems and Biological Organization.* Ed: E. Bienenstock, F. Fogelman Soulié, G. Weisbuch. NATO ASI Ser. F, V. 20. Springer-Verlag, 1986, xxi + 405 pp, $72.50. [ISBN: 0-387-16094-9] Forty papers from an interdisciplinary workshop emphasizing analogies between stability and diversity in biological systems and equilibrium properties of "spin glasses:" magnetic materials with random interactions between nearest neighbors. Contributions to automata theory, physical disordered systems, formal neural networks, combinatorial optimization, and models of biological organization. BC

Applications (Business), T(13: 1). *Mathematics for Business, Third Edition.* Stanley A. Salzman, Charles D. Miller. Scott Foresman, 1986, xxiii + 742 pp, $25.95. [ISBN: 0-673-18231-2] New edition (*First Edition*, TR, August-September 1978) includes more extensive treatment of payroll, bank reconciliation, insurance, greater detail on business statistics and a complete chapter on metric system. Many problems ranging from simple drill to more complex applications. Independence of chapters allows flexibility in course order. Includes appendix on calculator use. Easy-to-follow layout with goals spelled out and chapter reviews. GF

Applications (Economics), T, S*(15-16), L*. *A Guide to Econometrics, Second Edition.* Peter Kennedy. MIT Pr, 1985, xi + 238 pp, $9.95 (P). [ISBN: 0-262-61043-4] Written to supplement other texts on econometrics, intended to provide an overview not found elsewhere. Central theme: violations of five assumptions of classical linear regression. A leisurely and engaging style makes it well worth the price. BC

Applications (Linguistics), P, L. *Lecture Notes in Computer Science-231: NEWCAT: Parsing Natural Language Using Left-Associative Grammar.* Roland Hausser. Springer-Verlag, 1986, 540 pp, $34.80 (P). [ISBN: 0-387-16781-1] Departing from conventional "constituent structure analysis" of natural language (cf. school room sentence diagramming), Hausser proposes a left-associative approach (sentences built up word-by-word, left to right) in his NEW CATegorical approach, for computational efficiency. Linguistic theory, properties and examples of NEWCAT in German and English, LISP code for German parser DCAT/DLEX. RB

Applications (Medicine), S, P, L. *Computers in Medicine.* Ed: Michael Anbar. Computer Science Pr, 1987, vi + 314 pp, $29.95. [ISBN: 0-88175-080-8] An expository introduction to the impact of computers in medicine, consisting of self-contained chapters intended for a broad audience (practicing physicians, students, staff in training, computer sci-

entists/engineers, general readers). Computer systems organization; computer use/literacy; applications (computer graphics in diagnosis, patient monitoring, microprocessor-controlled prosthetics, decision making, record keeping); philosophical comments, future trends. RB

Applications (Physics), S(17).** *Hamilton's Principle in Continuum Mechanics.* A. Bedford. Res. Notes in Math., V. 139. Pitman, 1985, 106 pp, $29.95 (P). [ISBN: 0-273-08730-4] Detailed source on the use of Hamilton's Principle in a modern framework to derive equations governing mechanical behavior of continuous media. Introduction to variational methods using systems of particles. Also covers classical theories of fluids and solids and extensions to materials with microstructures and mixtures. Direct, self-contained, and well-written monograph of interest to mathematicians and physicists. MR

Applications (Physics), T(15: 2). *Physics of Waves.* William C. Elmore, Mark A. Heald. Dover, 1985, xiii + 477 pp, $9.95 (P). [ISBN: 0-486-64926-1] Undergraduate (junior) text introducing classical treatment of waves. Minimal amount of mathematical background required since much is developed, e.g., techniques in ordinary differential equations and partial differential equations, Fourier series, series solutions. Excellent exercises and generally clear, detailed text. Quantum mechanics is not covered. MR

Reviewers:

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INSTRUCTOR'S MANUAL
STUDENT STUDY GUIDE

THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 3 March 1987

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(ISSN 0002-9890)

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Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W , Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

Award for Distinguished Service to Professor Gail S. Young

KENNETH I. GROSS
National Science Foundation, Washington, DC 20550

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State University of New York at Binghamton, Binghamton, NY 13901

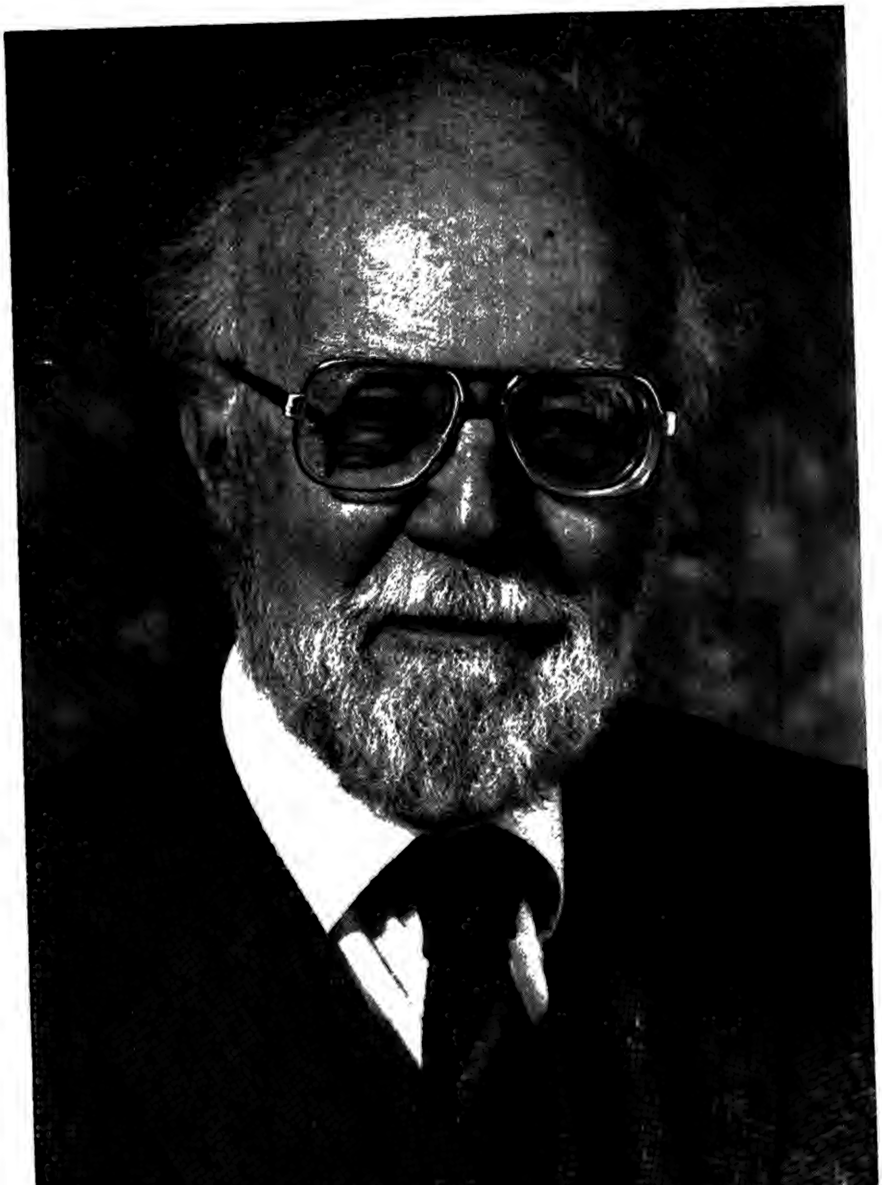
The 1987 Award for Distinguished Service to Mathematics is presented to Gail S. Young. Indeed, nothing seems more natural than to honor Gail.[1] His record of outstanding and devoted service to the mathematical community and to mathematics education is truly exceptional. But what makes Gail unique and special is that throughout his career he has emphasized human values in everything he has done. In enriching the lives of so many of us, he has advanced the entire profession. Deservedly, Gail has innumerable friends in the mathematical, educational, and academic communities, and beyond, all of whom will rejoice in this fitting tribute.

It is tempting to describe Gail Young today as the archetypal "elder statesman" of mathematics; but that description misses the point that he never strikes one as an elder! He is far too vigorous, too enthusiastic, too involved to qualify for any title suggesting withdrawal and dispassionate reflection. Our respect for Gail certainly includes an appreciation of his wisdom; but we have appeciated his wisdom for decades. He has not had to wait until now to achieve it, nor have we waited until now to evidence our recognition of it.

We list here certain significant details of Gail's life and career. But such a recital is still quite inadequate to capture the flavor and the meaning of Gail's role in American—and world—mathematics over the past thirty years. The whole man whom we know and respect is not to be found in the details about to be presented.

Gail Young was born in Chicago on October 3, 1915. He went as an undergraduate to Tulane University in 1935 and transferred to the University of Texas for his senior year. While at Tulane he was profoundly influenced by H. E. Buchanan, who started many mathematicians on their careers, and by Bill Duren, to whom Gail gives credit for introducing him to rigorous mathematics. He remained at Texas to do his Ph.D. under R. L. Moore, obtaining the degree in 1942. Until 1947 he taught at Purdue University. From there he went to the University of Michigan, attaining the rank of full professor. In 1959 he returned to Tulane University, where he stayed until 1970, serving as chairman for the years 1963–68. He was at the University of Rochester from 1970 to 1979, most of the time as chairman, moved to

[1] This Distinguished Service Award follows another recent testimonial. In August, 1985, a symposium on the occasion of Gail's seventieth birthday was held at the University of Wyoming. Much of what follows is excerpted from the preface of the proceedings of that symposium, entitled *The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics* (Springer-Verlag, 1986).



Gail S. Young

Case Western Reserve University in 1979 and to a visiting position at the University of Wyoming in 1981, where he has remained as vigorous, active, and effective as ever.

Gail's services to the mathematical community and to the cause of education, through his work within various professional organizations, are too numerous to list individually. In fact, several years ago, The Association for Women in Mathematics printed in its newsletter a list of the ten people in mathematics most active in committees. Gail placed third. The newsletter hastened to add that its readers should not be misled—this Gail was a man.

Some of his major positions are as follows. He was President of MAA (1969–70), chairman of its CUPM Teacher Training Panel (1964–68), and he went back to that committee as a member after his presidency. He served on many other MAA committees. For the AMS, he was a member of the original Committee on Employment and Educational Policy (1970–71, 1972–74), the Committee on Women in Mathematics (1974–76), the Committee on Human Rights of Mathematicians (1979–81), and others. Gail is a fellow of AAAS, served on its Council (1968–70, 1975) and was chairman of its mathematics component, Section A (1981–84). In that latter capacity, he initiated a continuing effort to establish a close relationship between the mathematical community and AAAS. He served several times as a member-at-large of CBMS and as an MAA representative (1966–71). Gail was instrumental in obtaining a large grant from the Ford Foundation to establish the CBMS Survey, and served as the first chairman of the Survey Committee (1964–72). He was a member of the NAS-NRC Committee on Applications of Mathematics (1964–67), and chairman of the U.S. Commission on Mathematical Instruction (1975–76). He was involved with the "New Math," largely through SMSG, and he has written extensively on problems in both elementary and secondary education. In a very different area was his chairmanship of the "Rochester Plan" (1975–78), an effort to reshape premedical and medical education at that institution, which had a national impact. Many other examples could be cited.

Wisdom, insight, and vision have been hallmarks of Gail's leadership. In 1965 he foresaw the employment decline for mathematicians that took place in the seventies, but he also warned us of the manpower shortage that would follow in the eighties, and which we are now experiencing. Early on, far in advance of current fashion, Gail emphasized the importance for the mathematical community of breaking down the artificial barriers that had been erected by the use of such terms as "pure" and "applied."

In addition to his exemplary record at the national level on which this award is based, Gail has embodied the qualities in research, teaching, and scholarship that we most admire. he has been an active and effective research mathematician. He has written over sixty papers in topology, n-dimensional analysis, complex variables, mathematics education, and "miscellanea." In addition, his fine book, *Topology*, coauthored with J. G. Hocking (Addison-Wesley, 1961), testifies to his skills as an

expositor. Gail's excellence as a teacher at all levels is a corollary to a remarkable intellectual breadth and depth coupled with a rare sensitivity to the needs of his students. He has had nineteen Ph.D. students.

The simple truth is that everything Gail has done—and he has done enormously much—has been well-done, useful, and important.

Our final word should be this: Gail, though now retired, is still an active member of our community and our profession—long may it be so, for we need him for his wisdom and we enjoy him for himself.

What Is Several Complex Variables?

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When I am asked what sort of mathematics I study, my stock response is "several complex variables." But the reaction this usually elicits makes me feel as though I have said "generalized theory of fluxions." Whereas "algebraic geometry" and "partial differential equations" arouse a glimmer of recognition (if not genuine understanding) in most mathematicians, several complex variables usually draws a blank. It is natural for the listener to suppose that we who work in the subject ran out of things to do in one complex variable, so now we are busy juggling multi-indices. The thesis of this article, formulated rather aggressively, is in fact

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*Author supported in part by a grant from the National Science Foundation. He would like to thank Steven R. Bell for several inspiring conversations on the topic of this paper.

quite the opposite: to limit oneself to the study of one complex variable is to do complex analysis with one eye closed.

Let me briefly elaborate on this last point. More than any other subject that I know, several complex variables (SCV for short) is proof that many different parts of mathematics can interact fruitfully. The symbiotic relationships that algebra, differential geometry, partial differential equations, algebraic geometry, and Banach algebras enjoy with SCV have led to major developments in all of these subjects. Does one need to be expert in all of these diverse fields in order to begin to appreciate what SCV is about? Fortunately the answer is no, and I intend to prove this point in the present article.

No article nor any book could introduce the reader to all the aspects of this subject. What I hope to do here is to provide some simple yet striking examples of how SCV differs from nineteenth century complex function theory. Many of these examples were known seventy years ago, but I intend to shed some modern perspectives on them. I will try to weave these examples into a cohesive essay, and to provide some views of current concerns in the subject. Along the way, we will see some new ways to think about one complex variable, which is of course yet one more reason for considering the broader perspective of SCV.

Judging from the number of items in the *Math Reviews*, one finds that SCV has grown nine-fold in the last twenty years or so. A particular high point occurred in the academic year 1980-1981 when there were five different complex analysis seminars every week in Princeton. I was lucky enough to have participated in this burst of activity and would like to share some of the excitement with the mathematical community at large. An expository article in the MONTHLY seems like an ideal place in which to do so.

1. Some Preliminaries. Complex analysis of several variables is done on the space consisting of the n-fold product of $\mathbb{C}$ with itself:

$$\mathbb{C} \times \cdots \times \mathbb{C}.$$

This space is denoted by the symbol $\mathbb{C}^n$. A typical element has the form

$$z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Since $\mathbb{C}$ is simply $\mathbb{R}^2$ with some additional algebraic structure, we realize that $\mathbb{C}^n$ is (topologically) $\mathbb{R}^{2n}$ with some additional algebraic properties. We have a natural way to identify points in $\mathbb{C}^n$ with points in $\mathbb{R}^{2n}$. This is described by the scheme

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \leftrightarrow (x_1 + iy_1, \dots, x_n + iy_n) \leftrightarrow (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}.$$

In particular, we measure distance in $\mathbb{C}^n$ in the customary Euclidean fashion: if $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are points in $\mathbb{C}^n$, then

$$|z - w| = \left(|z_1 - w_1|^2 + \cdots + |z_n - w_n|^2\right)^{1/2}.$$

By the same token, it is sometimes useful to think of $\mathbb{C}^n$ as a vector space over $\mathbb{C}$. If $z = (z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are elements of $\mathbb{C}^n$, then we can define an inner product by

$$\langle z, w\rangle = z_1 \cdot \overline{w}_1 + \cdots + z_n \cdot \overline{w}_n.$$

With this notion of inner product, the familiar notion of orthogonality can be used to aid in the study of function theoretic questions. Indeed, we shall use such geometric insight to guide our thoughts in Section 2.

Rather than use $\partial/\partial x_j$ and $\partial/\partial y_j$ for doing calculus on $\mathbb{C}^n$, it is more convenient to consider

$$\frac{\partial}{\partial z_j} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

and

$$\frac{\partial}{\partial \bar{z}_j} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Check for yourself that

$$\frac{\partial}{\partial z_j} z_k = \delta_{jk},$$

$$\frac{\partial}{\partial \bar{z}_j} \bar{z}_k = \delta_{jk},$$

$$\frac{\partial}{\partial z_j} \bar{z}_k = 0,$$

$$\frac{\partial}{\partial \bar{z}_j} z_k = 0.$$

Here δ_{jk} is the Kronecker delta. The real significance of $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ will become apparent after we define analytic (or holomorphic) functions.

Recall that there are several different ways to think about analyticity/holomorphy in one complex variable: two very important ones are the Cauchy-Riemann equations and local power series expansions. These points of view will be useful in SCV as well, as we shall now see.

What is an analytic function of several complex variables? A simple working definition is that if $\Omega \subseteq \mathbb{C}$ is an open connected set (a *domain*) and if $f\colon \Omega \to \mathbb{C}$, then we call f *analytic* if for each fixed $P = (p_1, \ldots, p_n) \in \Omega$ and each fixed

$j \in \{1, \ldots, n\}$, the single variable function

$$\mathbb{C} \ni z \mapsto f(p_1, \ldots, p_{j-1}, p_j + z, p_{j+1}, \ldots, p_n)$$

is analytic for z small. In other words, f is analytic if it is analytic (in the classical one variable sense) in each variable separately. As in one complex variable, the word "holomorphic" is used interchangeably with "analytic."

It is reassuring to know that no aberrations can occur: an f which is analytic according to our definition must (by a non-trivial theorem of F. Hartogs) be continuously differentiable to arbitrarily high order as a function of the $2n$ real variables $x_1, y_1, \ldots, x_n, y_n$. Like Goursat's version of the Cauchy Integral Theorem, this result of Hartogs is more an aesthetic than a useful one: the holomorphic functions which arise in practice are usually smooth by inspection. But Hartogs's result makes the theory cohesive.

If f is holomorphic on $\Omega \subseteq \mathbb{C}^n$, then f satisfies the *Cauchy-Riemann equations* in each variable separately: if $f = u + iv$ then

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}$$

and

$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j},$$

for $j = 1, \ldots, n$. Our new calculus notation makes the Cauchy-Riemann equations particularly easy to write down. For if $f = u + iv$ satisfies

$$\frac{\partial}{\partial \bar{z}_j} f = 0, \quad j = 1, \ldots, n,$$

then we may calculate as follows.

$$\begin{aligned} 0 = \frac{\partial}{\partial \bar{z}_j} f = \frac{\partial}{\partial \bar{z}_j}(u + iv) &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)(u + iv) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_j}u - \frac{\partial}{\partial y_j}v\right) + i\frac{1}{2}\left(\frac{\partial}{\partial x_j}v + \frac{\partial}{\partial y_j}u\right). \end{aligned}$$

Taking real and imaginary parts we see that f satisfies the Cauchy-Riemann equations in each variable. Since the calculation runs backwards too (try it!), we see that a continuously differentiable function of several complex variables is holomorphic if and only if

$$\frac{\partial}{\partial \bar{z}_j} f = 0, \quad j = 1, \ldots, n.$$

Typical examples of holomorphic functions are polynomials such as

$$p(z_1, z_2, z_3) = z_1 z_2 - (z_3)^2,$$

rational functions such as

$$r(z_1, z_2) = \frac{z_1}{(z_2)^2 + 1},$$

and convergent power series such as

$$q(z) = \sum_{k=0}^{\infty} (z_1 z_2)^k.$$

While the first example is a well-defined holomorphic function for all $z = (z_1, z_2)$, the second is only defined when $z_2 \neq \pm i$ and the third only when $|z_1 z_2| < 1$. To see that the power series truly defines a holomorphic function, one needs to check that a power series may be differentiated termwise on its domain of convergence. Such matters are best left to the reader, or see [**12**].

In general it is quite hard to see what is the (largest) domain of definition of a given holomorphic function. It is also very difficult in several complex variables to construct holomorphic functions with specified properties. Indeed, many of the most powerful tools of one complex variable (Mittag-Leffler and Weierstrass theorems, Blaschke products, inner-outer factorizations, the Riemann mapping theorem, conformal mapping, Rouché's theorem) are unavailable in SCV while others (the Cauchy Integral Formula, residues, harmonic functions) are considerably less useful. These obstructions have become major themes in the subject of SCV and have led to the development of powerful new tools.

2. A Fundamental Discovery of Hartogs. In 1906 Hartogs made a startling discovery which helped to establish SCV as a subject in its own right. The result concerns "natural boundaries" for holomorphic functions. Most students encounter the notion of natural boundary for the first time in the context of the Hadamard gap theorem: certain lacunary power series produce functions which are holomorphic on the open disc, continuous on the closed disc, yet cannot be analytically continued to any larger open set. The circle is said to be the *natural boundary* for such a function. Proceeding informally, we might ask whether similar functions exist on any open set in the complex plane. If they do, then they surely cannot be constructed using power series. Instead, the Mittag-Leffler theorem enables us to prove the following result:

THEOREM 1. *Let $\Omega \subseteq \mathbb{C}$ be an open set bounded by a simple closed curve. Then there exists a holomorphic f on Ω with the following property: if $\hat{\Omega}$ is any open set which strictly contains Ω, then there is no holomorphic F on $\hat{\Omega}$ such that F restricted to Ω equals f.*

Proof. Let $\mathscr{Q} = \{q_j\} \subseteq \Omega$ be a countable set which

(i) has no accumulation points in Ω,

and

(ii) accumulates at every boundary point of $\partial\Omega$.

(It is a good exercise in plane geometry to construct such a set—see Fig. 1.) By Weierstrass' theorem, there is a holomorphic function f on Ω whose zero set is precisely $\{q_j\}$.

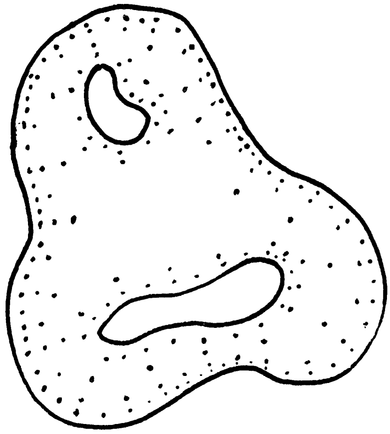

FIG. 1

Now if there were a $P \in \partial\Omega$ and $r > 0$ such that f continued analytically to $\Omega' \equiv \Omega \cup \{z: |z - P| < r\}$, then P, being an accumulation point of $\mathscr{Q}$, would be an interior accumulation point in Ω' of $f^{-1}(\{0\})$. See Fig. 2. Thus $f \equiv 0$, yielding a contradiction. ∎

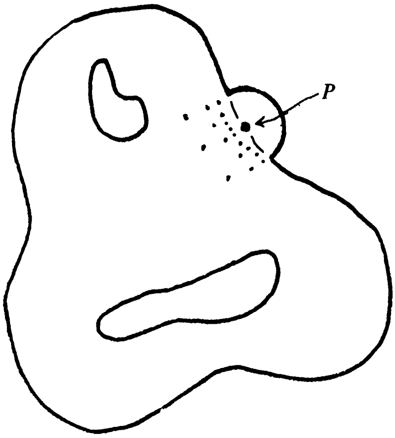

FIG. 2

Does a result like Theorem 1 hold true for holomorphic functions of several complex variables? Hartogs's discovery is that sometimes the answer is yes and sometimes no. In a moment we shall see why. First we introduce some notation:

If $P \in \mathbb{C}$ and $r > 0$, then let

$$D(P, r) \equiv \{z \in \mathbb{C}: |z - P| < r\},$$
$$\overline{D}(P, r) \equiv \{z \in \mathbb{C}: |z - P| \leqslant r\},$$
$$D \equiv D(0,1),$$
$$\overline{D} \equiv \overline{D}(0,1),$$

and if $P = (p_1, \ldots, p_n) \in \mathbb{C}^n$ and $r > 0$, then

$$D^n(P, r) \equiv \{z \in \mathbb{C}^n: |z_j - p_j| < r, j = 1, \ldots, n\},$$
$$\overline{D}^n(P, r) \equiv \{z \in \mathbb{C}^n: |z_j - p_j| \leqslant r, j = 1, \ldots, n\},$$
$$B(P, r) = \{z \in \mathbb{C}^n: |z_1|^2 + \cdots + |z_n|^2 < 1\},$$
$$\overline{B}(P, r) = \{z \in \mathbb{C}^n: |z_1|^2 + \cdots + |z_n|^2 \leqslant 1\}.$$

Now we return to our discussion. Let φ be the nonextendable holomorphic function on D whose existence is guaranteed by Theorem 1. Let $\Omega = D^2(0,1)$. Define a holomorphic function f on Ω by

$$f(z_1, z_2) = \varphi(z_1) \cdot \varphi(z_2).$$

It is immediate that f is a non-continuable holomorphic function of two complex variables on the domain Ω: there is no larger open set containing Ω to which f can be analytically continued.

Notice that the very same argument shows that every product domain in $\mathbb{C}^2$ exhibits this non-extension property. So we have found without much effort a large family of domains which, like domains in one complex variable, are the "natural domain" for some holomorphic function. Such a domain is called a *domain of holomorphy*.

If SCV were as trite as this last construction suggests, then the subject would bask in well-deserved obscurity. We begin to see some texture in the subject as we now turn to a domain which is *not* a domain of holomorphy.

THEOREM 2 (Hartogs [**9**]). *Let $r > 0$ and define*

$$\Omega = D^2(0, r) \setminus \overline{D}^2(0, r/2).$$

(See Fig. 3.) *If f is holomorphic on Ω, then there is a holomorphic function F on the domain*

$$\hat{\Omega} \equiv D^2(0, r)$$

such that $F\big|_{\Omega} = f$. (See Fig. 4.)

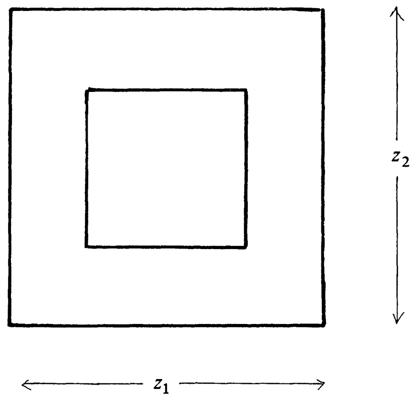

FIG. 3

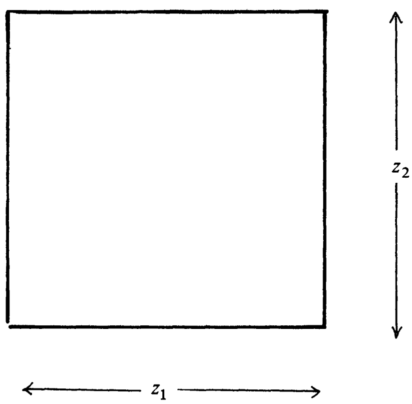

FIG. 4

Proof. For any fixed $|z_1| < r$ we may expand f in a Laurent series about zero in the z_2 variable:

$$f(z_1, z_2) = \sum_{k=-\infty}^{\infty} [a_k(z_1)] \cdot (z_2)^k.$$

Referring to Fig. 3, we see that the series surely converges for $r/2 < |z_2| < r$. Notice also that the formula

$$a_k(z_1) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(z_1, \zeta)}{\zeta^{k+1}}\, d\zeta$$

guarantees that a_k is a holomorphic function of z_1 (use Morera's theorem).

Referring again to Fig. 3, we now make the crucial observation that when $r/2 < |z_1| < r$, then $f(z_1, \cdot)$ is *holomorphic* on $D(0, r)$. Thus

$$a_k(z_1) = 0 \quad \text{when } k < 0 \quad \text{and} \quad r/2 < |z_1| < r.$$

By analytic continuation (the one variable result),

$$a_k \equiv 0 \quad \text{when } k < 0.$$

Thus we may define

$$F(z_1, z_2) = \sum_{k=0}^{\infty} [a_k(z_1)] \cdot (z_2)^k,$$

and F will have all the desired properties. ■

Several important phenomena came to the surface in this proof: first, the fact that our function f is holomorphic in each variable separately plays a crucial role; second, that there are enough dimensions for us to be able to "move around" and find an open set on which $a_k = 0$ for k negative.

We now see that something definitely new is going on in SCV, but we have not a clue as to how to identify which domains exhibit the phenomenon of Theorem 2, and which, like the product domains, support non-extendable holomorphic functions (and are thus domains of holomorphy). Consider the ball for example. It is not a product domain (it could conceivably be "equivalent" to one, but we shall see in Section 5 that such is not the case). Is it still a domain of holomorphy? Just to illustrate that we are dealing here with a fairly subtle problem, we now present the following result:

THEOREM 3. *There is a function f holomorphic on the unit ball $B = B(0,1) \in \mathbb{C}^2$ such that f cannot be analytically continued to any larger open set.*

Proof. If $P = (p_1, p_2)$ is in the boundary of the ball, $|p_1|^2 + |p_2|^2 = 1$, then define

$$\varphi_P(z) = \frac{1}{2}[(z_1 + p_1) \cdot \bar{p}_1 + (z_2 + p_2) \cdot \bar{p}_2].$$

It is instructive to notice that the formula defining φ_P is an instance of the inner product discussed at the beginning of Section 1:

$$\varphi_P(z) = \frac{1}{2} \cdot \langle z + P, P \rangle.$$

Linear algebra now tells us that φ_P will be largest when z (considered as a vector) points in the same direction as P and is as far from the origin as possible. Thus $\varphi_P(P) = 1$ and $|\varphi_P|$ is strictly smaller elsewhere in $\overline{B}$. A more quantitative way of saying this is that if $0 < r < s \leqslant 1$, then there is a constant $k = k(r, s)$ such that for any $z \in B(0, r)$ it holds that

$$0 < |\varphi_P(z)| < k < |\varphi_P(sP)|.$$

Let $\hat{a} = \{\hat{a}_j\}$ be a sequence of elements of B which accumulates at every point of ∂B. Let a be the sequence

$$\hat{a}_1, \hat{a}_1, \hat{a}_2, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \dots .$$

Then a contains all the elements of $\hat{a}$ and each element is repeated infinitely often. We will write $a = \{a_j\}$. For each a_j choose r_j as large as possible so that $D_j \equiv D^2(a_j, r_j) \subseteq B$. For each $j \in \{1, 2, 3, \dots\}$ define $K_j = \overline{B}(0, 1 - 1/j)$ and choose $z_j \in D_j \setminus K_j$. Let $p_j = z_j/|z_j|$ and observe that the function $s_j(z) = \varphi_{p_j}(z)$ has larger modulus at z_j than it does on K_j. If we define

$$c_j = \frac{1}{s_j(z_j)},$$

then

$$h_j(z) = c_j \cdot s_j$$

satisfies

$$h_j(z_j) = 1 \quad \text{and} \quad |h_j|\big|_{K_j} < t_j < 1$$

for some constant t_j. If positive integers N_j are chosen sufficiently large, then the functions

$$m_j(z) \equiv \left[h_j(z)\right]^{N_j}$$

satisfy

$$m_j(z_j) = 1 \quad \text{and} \quad |m_j|\big|_{K_j} < 2^{-j}.$$

Define

$$h(z) = \prod_{j=1}^{\infty} \left(1 - m_j(z)\right)^j.$$

Then the product converges uniformly on each K_j since

$$\sum j \cdot 2^{-j} < \infty.$$

Thus h is holomorphic and not identically zero, and h has a zero of order at least j at z_j. Since each $\hat{a}_j$ (and hence each D_j) is repeated infinitely often in the sequence a, each D_j contains points at which h vanishes to arbitrarily high order. Any analytic continuation of h to a neighborhood of a point $P \in \partial\Omega$ is a continuation to a neighborhood of some $\overline{D}_j$. Hence the domain of h would contain an accumulation point z_0 of a sequence of zeroes of increasing order of h. At z_0, h would vanish to infinite order. Hence h would be identically zero (by the one variable identity theorem). That contradicts the fact that h comes from a convergent infinite product.

■

It should be noticed that the special nature of the ball plays no role in the proof of Theorem 3. What *is* crucial is the existence, for each boundary point P of Ω, of functions φ_P which are "big" near P and "small" away from P. In fact domains of holomorphy are characterized by the fact that they possess such functions φ_P for each boundary point P.

It was a problem of long standing, called the *Levi* problem, to give a purely geometric description of domains of holomorphy. This problem was solved in the early 1950's by Oka, Bremermann, and others. The geometric notion which characterizes domains of holomorphy is called *pseudoconvexity*. The precise definition of pseudoconvexity is rather technical, but we can give an informal description as follows.

Consider the collection $\mathscr{E}$ of all domains which can be obtained from a convex domain by applying to it a holomorphic mapping:

$$(z_1,\dots z_n) = z \to \Phi(z) = (\varphi_1(z),\dots,\varphi_n(z)),$$

each φ_j holomorphic. Let $\mathscr{F}$ be the collection of domains Ω whose boundary can be paved (see in Fig. 5) by finitely many elements of $\mathscr{E}$. Here we say that $\partial\Omega$ is "paved" by the boundaries of domains Ω_j, $j = 1,\dots,k$, if $\partial\Omega = \cup(\partial\Omega \cap \partial\Omega_j)$.

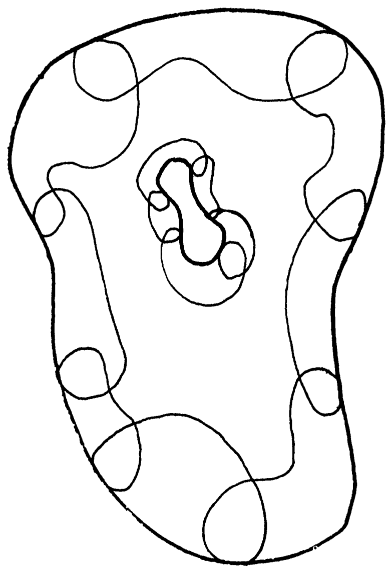

FIG. 5

Finally, let $\mathscr{G}$ be the collection of domains which can be obtained by increasing unions of elements of $\mathscr{F}$. The collection $\mathscr{G}$ coincides precisely with the collection of pseudoconvex domains, which in turn coincides with the collection of domains of holomorphy.

If Ω is pseudoconvex and $P \in \partial\Omega$, then it is possible (using deep techniques such as sheaf theory or partial differential equations) to construct a function φ_P which is large near P and small elsewhere. With these functions in hand, the proof of Theorem 3 can be mimicked to prove that Ω is a domain of holomorphy. This is the hard half of the Levi problem (see [**12**, p. 130] for a discussion of the Levi problem). As an exercise, the reader may wish to construct functions φ_P for a *convex* domain Ω and $P \in \partial\Omega$ and then verify that the proof of Theorem 3 may be successfully imitated to see that Ω is a domain of holomorphy. This is much easier than the general case.

3. Some Consequences of Hartogs's Theorem and Related Results. A fundamental fact about a non-constant holomorphic function f of one complex variable is that its zero set is discrete: if the zeroes of f accumulate in the interior of the domain of f, then $f \equiv 0$. Moreover, by Weierstrass' theorem, any discrete set can be the zero set of a non-constant holomorphic function. This is startlingly false in dimensions two and higher, as we shall now see.

THEOREM 4. *If f is holomorphic on a domain $\Omega \subseteq \mathbb{C}^n$, $n \geqslant 2$, then f has no isolated zeroes.*

Proof. Suppose that $P \in \Omega$ is an isolated zero of f. Let $D^n(P, r) \subseteq \Omega$ be a polydisc such that $\{z \in \mathbb{C}^n\colon f(z) = 0\} \cap D^n(P, r) = P$. See Fig. 6. Then $g(z) \equiv 1/f(z)$ is holomorphic on $D^n(P, r) \setminus \overline{D}^n(P, r/2)$. By Theorem 2, g continues analytically to $D^n(P, r)$. In particular, g is well defined at P, so f cannot vanish at P. ■

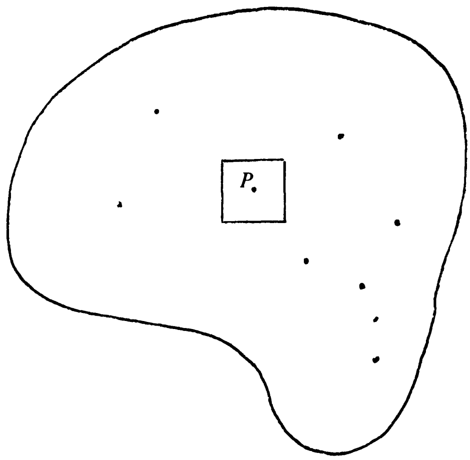

FIG. 6

Here is another result about zero sets which is of a similar flavor, but its proof is a bit different and shows once again how the existence of *several* complex dimensions can be exploited to obtain new results.

THEOREM 5. *Let f be holomorphic on Ω, a bounded domain in $\mathbb{C}^n$, $n \geqslant 2$. Define $\mathscr{Z} = \{z \in \Omega: f(z) = 0\}$. Then $\mathscr{Z}$ is either empty or non-compact in Ω.*

Proof. Suppose that $\mathscr{Z}$ is non-empty and is compact in Ω. Fix a point X outside Ω. Choose a point P in $\mathscr{Z}$ which is as far as possible from X. Let $\vec{v}$ be a unit vector in the direction $\overrightarrow{XP}$ and let $\vec{w}$ be a unit vector which is orthogonal to the vector $\vec{v}$ (in the sense of the inner product $\langle\, ,\rangle$). See Fig. 7. For $r > 0$, small and fixed, and for j sufficiently large, we define holomorphic functions of one complex variable by the formula

$$\varphi_j(\zeta) = f(P + (1/j)\vec{v} + r\zeta\vec{w}), \quad \zeta \in D.$$

Then each function φ_j is zero-free since the complex disc

$$\{P + (1/j)\vec{v} + r\zeta\vec{w}: \zeta \in D\}$$

lies in Ω but outside $\mathscr{Z}$ (provided that r is small enough). But the functions φ_j have the limit

$$\varphi_0(\zeta) = f(P + r\zeta\vec{w})$$

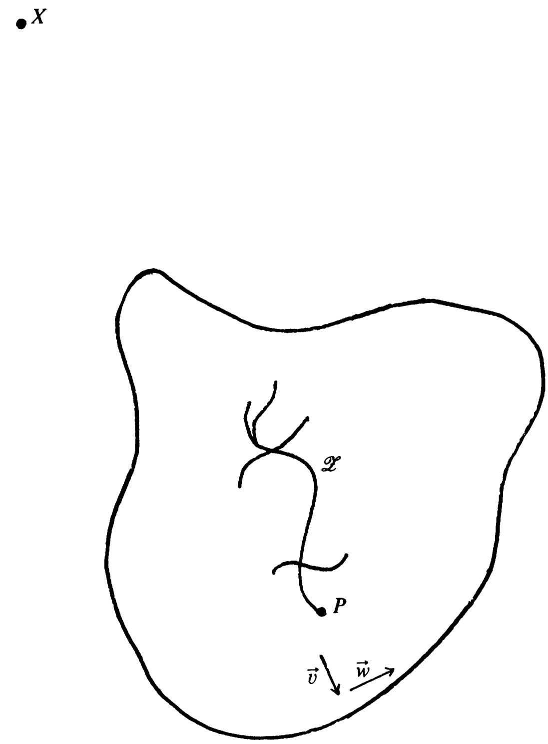

FIG. 7

FIG. 8

as $j \to \infty$. And the function φ_0 vanishes at $\zeta = 0$. By Hurwitz's theorem, we must conclude that $\varphi_0 \equiv 0$, and that is false (since P is as far as possible from X—see Fig. 8). Thus $\mathscr{X}$ cannot be compact. ∎

COROLLARY. *If f is holomorphic on a bounded domain Ω in $\mathbb{C}^n$, $n \geqslant 2$, then every level set of f escapes to $\partial\Omega$.*

Proof. Let α be in the range of f and set $g(z) = f(z) - \alpha$. Then the result follows by applying the theorem to g. ∎

Recall now the following form of the maximum modulus principle: if Ω is a smoothly bounded domain in $\mathbb{C}$ and f is non-constant, continuously differentiable on $\overline{\Omega}$, and holomorphic on Ω, then the maximum modulus of f occurs on the boundary and only on the boundary of Ω (we are assuming here a bit more than is necessary). This result holds in fact in any complex dimension, and the proof is the same as the classical one-variable proof. The maximum modulus principle suggests that functions like $f(z) = z^n$, $z \in C \subseteq \mathbb{C}$, are typical: the modulus of f on the boundary is greater than that in the interior. But notice that, in one variable, the open mapping theorem shows that it cannot be that the image of f is always

contained in the image of $f\big|_{\partial\Omega}$ (for f continuously differentiable on $\bar{\Omega}$ implies that $f(\partial\Omega)$ has no interior). The situation is just the opposite in dimensions two and higher:

THEOREM 6. *If $\Omega \subseteq \mathbb{C}^n$ is a bounded domain, $n \geqslant 2$, f is continuous on the closure $\bar{\Omega}$ of Ω, and f is holomorphic on Ω, then the image of $f\big|_{\partial\Omega}$ contains the full image of f on $\bar{\Omega}$.*

Proof. Let $P \in \Omega$. If $f(P) \notin f(\partial\Omega)$, set $w = f(P)$. Then $f^{-1}(w)$, being a closed, bounded set disjoint from $\partial\Omega$, is compact in Ω. This contradicts the corollary to Theorem 5. ■

We have seen now that zero sets of holomorphic functions of several complex variables are never discrete. Is there some way to describe zero sets of holomorphic functions in both $\mathbb{C}^1$ and $\mathbb{C}^n$ simultaneously and in the same language, so that the two theories do not seem so disparate? The answer is yes, and is best formulated in terms of complex dimension. Unfortunately, a rigorous treatment of this topic would take us far afield. Therefore we will content ourselves with the detailed discussion of a simple example, together with a few cultural remarks.

EXAMPLE. Consider the holomorphic function $\mathbb{C}^2$ defined by

$$f(z_1, z_2) = z_1 \cdot z_2.$$

Then

$$\begin{aligned}\mathscr{Z}(f) &= \{(z_1, z_2) \in \mathbb{C}^2 \colon z_1 \cdot z_2 = 0\} \\ &= \{(z_1, z_2) \in \mathbb{C}^2 \colon z_1 = 0\} \cup \{(z_1, z_2) \in \mathbb{C}^2 \colon z_2 = 0\}.\end{aligned}$$

In short, $\mathscr{Z}(f)$ is the union of two complex hyperplanes. In particular, $\mathscr{Z}(f)$ is a 1-dimensional complex surface (the number 1 being notable because it is one less than the dimension of the ambient space), except at the point $(0,0)$ of intersection of the two hyperplanes.

Compare the discussion of the preceding paragraph with the fact that the zero set of a holomorphic function in $\mathbb{C}^1$ is discrete. A discrete set in $\mathbb{C}^1$ is a set of complex dimension zero, which dimension is one lower than that of the ambient space.

In general one can prove, using a deep algebraic fact known as the Weierstrass Preparation Theorem, that the zero set of a non-constant analytic function in $\mathbb{C}^n$ is a complex hypersurface of dimension $(n - 1)$, except on a singular set of lower dimension. The discussion in the two preceding paragraphs illustrate this assertion. ■

Let us consider yet another interpretation of Theorem 2. It says that a holomorphic f of at least two variables never has isolated singularities. In particular, if f is holomorphic on a domain in $\mathbb{C}^2$ and if the set of singularities of f has complex dimension zero then the singularities of f form a discrete set; hence the singularities are isolated. Therefore they are removable. Inductively, if we have shown that a

singular set of complex dimension $(n-2)$ in $\mathbb{C}^n$ is removable, then let g be holomorphic on a domain Ω in $\mathbb{C}^{n+1}$, except on a singular set of dimension $(n-1)=(n+1)-2$. Let $\mathscr{P}$ be a complex hyperplane and consider g restricted to $\mathscr{P}$. Then, generically, the restricted function is a holomorphic function on $\mathscr{P}$ with a singular set of dimension $(n-2)$. By induction, this singularity is removable. Since this last assertion is true for generic hyperplanes $\mathscr{P}$, it can be argued that the full singularity of f is removable.

4. Inner Functions and Related Topics. An inner function f on the unit disc $D \subseteq \mathbb{C}$ is a bounded holomorphic function with almost everywhere radial boundary values on the unit circle having modulus 1. There are a great many inner functions on D. Any function of the form

$$\varphi(z) = z^k$$

is certainly inner, any Blaschke factor

$$B_a(z) = \frac{z-a}{1-\bar{a}z}$$

for $a \in D$ fixed and $z \in D$ is inner, and any Blaschke product

$$\prod_{j=1}^{\infty}\left(\frac{-\bar{a}_j}{|a_j|}\right)B_{a_j}(z)$$

is inner. Other inner functions may be obtained as exponentials of Cauchy integrals of singular measures. Indeed there are so many inner functions on the disc that the closed linear span of the inner functions in the uniform topology gives all bounded analytic functions (this is a deep theorem of D. Marshall [**14**]).

In the early 1960's Walter Rudin and A. G. Vitushkin posed the problem of determining whether there are non-constant inner functions on the ball in $\mathbb{C}^2$. It soon became apparent that if there are such functions, they must be highly pathological. Let us see why.

LEMMA. *A non-constant inner function on $D \subseteq \mathbb{C}$ cannot be bounded from zero.*

Proof. Let φ be non-constant and inner on D and suppose that

$$|\varphi(z)| \geqslant \mu > 0 \quad \text{on } D.$$

Then the function

$$\psi(z) \equiv \frac{1}{\varphi(z)}$$

is bounded and holomorphic. Since φ is the Poisson integral of its boundary values (which have unit modulus), it follows that $|\varphi(z)| < 1$ for all $z \in D$. But the same reasoning applies to $\psi(z)$. Thus $|\varphi(z)| \equiv 1$ on all of D. By the maximum modulus principle, φ must be constant. That is a contradiction. ∎

LEMMA. *If φ is a non-constant inner function on D, then the range of φ is dense in D.*

Proof. Let $a \in D$ be fixed and suppose that the range of φ omits an open disc centered at a. Then the function

$$B_a \circ \varphi$$

(where B_a is a Blaschke factor as defined above) is bounded from zero and is still inner. By the first lemma, that is impossible. ■

THEOREM 7. *Suppose that f is a non-constant inner function on the unit ball in $\mathbb{C}^2$. Then for any $P \in \partial B$ the cluster set*

$$\mathscr{C}(P) \equiv \{ w \in \mathbb{C}: \textit{there is a sequence } B \ni w_j \to P \textit{ such that } f(w_j) \to w \}$$

equals the entire closed disc $\overline{D}$.

Proof. For almost every complex hyperplane $\mathscr{P}$ which intersects B it holds that f restricted to $\mathscr{P} \cap B$ has unimodular boundary values. But then, by the second Lemma, this restriction has range which is dense in D. Fix $P \in \partial B$. Now let $\mathscr{P}_j$ be a sequence of these complex hyperplanes which approach P as shown in Fig. 9. Then the restriction of f to each $\mathscr{P}_j$ has range which is dense in D. But this simply means that $\mathscr{C}(P) = \overline{D}$. ■

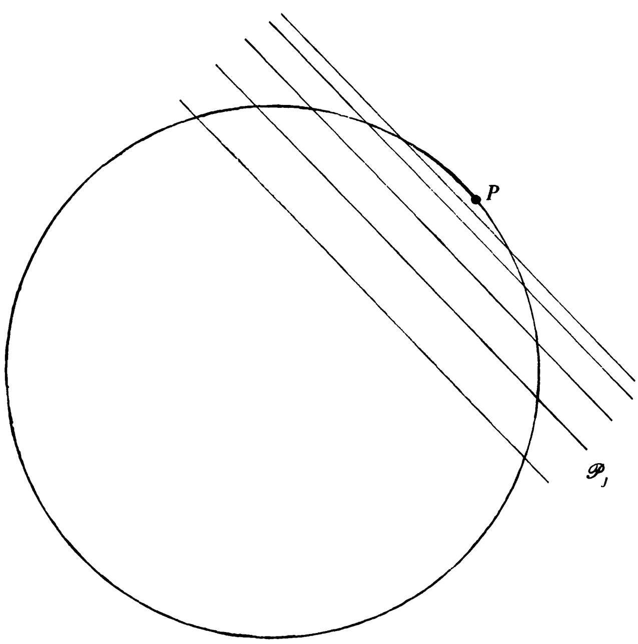

FIG. 9

We know from the Corollary to Theorem 5 that the level sets of a holomorphic function of several complex variables all escape to the boundary of the domain. But for a non-constant inner function in $\mathbb{C}^n$, $n \geqslant 2$, we see that the behavior is genuinely pathological: a dense collection of level sets escapes to *every* point in the boundary of the domain B. It thus came as a shock when in 1981 it was discovered that non-constant inner functions exist. (Relevant papers are **[1]**, **[8]**, **[13]**. See **[17]** for a discussion of the rather delicate priority question.) Indeed there are enough inner functions so that their closed linear span, in a suitably weak topology, generates all the bounded holomorphic functions on the ball.

While the construction of inner functions on the ball is elementary, it is extremely technical and ingenious and we cannot reproduce it here. But we have described these results to bring out the fact that relatively basic phenomena are still being discovered in the subject of SCV.

We conclude this section by considering a factorization problem. If Ω is a bounded domain in $\mathbb{C}^n$, $P \in \Omega$, f is holomorphic on Ω, and $f(P) = 0$, then can we "factor out" the zero?

In one complex dimension, the answer to this question is easily "yes": If f is not identically zero, then there is a unique integer $k > 0$ such that

$$f(z) = (z - P)^k \cdot g(z)$$

for some holomorphic g on Ω such that g does not vanish at P.

In more than one dimension, matters are more complicated. For one thing, the zero cannot be isolated. In addition, as we have seen in previous considerations, our ability to answer this question depends on the shape of the domain. In case $\Omega = B \subseteq \mathbb{C}^2$ then it is elementary to see that $f(z)$ is given by a convergent power series:

$$\sum_{j,k=0}^{\infty} a_{jk}(z_1)^j(z_2)^k.$$

Suppose for simplicity that $P = 0$. Then $f(P) = 0$ means that $a_{00} = 0$. Hence

$$\begin{aligned} f(z_1, z_2) &= z_1 \cdot \left(\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} a_{jk}(z_1)^{j-1}(z_2)^k \right) + z_2 \cdot \left(\sum_{k=1}^{\infty} a_{0k}(z_2)^{k-1} \right) \\ &\equiv z_1 \cdot g_1(z_1, z_2) + z_2 \cdot g_2(z_1, z_2). \end{aligned}$$

This is a natural generalization for the ball in two dimensions of the familiar result in one variable. But what of more general domains? If Ω is any domain (for simplicity assume that it is in $\mathbb{C}^2$), $P \in \Omega$, f is holomorphic on Ω and vanishes at P, then we can restrict attention to a small ball centered at P and decompose f as

$$f(z_1, z_2) = z_1 g_1(z_1, z_2) + z_2 g_2(z_1, z_2)$$

on this small ball. But what about a decomposition on the *entire domain* Ω?

It turns out that the decomposition can be achieved whenever Ω is a domain of holomorphy, but the proof is essentially equivalent to the solution of the Levi

problem. In fact this is one of the unifying features of SCV: the construction of functions (in this case the g_j's) is very hard, can usually only be performed on domains of holomorphy, and involves (or in many cases is equivalent to) the solution of the celebrated Levi problem. One of the important features of the recent solution of the inner functions problem is that it gives a very powerful technique for constructing non-trivial holomorphic functions which *does not* use the solution of the Levi problem. It should prove to be an important tool in years to come.

5. Holomorphic Mappings. A *holomorphic mapping* of a domain $\Omega \subseteq \mathbb{C}^n$ to a domain $\Omega' \subseteq \mathbb{C}^m$ is a function

$$\Phi(z_1,\dots,z_n) = (\varphi_1(z_1,\dots,z_n),\dots,\varphi_m(z_1,\dots,z_n)),$$

where each φ_j is a complex-valued holomorphic function in the usual sense. In this section we will be concerned with *biholomorphic mappings*: mappings which are holomorphic, one-to-one, onto, and have a holomorphic inverse (this last requirement is redundant, but we include it for simplicity and clarity). Merely for topological reasons it will be the case for biholomorphic maps that $m = n$.

When $n = 1$, biholomorphic mappings are just conformal mappings. And the geometry of conformality is a considerable aid in studying these maps. As soon as $n \geqslant 2$ then a mapping is never conformal (in the sense of preserving angle and length infinitesimally) unless it is rational. Thus our mappings have less classical geometry in them, but they are clearly the right "functors" to use in complex function theory: if there is a biholomorphic map $\Phi\colon \Omega \to \Omega'$ then, under composition, holomorphic functions on Ω' may be pulled back to Ω and vice versa.

The most striking result about conformal mapping in the classical one variable theory is the Riemann mapping theorem: a simply connected proper subdomain of the plane is biholomorphic to the disc. One of the earliest results in several complex variables, due to Poincaré, is that the analogous result in $\mathbb{C}^2$ is false.

THEOREM 8. *The ball B and the bidisc $D^2(0,1)$ in $\mathbb{C}^2$ are not biholomorphic to each other.*

Proof. We introduce a biholomorphic invariant which distinguishes B from D^2. If $\Omega \subseteq \mathbb{C}^2$ is a bounded domain containing a point P, then consider

$$X(\Omega, P) \equiv \left\{ \xi \in \mathbb{C}^2 \,\middle|\, \begin{array}{l}\text{there is a holomorphic mapping } \varphi\colon D \to \Omega \text{ satisfying} \\ \varphi(0) = P \text{ and } \varphi'(0) = \xi\end{array} \right\}.$$

Here $\varphi\colon D \to \Omega$ is an ordered pair of functions, $\varphi(\zeta) = (\varphi_1(\zeta), \varphi_2(\zeta))$, and $\varphi'(\zeta)$ denotes $(\varphi_1'(\zeta), \varphi_2'(\zeta))$.

If $\Phi\colon \Omega \to \Omega'$ is a holomorphic map, then the Jacobian matrix

$$\operatorname{Jac}\Phi = \left(\frac{\partial \varphi_j}{\partial z_k}\right)$$

maps $X(\Omega, P)$ to $X(\Omega', \Phi(P))$ (this is just an exercise with the chain rule). If Φ is

*bi*holomorphic, then the map of $X(\Omega, P)$ to $X(\Omega', \Phi(P))$ must be linear, one-to-one and onto (just examine Φ^{-1}).

Now suppose that Φ is a biholomorphic map of B to D^2. By composing Φ with suitable Möbius transformations of the disc, we may suppose that Φ maps 0 to 0. Thus $\operatorname{Jac}\Phi$ is a linear bijection of $X(B,0)$ to $X(D^2,0)$. We claim that

$$X(B,0) = \overline{B}$$

and

$$X(D^2,0) = \overline{D}^2.$$

This would certainly complete the proof. There could be no *linear*, one-to-one, onto mapping of $\overline{B}$ to $\overline{D}^2$ since the ball has smooth boundary while $\partial \overline{D}^2$ has corners.

Verification of the claim amounts to judicious application of the one variable Schwarz's Lemma. For the first assertion, notice that if $\xi \in \overline{B}$, then the map

$$\varphi(\zeta) \equiv \zeta\xi$$

maps D into B and satisfies $\varphi(0) = 0$ and $\varphi'(0) = \xi$. Therefore $\overline{B} \subseteq X(B,0)$.

Conversely, if $\xi \in X(B,0)$, then let $\varphi\colon D \to B$ satisfy $\varphi(0) = 0$ and $\varphi'(0) = \xi$. If ρ is the complex linear projection of B to $\{(z_1,0)\colon |z_1| < 1\}$ and σ is any unitary rotation, then $\rho \circ \sigma \circ \varphi$ maps D to D and takes 0 to 0. Schwarz's Lemma then gives that $|(\rho \circ \sigma \circ \varphi)'(0)| \leqslant 1$. Using the chain rule and writing this out gives $|\rho \circ \sigma(\xi)| \leqslant 1$. Since σ was arbitrary, it follows that $|\xi| \leqslant 1$. Thus $X(B,0) \subseteq \overline{B}$. So $X(B,0) = \overline{B}$.

The proof that $X(D^2,0) = \overline{D}^2$ is left as an exercise for the reader. ■

Poincaré's remarkable discovery (obtained, incidentally, by entirely different methods), led to the general question of determining when two domains are biholomorphically equivalent. The theorem shows that even in the topologically trivial case there are subtle obstructions. Poincaré initiated a program, in the case of smoothly bounded domains, to calculate differential geometric invariants in the boundaries of domains which would behave canonically under biholomorphic mappings. These could then be used, in principle, to classify domains in $\mathbb{C}^n$.

In order for such a program to be successful, one would have to know that biholomorphic mappings extend smoothly to the boundaries of the relevant domains. That such is the case in one complex variable is an old, but still rather difficult, result (see **[2]**). In several complex variables, theorems of this nature (for several special classes of domains) have only come about in the last fifteen years (see [3], [4]). The proofs of these results use an enormous amount of machinery from geometry, algebra, differential equations, and analysis. It is still an important open question whether biholomorphic maps of *arbitrary* smoothly bounded domains in $\mathbb{C}^n$ extend smoothly to the boundary.

6. Concluding Remarks. In this article I have touched on several important and central themes in the subject of several complex variables: domains of holomorphy,

pseudoconvexity, analytic varieties, removable singularities, inner functions, factorization of holomorphic functions, and biholomorphic mappings. The list of important topics which I have omitted is considerably longer. But my aim has been to give the reader just a taste of the subject.

While I have been careful in this article to attribute most of the theorems presented, I have not attempted to give a complete history of the subject. Many important names (H. Cartan, Grauert, Oka, etc.) have not even been mentioned. The reader who wishes to have more detailed bibliographical information should consult [**5**], [**6**], [**7**], [**10**], [**11**], [**12**], and [**15**].

By the same token, many significant aspects of SCV, including differential geometry, sheaf theory, commutative ring theory, partial differential equations, and probability, have been ignored in this article. The reader who has been tantalized will find that the references in the last paragraph will provide further information on these topics.

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The Jordan Canonical Form: an Old Proof

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To my way of thinking, it's a marvelously simple proof. It's over 50 years old, and when one uses the modern language of graph theory, it can be made very visual. The most difficult part of the proof is in achieving triangular form. After that the details are extremely easy to follow: elementary row operations followed by the corresponding elementary column operations. Thus for students who are more familiar with Gaussian elimination than with other aspects of linear algebra, it is a good way to introduce the Jordan canonical form.

So where is this proof and why isn't it a standard proof? (I think it should be and I hope to convince you.) It's in a classical book [**8**] by H. W. Turnbull and A. C. Aitken entitled *An Introduction to the Theory of Canonical Matrices*. This book is well known to matrix theorists, so that one cannot claim that it's in an obscure book that has long been forgotten. The reason that it has been ignored may be due to the changes in mathematical rigor that have occurred since 1932 and the lack of the formalism provided by graph theory at that time. (König's classical book entitled *Theorie der Endlichen und Unendlichen Graphen* was published in 1936, although his classical paper [**6**] "Graphen und Matrizen" was published in 1931.) At first reading, Turnbull and Aitken's proof is somewhat obscure, and it is not clear that their proof is general. They refer to *chains* of nonzero elements in a matrix which now we would call *paths in the digraph associated with the matrix*. Whatever the reason, I hope to revive their proof by publication of this article.

In a recent article [**2**] in this journal, Fletcher and Sorensen describe a proof for the existence of the Jordan canonical form which is algorithmic in nature. This proof was adopted in [**5**]. The proof, like the one to be given here, proceeds in three steps: (I) reduce to upper triangular form; (II) further reduce to the case in which all eigenvalues are equal; (III) use induction to reduce an upper triangular matrix with equal eigenvalues to Jordan canonical form. As pointed out in [**2**], the only nonconstructive step is I. In [**2**] step II is accomplished by solving, using induction, a linear matrix equation of the form $AX - XA = S$, while step III is accomplished by induction and matrix factorizations. The Turnbull-Aitken approach accomplishes

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*Partially supported by National Science Foundation Grant No. DMS-8320189.

steps II and III using only induction and elementary row and column operations. Another elementary derivation of the Jordan canonical form is due to Filippov [**1**] (also described by Strang [7]). In this proof, given a matrix A of order n, one searches for Jordan strings of vectors $w_1, w_2, w_3, \ldots$ where

$$Aw_1 = \lambda w_1, \quad Aw_2 = \lambda w_2 + w_1, \quad Aw_3 = \lambda w_3 + w_2, \ldots .$$

The existence of n vectors which can be arranged into Jordan strings is accomplished by restricting A to its column space (or, when A is not singular, by restricting $A - \lambda I$ to its column space, where λ is any one of the eigenvalues of A) and using induction. The n vectors then become the columns of a matrix which brings A to Jordan canonical form. The only technical part of the proof is the verification of the linear independence of the n vectors.

1. Combinatorial structure. Our discussion is quite leisurely. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be n by n complex matrices (or n by n matrices over an algebraically closed field). Then A is *similar* to B provided there is an invertible matrix P such that $B = P^{-1}AP$. The idea of the Jordan canonical form is to find as simple as possible matrix B which is similar to A. Simple in this context is intended to mean that the structure of the nonzero off-diagonal entries of B is simple. This structure is taken to be the *digraph* $D(B)$ *of* B which is defined as follows. The *vertices* of $D(B)$ are the integers $1, 2, \ldots, n$ (corresponding simultaneously to the rows and columns of B). The *arcs* of $D(B)$ are certain ordered pairs of distinct vertices. Precisely, there is an arc (i, j) from i to j provided $i \neq j$ and $b_{ij} \neq 0$. An example of a matrix and its digraph is pictured in Fig. 1.

$$\begin{bmatrix} 3 & 1 & 0 & 2 \\ -4 & 0 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

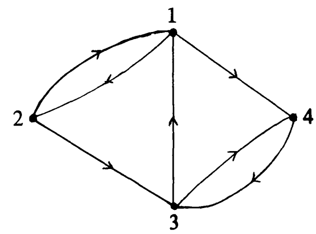


FIG. 1. **A matrix and its digraph.**

The digraphs with the simplest structure are those with no arcs, that is, consist of a number of isolated vertices. We say that these digraphs have the *trivial structure*. They are the digraphs which are associated in the above way with *diagonal matrices*, matrices all of whose off-diagonal entries equal 0. Such a matrix and its digraph are illustrated in Fig. 2. As we all know, not every matrix is similar to a diagonal matrix, that is, not every matrix is similar to one with trivial structure. But Jacobi's theorem of 1837 tells us that we can always achieve an acyclic structure.

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

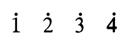


FIG. 2. A diagonal matrix and its digraph.

THEOREM 1.1 (Jacobi). *The n by n complex matrix A is similar to an upper triangular matrix T. The diagonal entries of T are the n eigenvalues of A, and T can be chosen so that these eigenvalues appear on its main diagonal in any specified order.*

(A proof of Jacobi's theorem can be found in almost every book on linear algebra or matrix theory. Schur's theorem asserts that A is unitarily similar to an upper triangular matrix.) What kind of digraphs do upper triangular matrices have? If in the picture of the digraph of an upper triangular matrix we arrange the vertices in a column with vertex 1 on top and vertex n on bottom, then all the arcs point downwards. A triangular matrix and its digraph are pictured in this way in Fig. 3.

$$\begin{bmatrix} 3 & -1 & 2 & 8 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

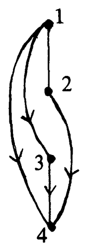


FIG. 3. An upper triangular matrix and its diagraph.

A *cycle* in a digraph D is a sequence $(i_1, i_2, i_3, \ldots, i_k, i_1)$ of $k+1$ vertices $(k \geqslant 2)$ where $i_1, i_2, \ldots, i_k$ are distinct and where (i_1, i_2), $(i_2, i_3), \ldots, (i_{k-1}, i_k)$, (i_k, i_1) are all arcs of D. A digraph with no cycles is called *acyclic*. Since in the picture of an upper triangular matrix all arcs point downwards, *the digraph of an upper triangular matrix is acyclic*. The converse is not true in the strict sense. A matrix can have an acyclic digraph without being upper triangular (or lower triangular). Such a matrix is pictured in Fig. 4. But it can be made upper triangular by means of a *permutation similarity*: *if the matrix B has an acyclic digraph, then there is a permutation matrix P such that $P^{-1}BP$ is an upper triangular matrix.* Put another way, if B has an acyclic digraph, then it is possible to reorder the rows and the columns in the same way and obtain an upper triangular matrix. By putting the rows and columns of the matrix in Fig. 4 in the order 3, 2, 1, 4, we obtain the upper triangular matrix in Fig. 3. When the permutation matrix P corresponds to a transposition, we say that $P^{-1}BP$ is obtained from B by an *elementary permutation similarity*. Since every permutation is a product of transpositions, every permutation

similarity can be accomplished by a sequence of elementary permutation similarities. We shan't bother to prove that a matrix B with acyclic digraph can be made upper triangular by means of a permutation similarity. It is not needed for the proof of the Jordan canonical form, can be found in many books (e.g. [4]), and can be left as an exercise for those interested (show there must exist a vertex with no arcs coming into it and then use induction). We can now give a *combinatorial statement of Jacobi's theorem*: *The n by n matrix A is similar to a matrix B whose digraph is acyclic.*

$$\begin{bmatrix} 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 3 \\ 2 & -1 & 3 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

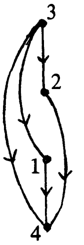


FIG. 4. A non-triangular matrix with an acyclic digraph.

The Jordan canonical form shows that the digraph can be made significantly simpler. For this we need some additional concepts from the theory of digraphs. A *path* in a digraph D is a sequence $(i_1, i_2, \ldots, i_k)$ of k distinct vertices such that $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ are all arcs of D. The *length of the path* is $k - 1$, the number of its arcs, while the *size of the path* is k, the number of its vertices. We allow the possibility of $k = 1$, a path of length 0 and size 1. A digraph whose set of vertices and set of arcs are, respectively, $\{i_1, i_2, \ldots, i_k\}$ and $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)\}$ will be called a *path-digraph*. A path-digraph is a special kind of acyclic digraph. An example of a matrix whose digraph is a path-digraph is illustrated in Fig. 5.

$$B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

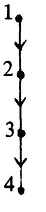


FIG. 5. A matrix B and its path-digraph.

Every matrix whose digraph is a path can be brought to this form where the nonzero off-diagonal entries are precisely those immediately above the main diagonal, by means of a permutation similarity (the order of the rows and columns is the order of the vertices of the path).

The matrix B in Fig. 5 can be made *numerically* simpler without changing its structure by means of a *diagonal similarity*. An *elementary diagonal matrix* is a diagonal matrix with nonzero diagonal entries most of which are different from 1. Every invertible diagonal matrix is a product of elementary diagonal matrices. The n by n diagonal matrix with diagonal entries equal to $d_1, d_2, \ldots, d_n$ will be denoted by $\operatorname{diag}(d_1, d_2, \ldots, d_n)$. Let $D_1 = \operatorname{diag}(1, 1/2, 1, 1)$. Then with B as in Fig. 5,

$$D_1^{-1}BD_1 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & 14 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

If we now let $D_2 = \operatorname{diag}(1, 1, 1/14, 1)$ and $D_3 = \operatorname{diag}(1, 1, 1, 1/42)$, and then let $D = D_1 D_2 D_3 = \operatorname{diag}(1, 1/2, 1/14, 1/42)$, we see that B is diagonally similar to

$$D^{-1}BD = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

The above reduction applies quite generally and this enables us to conclude: *The digraph of an n by n matrix B is a path-digraph if and only if it is possible to attain a matrix C of the form*

$$C = \begin{bmatrix} \lambda_1 & 1 & & & \text{\Large 0} \\ & \lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \text{\Large 0} & & & & \lambda_n \end{bmatrix}$$

by means of a permutation similarity followed by a diagonal similarity.

Given the n by n matrix A, the Jordan canonical form A_J of A has a more complicated structure than that of C. The digraph of A_J is not in general a path-digraph. Rather it consists of a collection of paths which pairwise have no vertex in common, that is a collection of *pairwise vertex-disjoint paths*. Each path is associated with exactly one eigenvalue of A, like the matrix C when $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. But two different paths may be associated with the same eigenvalue. For example, a matrix in Jordan canonical form and its digraph are illustrated in Fig. 6. The digraph of the matrix consists of four pairwise vertex-disjoint paths. The first three of these correspond to the eigenvalue 5 while the fourth corresponds to the eigenvalue 0.

$$\left[\begin{array}{ccc|cc|c|ccc}
5 & 1 & 0 & & & & & & \\
0 & 5 & 1 & & & & & & \\
0 & 0 & 5 & & & & & & \\
\hline
 & & & 5 & 1 & & & & \\
 & & & 0 & 5 & & & & \\
\hline
 & & & & & 5 & & & \\
\hline
 & & & & & & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & 1 \\
 & & & & & & 0 & 0 & 0
\end{array}\right]$$

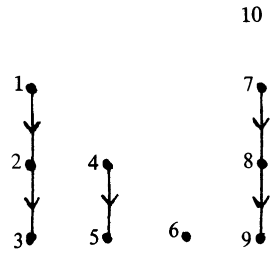


FIG. 6. A matrix in Jordan canonical form and its digraph (unspecified entries of the matrix are 0).

We can now state the existence of the Jordan canonical form of the n by n matrix A as follows.

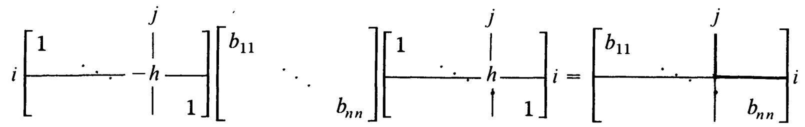
THEOREM 1.2 (Combinatorial statement of Jordan canonical form). *The n by n matrix A is similar to a matrix A_J whose digraph is a collection of pairwise vertex-disjoint paths where each path is associated with exactly one eigenvalue.*

We have already introduced two kinds of similarities, namely (elementary) permutation similarities and (elementary) diagonal similarities. To prove Theorem 1.2 we need one additional kind of similarity. Let i and j be integers with $1 \leqslant i, j \leqslant n$ and $i \neq j$, and let h be any nonzero complex number. Let P be the n by n matrix which is obtained from the n by n identity matrix by putting h in the (i, j)-position. (Thus the digraph of P has exactly one arc, namely (i, j).) The matrix P is invertible and P^{-1} is obtained from P by replacing h with $-h$. The matrix $C = P^{-1}BP$ is obtained from B by adding h times column i of B to column j and then $-h$ times row j of BP to row i. We say that C is obtained from B by an *elementary combination similarity*. Suppose now $B = [b_{kl}]$ is an upper triangular matrix and $i < j$. Then row j of the matrix PB above equals row j of B, and hence C is obtained from B by adding h times column i of B to column j and $-h$ times row j of B to row i *in either order*. Moreover the matrix $C = [c_{kl}]$ is also an upper triangular matrix which differs from B only in those positions of row i in columns $j, j+1, \ldots, n$ and of column j in rows $1, 2, \ldots, i$. This is illustrated in Fig. 7. The (i, j)-entry of C is given by

$$c_{ij} = b_{ij} + h(b_{ii} - b_{jj}).$$

Thus if $b_{ii} \neq b_{jj}$, we may choose $h = -b_{ij}/(b_{ii} - b_{jj})$ and then $c_{ij} = 0$.

Since an invertible matrix can be reduced to the identity matrix by a sequence of elementary row operations (interchange two rows, multiply a row by a nonzero number, add a multiple of one row to a different row), it follows that the matrix A is similar to the matrix B if and only if B can be obtained from A by a sequence of elementary permutation, diagonal, and combination similarities.



$$P^{-1}BP = C$$

FIG. 7. An elementary combination similarity of an upper triangular matrix.

2. The Jordan Canonical Form. We now turn to the proof of the existence of a Jordan canonical form for the n by n complex matrix A. The section on combinatorial structure is not necessary for presentation or understanding of the proof, although we feel that it enhances and enriches it. We have given a combinatorial statement of the Jordan canonical form. The usual way to describe it is as follows. Define a *Jordan block* to be a k by k matrix of the form

$$J_k(a) = \begin{bmatrix} a & 1 & & & 0 \\ & a & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & a \end{bmatrix}$$

where $J_1(a) = [a]$.

THEOREM 2.1 (Classical statement of Jordan Canonical Form). *The n by n complex matrix A is similar to a matrix which is a direct sum of Jordan blocks.*

Proof. The reduction is in three steps, the first of which was already given in the last section.

I. By Jacobi's (or Schur's) Theorem (Theorem 1.1), there exists an invertible matrix P such that $T = P^{-1}AP$ is an upper triangular matrix where equal diagonal entries occur consecutively.

Since similarity is a transitive relation, we now deal exclusively with T.

II. Suppose T has at least two different entries on its main diagonal. Then we may write

$$T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix},$$

where T_1 is an upper triangular matrix each of whose diagonal entries is some constant a, and T_2 is an upper triangular matrix none of whose diagonal entries equals a. Using elementary combination similarities, we may make each entry of X equal to 0 *with no change in T_1 or T_2*. We do this row by row, starting with the last row of X, and treating each entry of the current row beginning with its first. *Recall Fig. 7.* Thus T is similar to the direct sum of T_1 and T_2. If T_2 does not have a

constant main diagonal, we repeat this step on T_2. Eventually we find that T is similar to a direct sum of upper triangular matrices each of which has a constant main diagonal. The reduction of T to Jordan canonical form will be complete once each of the upper triangular matrices in the direct sum is reduced to Jordan canonical form. This now enables us to assume that T has a constant main diagonal, that is, all eigenvalues of T are equal.

III. Now T can be taken to be an n by n upper triangular matrix of the form

$$\begin{bmatrix} a & & & & * \\ & a & & & \\ & & \ddots & & \\ & & & \ddots & \\ & 0 & & & a \end{bmatrix}.$$

That this matrix can be reduced to Jordan canonical form is established by induction on n. When $n = 1$, $T = [a]$ which is already in Jordan canonical form. Suppose $n = 2$ so that

$$T = \begin{bmatrix} a & p \\ 0 & a \end{bmatrix}.$$

If $p = 0$, T is in Jordan canonical form. Otherwise, an elementary diagonal similarity transforms T to

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},$$

which is in Jordan canonical form. We now suppose $n > 2$ and proceed by induction. By the inductive hypothesis, the leading $(n-1)$ by $(n-1)$ principal submatrix S of T can be transformed by a similarity to Jordan canonical form F: $Q^{-1}SQ = F$. If we now let P be the direct sum of Q with the 1 by 1 identity matrix, we obtain that $P^{-1}TP = T_1$, where

$$T_1 = \left[\begin{array}{ccc|c} & & & * \\ & F & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & a \end{array}\right].$$

First suppose that there is a nonzero entry h in the last column of T_1 which is in the same row as a 1 of one of the Jordan blocks of F. For instance, we might have

$$T_1 = \left[\begin{array}{cc|cccc|c} a & 1 & & & & & * \\ 0 & a & & 0 & & & * \\ \hline & & a & 1 & 0 & 0 & * \\ & 0 & 0 & a & 1 & 0 & h \\ & & 0 & 0 & a & 1 & * \\ & & 0 & 0 & 0 & a & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a \end{array}\right]. \tag{2.1}$$

We show we can eliminate entries like h which are in the last column and opposite a 1. Indeed the elementary combination similarity which adds $-h$ times column 5 of T_1 to column 7 and adds h times row 7 to row 5 replaces h with 0 and *does not change any other entry of* T_1. Thus a sequence of elementary combination similarities gets us to an upper triangular matrix T_2 which has the same form as T_1 with F as its leading $(n-1)$ by $(n-1)$ principal submatrix but has at most one nonzero off-diagonal entry in each row. For the matrix T_1 in (2.1) above, the matrix T_2 is given by

$$T_2 = \left[\begin{array}{cc|cccc|c} a & 1 & & & & & 0 \\ 0 & a & & \multicolumn{2}{c}{\mathbf{0}} & & s \\ \hline & & a & 1 & 0 & 0 & 0 \\ & & 0 & a & 1 & 0 & 0 \\ \multicolumn{2}{c|}{\mathbf{0}} & 0 & 0 & a & 1 & 0 \\ & & 0 & 0 & 0 & a & t \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a \end{array}\right]. \tag{2.2}$$

If the last column of T_2 has no nonzero off-diagonal entries, then T_2 is in Jordan canonical form. So suppose T_2 has at least one nonzero off-diagonal entry in column n.

We now show that we can replace with 0 all nonzero off-diagonal entries in column n of T_2 except for one, *without changing any other entry of* T_2. We do this by showing how for each pair of nonzero off-diagonal entries in column n of T_2, one of them (the one across the Jordan block of smaller size) can be made 0 by means of a sequence of elementary combination similarities.

The digraph of T_2 is fairly simple. It consists, in general, of a number of pairwise vertex-disjoint paths and, entirely vertex-disjoint from them, a number of other paths which are pairwise vertex-disjoint *except* for the fact that they all terminate at vertex n (corresponding to column n). The total number of paths equals the number of Jordan blocks of F. The number of paths terminating at vertex n equals the number of nonzero off-diagonal entries in column n of T_2. The paths which don't terminate at vertex n already correspond to Jordan blocks of T_2 (recall the combinatorial statement of the Jordan canonical form, Theorem 1.2). If there is such a path, then the inductive hypothesis immediately applies. Thus we may assume that all the paths of the digraph of T_2 terminate at vertex n. These paths are in one-to-one correspondence with the nonzero off-diagonal entries of column n.

We refer to T_2 in (2.2) for our description, but the procedure as will be clear is quite general. The path of largest size terminating at $n = 7$ is the path $(3,4,5,6,7)$ whose last arc is labelled (t). (If both paths have the same size, then we arbitrarily pick one.) Using an elementary diagonal similarity we can assume $t = 1$. Consider the path $(1,2,7)$ whose last arc is labelled (s). We successively perform the following elementary combination similarities (recall that we've taken t to be 1):

(i) Add $-s$ times row 6 to row 2 and then add s times column 2 to column 6.
(ii) Add $-s$ times row 5 to row 1 and then add s times column 1 to column 5.

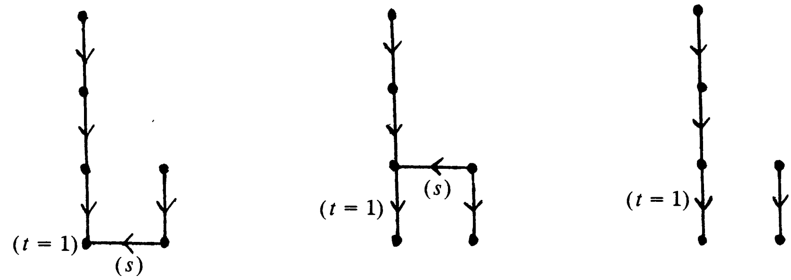


FIG. 8. The digraph of T_2 in (2.2) and those that result when applying steps (i) and (ii).

The digraphs of the initial matrix T_2, and those of the matrices resulting after each of the above steps, are pictured in Fig. 8.

Thus by elementary combination similarities we have succeeded in replacing s with 0 *with no other change in the matrix* T_2.

Hence the matrix T_2 in (2.2) is similar to the matrix T_3 that results when s is set equal to 0. The 4 by 4 Jordan block of F of largest order which corresponds to the path of largest size 4 terminating at $n = 7$ now joins through the entry $t = 1$ with the last diagonal entry a to give a 5 by 5 Jordan block. More precisely, the matrix T_3 is permutation similar to the direct sum of the Jordan blocks $J_3(a)$, $J_3(a)$, $J_5(a)$ and $J_2(a)$.

If the Jordan canonical form of the matrix S has more than two Jordan blocks so that the matrices T_1 and T_2 have more than three blocks, then the preceding strategy can be applied to the nine blocks making up the lower right corner. Hence we have achieved Jordan canonical form and the induction, and therefore proof, is complete. □

3. Coda. Appealing to induction and Gaussian elimination, we have described two of the three main steps in the reduction of a matrix to Jordan canonical form. The first step, namely reduction to triangular form, can also be argued inductively. The entire proof of the existence of a Jordan canonical form of a matrix can be argued from beginning to end using the basic tools of induction and Gaussian elimination. The Turnbull-Aitken approach to the Jordan canonical form therefore is quite suitable for a first course in linear algebra.

The point of view taken in many discussions of the Jordan canonical form is that of change of basis of a vector space V so that the matrix representation of a linear transformation T acting on V has Jordan canonical form. This point of view can be accommodated in the approach given here by labelling the vertices of the digraph of A by the basis vectors $v_1, v_2, \dots, v_n$ rather than by the integers $1, 2, \dots, n$.

It is of course a formidable task to actually compute the Jordan canonical form of a complex matrix of order n. The numerical problems associated with the task are discussed in Golub and van Loan [3]. However, if a matrix is in triangular form with rational entries, then it is an interesting programming exercise (using fixed point

arithmetic, representing fractions as ordered pairs of integers) to complete the reduction to Jordan canonical form.

Finally it should be pointed out that there is no general relationship between the digraph of a triangular matrix with constant main diagonal and the Jordan canonical form of the matrix. For instance the matrices A and B given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

have the same digraph (the nonzero entries occupy the same positions in both matrices), yet they have different Jordan canonical forms (A is nilpotent of index 2 while B is nilpotent of index 3).

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THE EDITOR'S CORNER

Finite Lists of Obstructions

HERBERT S. WILF

This second article of the 'Editor's Corner' series reports on a new and very powerful theory that has recently been developed by Neil Robertson and Paul Seymour, of Ohio State University and Bell Communications Research, respectively. This theory can be thought of as describing what it 'really is' about the family of planar graphs that makes Kuratowski's theorem true in that family, and consequently obtaining far-reaching generalizations of that theorem.

Their work is being written up in a sequence of about 18 papers, whose total length in print may run to about 500 pages. Of these papers, at this moment 5 have already appeared ([**2**]—[**6**] below) and 7 others are in various stages of completion.

Hence one should not expect this column even to be able to summarize all of their principal results. We will try, however, to explain some of the ancestry of the work, state some of the main theorems, and discuss a few applications. We suggest that those who wish to learn more about it begin with [**1**].

We begin by recalling Kuratowski's theorem.

A *graph* is a finite list of *vertices* together with a finite list of (unordered) pairs of vertices, called *edges*. We allow *loops* (i.e., edges (v, v)) and multiple edges (i.e., edges can be repeated). A graph is *planar* if it embeds in the plane,* which is to say, if we can draw a picture of the graph in the plane in which the vertices are points, the edges are lines (simple arcs), and two of these lines never cross except at vertices of the graph.

Of the graphs in Figure 1, the first is planar, the next two are not.

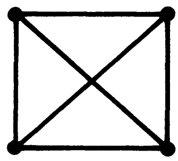

FIGURE 1(a)

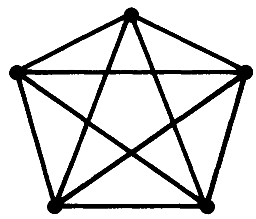

FIGURE 1 (b)

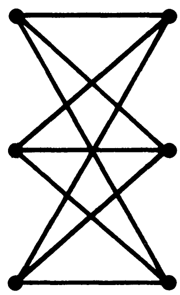

FIGURE 1(c)

Kuratowski's theorem asserts that, in a certain sense, there are only two reasons why a graph might fail to be planar, i.e., there are just two 'obstructions' to embedding in the plane. Again roughly speaking, the two obstructions are the graphs shown in Figures 1(b), 1(c).

To speak less roughly, let's define a *minor* of a graph G. We will say that a graph H is a minor of G if we can obtain H from G by a sequence of operations of the following two kinds:

(a) replace a graph by a subgraph of itself or

(b) contract an edge (i.e., identify its two endpoints).

If we accept that the graphs of Figures 1(b), 1(c) (the 'Kuratowski graphs') are not planar, then it is very clear that a graph that contains either of these graphs as a minor is also not planar.

Considerably less clear, but true, is the fact that *every nonplanar graph contains at least one of the two Kuratowski graphs as a minor*, and that is Kuratowski's theorem. In fact, Kuratowski's theorem was originally proved for a slightly different sense of the word 'contains.' We say that *G topologically contains H* if some subgraph of G is

* The 'plane' can be replaced by the 'sphere' throughout the sequel, if desired.

isomorphic to a subdivision of H, where to 'subdivide' a graph is to replace its edges by paths that are vertex disjoint except at endpoints. It isn't hard to show that the two formulations of Kuratowski's theorem are equivalent.

There is no really simple proof of the theorem. See, for example, **[7]** for a complete proof.

It is in that sense that there are only two obstructions to embedding in the plane, or on the sphere.

It is perhaps worthy of remark at this point to observe that Kuratowski's theorem provides no obvious fast algorithm for deciding planarity of a given graph, since it isn't apparent how to check whether one of the forbidden configurations appears as a minor. Fast ways are known, however, to check for planarity **[8]**.

Now what happens if we go to the torus (= sphere + one handle)? Is there just a finite list of obstructions to embedding a graph there? More generally, if we go to a surface of genus g, then what are the obstructions to graphical embedding? Are there only finitely many?

Until the work of Robertson and Seymour these questions had not been answered. Some similar questions have been answered fairly recently. Thus Archdeacon **[9]** and Glover, Huneke and Wang **[10]** showed that there are only finitely many obstructions to embedding in the projective plane (in fact, that there are 35 minor-minimal graphs that do not embed in the projective plane). Archdeacon and Huneke showed that the same holds for all nonorientable surfaces.

Now, however, the above questions have been answered for all surfaces, and in the affirmative.

THEOREM 1. (Robertson, Seymour). *Fix an integer $g \geq 0$. There is a finite list $\mathscr{L} = \mathscr{L}(g)$, of graphs, with the following property: a graph G embeds on a surface of genus g if and only if it does not contain, as a minor, any of the graphs on the list $\mathscr{L}$.*

Their work actually goes a good deal further than the above, and indeed their main theorems aren't only about embedding graphs on surfaces. Instead, they begin with a family F of graphs that is describable as the class of all graphs that don't contain any members of some fixed, finite list as minors.

Obviously any such family F is *minor-closed*, i.e., contains every minor of every graph in the family.

One version of Robertson-Seymour's main theorem is the assertion that the condition is sufficient also, that is, *every minor-closed family of graphs is describable as the set of all graphs that don't contain, as a minor, any of the graphs in a certain fixed, finite list of graphs.*

Example. The family of all forests is obviously minor-closed. What is its 'fixed, finite list' of excluded minors?

Clearly the above theorem implies the results stated about embeddability of graphs on surfaces of fixed genus, because the set of all such graphs is evidently

minor-closed. It is quite remarkable that the seemingly genus-specific result involves nothing more than the invariance of the set of graphs considered under the two simple surgical operations of (a) go to subgraph and (b) contract an edge.

One might think that a way 'around' the theorem would be to consider the list of all graphs that do not contain any minor from a certain *infinite* list of graphs. Their work, however, shows that the infinite list wouldn't have been necessary; a finite one would have done the same job. Here are a few of the necessary ideas, in precise form.

If S is a given set, then a *partial order* for S is a transitive, reflexive, antisymmetric relation. If the relation is only transitive and reflexive but not necessarily antisymmetric then it is a *quasi-order*. A *chain* in a quasi-order is a sequence of elements of S that is totally ordered; i.e., for every pair a, b of distinct elements of the sequence we have either $a \preceq b$ or $b \preceq a$. An *antichain* in a quasi-order is a sequence of elements of S no two of which are comparable; i.e., neither $a \preceq b$ nor $b \preceq a$ is true.

On the set of all graphs, the relation 'G is a minor of H' is a partial order; the relation 'G is topologically contained in H' is a quasi-order. Both, of course, are quasi-orders.

A family of graphs is *well quasi-ordered* (wqo) if it contains no infinite antichain and no infinite descending chain. The class of all graphs clearly has no infinite descending chain, in the 'minor' ordering. That it contains no infinite antichain was conjectured years ago by K. Wagner, and this has now been proved by Robertson and Seymour. One may say that the class of all graphs, ordered by 'is a minor of,' is well quasi-ordered.

It has been known since the work of Kruskal **[11]** over 20 years ago that the class of all trees is a wqo under 'is topologically contained in.' In more detail, Kruskal's theorem asserts that if we are given any infinite sequence of (rooted) trees, then there are two trees T_i, T_j in the sequence such that $i < j$ and T_i is topologically contained in T_j.

The key to all of the above problems at once lies in proving that the class of all graphs, ordered by minors, is a wqo; and that is exactly what Robertson and Seymour have done.

To sum up, the property of the set of planar graphs that makes Kuratowski's theorem work is that it is closed under the operations of replacing a graph by a subgraph and of contracting an edge, and that's all.

As an unsolved problem we mention the following. Despite the apparent difficulty of converting Kuratowski's theorem into an algorithm, planar graphs can be recognized in linear time **[8]**. Can graphs of genus $\leq g$ be recognized in linear time?

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NOTES

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A Counterexample to a Conjectured Hermitian Matrix Inequality

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Dedicated to Professor Zirô Takeda with respect and affection.

At the end of [**1**], page 540, in a recent issue of this MONTHLY, the following conjecture is cited.

PROBLEM. Let A, B and C be nonnegative Hermitian matrices such that $A \leqslant C$, $B \leqslant C$. Is it true that

$$(A^2 + B^2)^{1/2} \leqslant \sqrt{2}\,C?$$

We present a very easy counterexample to this conjecture as follows.

Take A, B and C as follows:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and

$$C = A + B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Obviously A, B and C satisfy the hypotheses of the problem and the computer shows

$$D = \sqrt{2}\, C - (A^2 + B^2)^{1/2} = \begin{pmatrix} 1.250757\ldots & -0.699400\ldots \\ -0.699400\ldots & 0.193951\ldots \end{pmatrix}.$$

The eigenvalues of D are $-0.154214\ldots$ and $1.598922\ldots$. Consequently

$$(A^2 + B^2)^{1/2} \nleq \sqrt{2}\, C.$$

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An Asymptotic Approximation Connected With the Golden Number

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1. Introduction. The golden number $\theta = (1 + \sqrt{5})/2$ is famous since antiquity for its association with the division of a straight line into the golden section and with Euclid's construction of the regular pentagon (Archibald [**2**]). The number θ is a remarkable algebraic number, on account of its additive and geometrical properties, and makes its appearance in analysis as the limit of several well-known infinite sequences. Examples are the infinite continued fraction with partial quotients all equal to unity and the limiting value of the ratio of successive terms of the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$. The subject of this note is the repeated square root sequence $\{u_n\}$ defined by

$$u_1 = 1, \quad u_n = (1 + u_{n-1})^{1/2}, \quad (n \geqslant 2) \tag{1}$$

the nth term of which can be represented in the form of $n - 1$ nested square roots as

$$u_n = \sqrt{1 + \cdots + \sqrt{1 + \sqrt{1 + 1}}}\ . \tag{2}$$

The limit of this sequence as $n \to \infty$ is readily verified to be the positive root of the quadratic equation (Hardy [6], Courant and Robbins [4], Coxeter [5])

$$\theta^2 = 1 + \theta \quad \text{or} \quad \theta = (1 + \sqrt{5})/2 = 1.6180339887\ldots.$$

Here, we shall not be concerned with the actual limiting value of (1) but with the considerably more subtle problem of the determination of the manner in which u_n approaches the limit θ. In this way it is possible to estimate the value of the expression involving nested radicals in (2) for large, but finite, values of n. This problem is a simple illustration of the important theory of iterated functions and affords a both instructive and interesting example in asymptotic analysis.

The convergence of u_n is found numerically to be quite rapid and, for illustrative purposes, values of u_n up to $n = 25$, together with the difference

$$v_n = \theta - u_n,$$

are presented in Table 1. A closer inspection of the difference sequence $\{v_n\}$ reveals that $\ln v_n$ decreases approximately linearly with n for $n \gtrsim 5$. This suggests that, for large n, v_n is described by a formula of the form $v_n \sim a/b^n$, where a and b are constants. In the first part of this note, we shall describe a simple asymptotic estimate for v_n as $n \to \infty$ and show how the constants a and b may be determined in terms of the golden number. Generalization of the repeated square root sequence by putting $u_1 = x$, where x is a real number, then enables us to obtain an estimate for the limiting value of the product of the successive convergents of (1). In a special case this is shown to lead to a rederivation of Vieta's expression for π.

Table 1

n	u_n	$v_n = \theta - u_n$
1	1.0000000000	0.618034
2	1.4142135624	0.203820
3	1.5537739740	0.642600×10^{-1}
4	1.5980531825	0.199808×10^{-1}
5	1.6118477541	0.618623×10^{-2}
10	1.6180165422	0.174465×10^{-4}
15	1.6180339396	0.491611×10^{-7}
20	1.6180339886	0.138527×10^{-9}
25	1.6180339887	0.390342×10^{-12}

2. An asymptotic approximation. To determine an asymptotic formula for v_n as $n \to \infty$, we apply the theory of iterated functions as described in de Bruijn [3]. We express v_n in the form

$$\begin{aligned} v_n &= f(v_{n-1}) = f_2(v_{n-2}) = \cdots = f_n(v_1) \quad (n \geqslant 2) \\ f(v) &= \theta - \sqrt{\theta^2 - v}, \end{aligned} \tag{3}$$

where f_n denotes the nth iterate of f and $v_1 = \theta - 1 = 1/\theta$. Considered as a function of the real variable v for $|v| < \theta^2$, the function $f(v)$ satisfies $|f(v)| < |v|/\theta$

($v \neq 0$), whence we obtain the upper bound

$$v_n < v_{n-1}/\theta < v_{n-2}/\theta^2 < \cdots < v_1/\theta^{n-1} = \theta^{-n} \quad (n \geqslant 2). \tag{4}$$

For $|v| < \theta^2$, the power series expansion of $f(v)$ is

$$f(v) = a_1 v + a_2 v^2 + a_3 v^3 + \cdots, \quad (|v| < \theta^2) \tag{5}$$

where the coefficients a_k $(k = 1, 2, \ldots)$ are given by

$$a_k = (-)^{k+1}\binom{\frac{1}{2}}{k}\theta^{1-2k} = (2\theta)^{1-2k}\frac{(2k-2)!}{k!(k-1)!}.$$

If we now write $f(v) = a_1 v\{1 + R(v)\}$, where $a_1 R(v) \equiv a_2 v + a_3 v^2 + \cdots$, the ratio between successive terms in the sequence is

$$v_{r+1}/v_r = a_1\{1 + R(v_r)\} \quad (r = 1, 2, \ldots).$$

Hence

$$v_n/v_1 = a_1^{n-1}\prod_{r=1}^{n-1}\{1 + R(v_r)\}. \tag{6}$$

As $n \to \infty$, the product on the right-hand side of (6) converges absolutely to a finite limit, since, from (4), $R(v_r) = O(\theta^{-r})$ for $r \to \infty$ and the sum $\Sigma_{r=1}^{r=\infty} R(v_r)$ is absolutely convergent. Therefore we can define an analytic function $F(v)$ such that at the points v_m, $m = 1, 2, \ldots$,

$$F(v_m) \equiv v_m \prod_{r=m}^{\infty}\{1 + R(v_r)\}.$$

The limiting value of v_n/a_1^{n-1} as $n \to \infty$ may then be expressed by

$$v_1 \prod_{r=1}^{\infty}\{1 + R(v_r)\} = F(v_1) = F(1/\theta). \tag{7}$$

The desired asymptotic formula for v_n is therefore

$$v_n \sim a_1^{n-1}F(1/\theta) = 2K/(2\theta)^n, \quad n \to \infty \tag{8}$$

where we have defined the constant K by

$$K \equiv \theta F(1/\theta).$$

To determine the value of K, we relate the function F at two consecutive points v_m and v_{m+1}. Recalling that $v_{m+1}/v_m = a_1\{1 + R(v_m)\}$, we find

$$\begin{aligned} F(v_{m+1}) &= v_{m+1}\prod_{r=m+1}^{\infty}\{1 + R(v_r)\} = a_1 v_m \prod_{r=m}^{\infty}\{1 + R(v_r)\} \\ &= a_1 F(v_m). \end{aligned}$$

Since, from (3), $v_{m+1} = f(v_m)$, it then follows that

$$F\left(\theta - \sqrt{\theta^2 - v_m}\right) = \frac{1}{2\theta}F(v_m).$$

Hence, considered as a function of the variable v, the limit function $F(v)$ must satisfy the functional equation

$$F(v) = 2\theta F\left(\theta - \sqrt{\theta^2 - v}\right), \quad \left(|v| < \theta^2\right) \tag{9}$$

subject to the conditions $F(0) = 0$ and $F'(0) = 1$.

This functional equation may be solved in the form of a power series by expanding $F(v)$ as

$$F(v) = v + b_2 v^2 + b_3 v^3 + \cdots \quad \left(|v| < \theta^2\right). \tag{10}$$

Inserting the expansions for $f(v)$ and $F(v)$ in (9) and equating successive coefficients of v^r, we can then determine recursively the coefficients $b_2, b_3, \dots$ in terms of $a_1, a_2, \dots$. With $c_r \equiv a_1(1 - a_1^{r-1})b_r$, we obtain

$$\begin{gathered} c_2 = a_2, \quad c_3 = a_3 + 2a_1a_2b_2, \quad c_4 = a_4 + 3a_1^2a_2b_3 + \left(2a_1a_3 + a_2^2\right)b_2, \\ c_5 = a_5 + 4a_1^3a_2b_4 + \left(3a_1^2a_3 + 3a_1a_2^2\right)b_3 + \left(2a_1a_4 + 2a_2a_3\right)b_2, \dots . \end{gathered} \tag{11}$$

From (10) and (11), the constant K may then be expressed in terms of the golden number as

$$\begin{aligned} K = 1 &+ \frac{1}{2\theta(2\theta - 1)} + \frac{1}{\theta(2\theta - 1)(4\theta^2 - 1)} + \frac{20\theta^2 + 1}{4\theta^2(2\theta - 1)(4\theta^2 - 1)(8\theta^3 - 1)} \\ &+ \frac{56\theta^3 + 4\theta + 3}{\theta(2\theta - 1)(4\theta^2 - 1)(8\theta^3 - 1)(16\theta^4 - 1)} \\ &+ \frac{224\theta^4(12\theta^3 + \theta + 1) + 112\theta^3 + 6\theta + 1}{2\theta^2(2\theta - 1)(4\theta^2 - 1)(8\theta^3 - 1)(16\theta^4 - 1)(32\theta^5 - 1)} + \cdots \\ &\simeq 1.098630. \end{aligned}$$

The values of v_n determined by this formula may be seen to agree closely for large values of n with those given in Table 1. It is surprising that, even for values of n as low as $n = 5$, the percentage error involved in using (8) is less than 0.1%. For comparison, the value of the constant K has been calculated numerically from both (1) and (3) and is found to have the value accurate to the eighth decimal figure 1.09864196. We remark that the formula (8) is only the leading term of the asymptotic expansion of v_n in descending powers of $(2\theta)^n$. The interested reader may find details about the refinement of this approximation described in de Bruijn [3].

3. A generalization of the repeated square root sequence. The repeated square root sequence (1) may be generalized by taking $u_1 = x$, where x denotes a real variable, and defining an iterated function $u_n(x)$ by

$$u_1 = x, \quad u_n(x) = \left\{\lambda + \mu u_{n-1}(x)\right\}^{1/2}, \qquad (n \geqslant 2) \tag{12}$$

where λ and μ are real constants. For convenience we shall only consider values of x for which (12) is real. If $\mu^2 + 4\lambda \geqslant 0$, the limit of the sequence, when it exists, is

real and independent of x and is given by

$$u_\infty = \frac{1}{2}\mu + \frac{1}{2}(\mu^2 + 4\lambda)^{1/2}.$$

The interval of x for which the sequence (12) converges to u_∞ may be determined by graphical representation of the iteration function $g(u) = (\lambda + \mu u)^{1/2}$, as illustrated, for example, in Laidler and Landau [9]. It is well known (e.g., Kreyszig [8]) that for the sequence $u_n = g(u_{n-1})$ to converge it is necessary that $|g'(u_\infty)| < 1$. Since $g'(u_\infty) = \mu/2u_\infty$, it is seen that when $\lambda > 0$ and $\mu > 0$, the sequence (12) converges for $x > -\lambda/\mu$, the convergence being respectively monotonically increasing or decreasing according as $x < u_\infty$ or $x > u_\infty$. When $\lambda < 0$ and $\mu > 0$, with $\mu^2 + 4\lambda \geqslant 0$, (12) converges in a similar manner provided $x > \{\mu - (\mu^2 + 4\lambda)^{1/2}\}/2$. The case $\lambda > 0$ and $\mu < 0$ is more involved since convergence to u_∞ (when $\lambda > 3\mu^2/4$ so that $|g'(u_\infty)| < 1$) is now oscillatory. We do not discuss this case any further here, but simply remark that the convergence properties of (12) are ideally suited for numerical experimentation with an ordinary pocket calculator.

The difference sequence $\{v_n(x)\}$, defined by $v_n(x) = u_\infty - u_n(x)$, is given as in (3), where now

$$f(v) = u_\infty - \left(u_\infty^2 - \mu v\right)^{1/2}.$$

When $|\mu v| < u_\infty^2$, the coefficients a_k in (5) are

$$a_k = \mu^k (2u_\infty)^{1-2k} \frac{(2k-2)!}{k!(k-1)!}.$$

Proceeding as in Section 2, we can then show that the asymptotic approximation for $v_n(x)$ takes the form, provided $|a_1| < 1$,

$$v_n(x) \sim (\mu/2u_\infty)^{n-1} F(u_\infty - x), \quad n \to \infty \quad \left(|\mu v| < u_\infty^2\right) \tag{13}$$

where the limit function $F(v)$ satisfies the functional equation

$$(\mu/2u_\infty) F(v) = F\left(u_\infty - \sqrt{u_\infty^2 - \mu v}\right) \tag{14}$$

with, as before, $F(0) = 0$, $F'(0) = 1$. The coefficients $b_2, b_3, \ldots$ in the expansion (10) are given in terms of the a_k above by (11).

These asymptotic results may be employed to estimate the value of the product of the successive convergents of (12) as $n \to \infty$. By noticing that the derivative of $u_n(x)$ with respect to x is

$$u_n'(x) = \frac{1}{2}\mu u_{n-1}'(x)/u_n(x) = (\mu/2)^{n-1} x \Big/ \prod_{r=1}^{n} u_r(x),$$

we find, using (13),

$$\lim_{n\to\infty} u_\infty^{-n} \prod_{r=1}^{n} u_r(x) = x/\{u_\infty F'(u_\infty - x)\}, \quad \left(|\mu v| < u_\infty^2\right) \tag{15}$$

where the prime denotes differentiation with respect to the argument. The limit of

the product of the successive convergents of (1), corresponding to $\lambda = \mu = 1$, is therefore given by

$$\lim_{n\to\infty} \theta^{-n} \prod_{r=1}^{n} u_r(1) = 1/\{\theta F'(1/\theta)\}. \tag{16}$$

By differentiation of the expansion for $F(v)$, we find

$$\begin{aligned}\theta F'(1/\theta) = \theta &+ \frac{1}{\theta(2\theta-1)} + \frac{3}{\theta^2(2\theta-1)(4\theta^2-1)} \\ &+ \frac{20\theta^2+1}{\theta^4(2\theta-1)(4\theta^2-1)(8\theta^3-1)} \\ &+ \frac{5(56\theta^3+4\theta+3)}{\theta^4(2\theta-1)(4\theta^2-1)(8\theta^3-1)(16\theta^4-1)} \\ &+ \frac{3\{224(12\theta^3+\theta+1)+112\theta^3+6\theta+1\}}{\theta^6(2\theta-1)(4\theta^2-1)(8\theta^3-1)(16\theta^4-1)(32\theta^5-1)} + \cdots .\end{aligned}$$

As a numerical verification, the sum of the first six terms in the above expansion yields the value of $\{\theta F'(1/\theta)\}^{-1} = 0.50953$. This may be compared with the numerically calculated value of the left-hand side of (16) of $0.509490\ldots$.

To conclude, we mention two special cases in which the functional equation (14) can be solved explicitly, so that the limits (13) and (15) may be expressed in closed form. The trivial case $\lambda = 0$ and $\mu > 0$ has the solution $F(v) = -\mu \ln\{1-(v/\mu)\}$, and the corresponding limit in (15) may easily be verified to be $(x/\mu)^2$. A more interesting situation is $\lambda = 2\mu^2$, $\mu > 0$, where $u_\infty = 2\mu$ and $x > -2\mu$ for convergence. The functional equation (14) in this case may be expressed more conveniently in the form

$$F'(v) = \{1-(v/2u_\infty)\}^{-1/2} F'\Big(u_\infty\big\{1-\sqrt{1-(v/2u_\infty)}\big\}\Big),$$

where the prime again denotes differentiation with respect to the argument concerned. From the quadratic transformation properties of the Gauss hypergeometric function (see Abramowitz and Stegun [**1**]), this equation may be seen to have the solution

$$F'(v) = {}_2F_1\left(1,1;\frac{3}{2};v/2u_\infty\right),$$

which satisfies the condition $F'(0) = 1$. On account of the special values of the parameters, this hypergeometric function may be expressed in terms of elementary functions, so that

$$\begin{aligned}F'(v) &= (v/2u_\infty)^{-1/2}\frac{\arcsin\{(v/2u_\infty)^{1/2}\}}{\sqrt{1-(v/2u_\infty)}} = \frac{\arccos(1-v/u_\infty)}{\sqrt{1-(1-v/u_\infty)^2}} \quad (v>0) \\ &= \frac{\operatorname{arccosh}(1-v/u_\infty)}{\sqrt{(1-v/u_\infty)^2-1}} \quad (v<0).\end{aligned} \tag{17}$$

Thus, in the particular case $\lambda = \mu = 1/2$ (that is $u_\infty = 1$), we find from (15) the algebraic form of Euler's product for the sine function (Hobson [7])

$$\sqrt{\frac{1}{2}+\frac{1}{2}x}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}x}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}x}}}\cdots$$

$$=\begin{cases}\dfrac{\sqrt{1-x^2}}{\arccos x}, & (|x|<1),\\ 1, & (x=1),\\ \dfrac{\sqrt{x^2-1}}{\operatorname{arccosh} x}, & (x>1).\end{cases}$$

In the special case $x = 0$ we recover Vieta's famous expression for π

$$\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots = 2/\pi.$$

Integration of (17) between the limits 0 and v (> 0), to find $F(v) = \{\arccos(1 - v)\}^2/2$, then shows that the limit (13) in this case may be expressed exactly as

$$v_n(x) \sim 2^{1-2n}\arccos^2 x, \quad n \to \infty \quad (|x| < 1;\ \lambda = \mu = 1/2).$$

The reader will notice, however, that other values of λ and μ (> 0) satisfying $\lambda = 2\mu^2$ can always be represented as the particular case with $\lambda = \mu = 1/2$, since (12) may then be rewritten as

$$\tilde{u}_n = \left(\frac{1}{2} + \frac{1}{2}\tilde{u}_{n-1}\right)^{1/2}, \quad \tilde{u}_n \equiv u_n/2\mu.$$

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Duality in Nonnormal Quartic Fields

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In this note the 'incredible identities' of D. Shanks [3] which equate pairs of algebraic numbers such as $\sqrt{5} + \sqrt{22 + 2\sqrt{5}}$ and $\sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 + 10\sqrt{29}}}$ are shown to be really quite credible after all. Behind them lies a simple and elegant theory of quartic extensions of the field of rational numbers Q (extensions of Q obtained by adding the roots of an irreducible fourth order polynomial).

Let's start with recalling the definition of a nonnormal extension, since it is this type of quartic extension with which we are concerned. Two rational extensions are called conjugate over Q if they may both be generated by roots of the same minimal polynomial over Q. That is, $Q(\theta)$ and $Q(\theta')$ are conjugate if there exists an irreducible polynomial $f(x)$ in $Q[x]$ such that $f(\theta) = f(\theta') = 0$. A normal extension is one that coincides with all its conjugates. Similarly, a normal polynomial is an irreducible polynomial each of whose roots generates the same field, and two different polynomials whose roots generate the same set of conjugate fields are termed equivalent, all equivalent polynomials defining an equivalence class. Every field is the subfield of a normal field, in particular of its normal closure or splitting field $Q(\theta, \theta', \dots)$ where $\theta, \theta', \dots$ are all the roots of an irreducible polynomial. When we talk of the Galois group of an extension $Q(\theta)$ we mean the group of automorphisms of θ and all its conjugates, which has the same order as the degree of the splitting field over Q.

We first show that all nonnormal quartic extensions of Q have dihedral Galois groups and biquadratic minimal polynomials (irreducible polynomials of the form $x^4 + ax^2 + b$, $a, b \in Q$). Then we describe the equivalence classes of nonnormal biquadratic polynomials. Finally we take a closer look at the complete set of quartic fields having a given dihedral Galois group and find that there are exactly four: two pairs of conjugate extensions over Q. They are of the form

$$Q(\theta), Q(\theta'), Q(\theta + \theta'), Q(\theta - \theta') \quad \text{where } \theta^2 = \frac{a + \sqrt{d}}{2} \left(a, d \in Q, \sqrt{d} \notin Q\right).$$

It is the relationship between these fields that lies behind the 'incredible identities.' Our understanding of these fields may be enhanced by the following alternative characterisations.

THEOREM. *The following statements are equivalent.*

(i) *K is a nonnormal quartic;*

(ii) *K has a nonnormal biquadratic minimal polynomial;*

(iii) *K has a dihedral Galois group.*

Proof (i) $\Rightarrow$ (ii): Let K have quadratic subfield $Q(\sqrt{t})$ where t is a square-free integer, so $K = Q(\sqrt{\alpha})$ where $\alpha \in Q(\sqrt{t})$. If $\alpha = r + s\sqrt{t}$ for r, $s \in Q$ then the minimal polynomial for $\sqrt{\alpha}$ over $Q(\sqrt{t})$ is $x^2 - (r + s\sqrt{t})$ and that for $\sqrt{\alpha}$ over Q is

$$\left(x^2 - (r + s\sqrt{t})\right)\left(x^2 - (r - s\sqrt{t}\right)$$

since $\sqrt{t}$ has conjugate $-\sqrt{t}$ over Q. This is of the form $x^4 + ax^2 + b$ and it is irreducible over Q and nonnormal since K is a quartic, nonnormal extension of Q.

(ii) $\Rightarrow$ (iii): Let the generator of K have minimal polynomial $x^4 + ax^2 + b$ with roots $\pm\sqrt{\alpha}$, $\pm\sqrt{\bar{\alpha}}$ where

$$\alpha = \frac{-a + \sqrt{d}}{2} \quad \text{and} \quad \bar{\alpha} = \frac{-a - \sqrt{d}}{2}$$

for $d = a^2 - 4b$. If $K = Q(\sqrt{\alpha})$ then $K' = Q(\sqrt{\bar{\alpha}})$ is its only distinct conjugate and their splitting field $Q(\sqrt{\alpha}, \sqrt{\bar{\alpha}})$ has degree 8 over Q. Hence its Galois group is isomorphic to a Sylow 2 subgroup of S_4, which is isomorphic to D_4.

(iii) $\Rightarrow$ (i): Since the Galois group G of K is isomorphic to D_4 we could represent G in the form

$$\left\{\sigma, \tau \mid \sigma^4 = \tau^2 = e, \quad \tau^{-1}\sigma\tau = \sigma^3\right\}.$$

But G may also be regarded as a permutation group on the roots θ_1, θ_2, θ_3, θ_4 which generate K and its conjugates. Suppose we have numbered the roots in such a way that $\sigma = (1234)$. Since σ commutes only with its powers, the equation $x^{-1}\sigma x = \sigma^3$ has precisely four solutions in S_4: if τ is one solution, the other three are $\sigma\tau$, $\sigma^2\tau$ and $\sigma^3\tau$. But all four solutions are in G so we may take τ to be any solution, say $\tau = (13)$, and now we may write down the whole of G as

$$\{e, (1234), (13)(24), (1432), (13), (14)(32), (24), (12)(34)\}.$$

Now consider the subgroups of G which fix θ_1, θ_2, θ_3 and θ_4: $\{e, (24)\}$ fixes both θ_1 and θ_3, so we may put $K = Q(\theta_1) = Q(\theta_3)$, $\{e, (13)\}$ has fixed field $K' = Q(\theta_2) = Q(\theta_4)$ and both are subgroups of

$$H = \{e, (13), (24), (13)(24)\},$$

which itself fixes the element $\gamma = \theta_1\theta_3 - \theta_2\theta_4$. Thus K and K' are quadratic extensions of $Q(\gamma)$, which itself is a quadratic extension of Q because the whole of G fixes γ^2.

This proves the theorem, but a further point arises from its proof: that $Q(\gamma)$ is the unique quadratic subfield of K since H is the only subgroup of index 2 in G. We'll need this later for a closer examination of the structure of these fields.

Of the three criteria in the theorem the second is the most tangible. So let's take a closer look at biquadratic polynomials... and discover that it is possible to spot whether two such polynomials have roots which generate the same quartic exten-

sions by a short examination of their coefficients. Let

$$\alpha^2 + a\alpha + b = \beta^2 + a'\beta + b' = 0 \quad \text{where} \quad Q(\sqrt{\alpha}) = Q(\sqrt{\beta}).$$

Then how are a and b related to a' and b'? Since $Q(\sqrt{d})$ is the unique quadratic subfield of $Q(\sqrt{\alpha})$, $Q(\sqrt{\alpha}) = Q(\sqrt{\beta})$ if and only if $\beta = \gamma^2\alpha$ for some $\gamma \in Q(\sqrt{d})$. We may write γ as $u + v\alpha$ for $u, v \in Q$ since $Q(\sqrt{d}) = Q(\alpha)$, so $\beta = (u + v\alpha)^2\alpha$; and calculating the coefficients of a quadratic in terms of its roots gives

$$a' = -au^2 + 2(a^2 - 2b)uv + (3ab - a^3)v^2$$

and

$$b' = (u^2 - auv + v^2b)^2 b.$$

After the transformation $w = u - av$ we conclude that all nonnormal irreducible biquadratic polynomials of the form

$$x^4 + (aw^2 + 4bwv + abv^2)x^2 + b(w^2 + awv + bv^2)$$

$w, v \in Q$ are equivalent. All we need now is an easy criterion for deciding when such polynomials are nonnormal and irreducible.

LEMMA. *Let $f(x) = x^4 + ax^2 + b$ where $a, b \in Q$ and put $d = a^2 - 4b$. Then $f(x)$ is irreducible and nonnormal if and only if none of $\sqrt{b}$, $\sqrt{d}$ or $\sqrt{d/b}$ are rational.*

Proof. Capelli's theorem [2] states that the rational polynomial $g(h(x))$ is reducible over Q if and only if either $g(x)$ is reducible over Q or $h(x) - \theta$ is reducible over $Q(\theta)$, where θ is a root of $g(x)$. Applying this result to $f(x) = g(x^2)$ shows that $f(x)$ is irreducible if and only if neither $\sqrt{d}$ is rational nor $(-a \pm \sqrt{d})/2$ is a square in $Q(\sqrt{d})$. After a short calculation (equating coefficients in the identity $\alpha = (u + v\sqrt{d})^2$) this second condition reduces to the condition that $\sqrt{b} \neq \frac{1}{2}(a + 4u^2)$ for any $u \in Q$. If $f(x)$ is irreducible its roots define a pair of quartic extensions $Q(\sqrt{\alpha})$ and $Q(\sqrt{\bar{\alpha}})$ where $\alpha = (-a + \sqrt{d})/2$ and $\bar{\alpha} = (-a - \sqrt{d})/2$, which possess the common quadratic subfield $Q(\sqrt{d})$. Now $f(x)$ is normal if and only if $Q(\sqrt{\alpha}) = Q(\sqrt{\bar{\alpha}})$, that is if and only if $\alpha = \gamma^2\bar{\alpha}$ for some $\gamma \in Q(\sqrt{d})$. But if this is the case then $b = (\gamma\bar{\alpha})^2$ and so $\sqrt{b} \in Q(\sqrt{d})$. Conversely, if $\sqrt{b} \in Q(\sqrt{d})$ then $\alpha\bar{\alpha} = \lambda^2$ for some $\lambda \in Q(\sqrt{d})$, and $\alpha = \gamma^2\bar{\alpha}$ where $\gamma = \lambda/\bar{\alpha}$. Thus $f(x)$ is normal if and only if $\sqrt{b} \in Q(\sqrt{d})$ in which case either $\sqrt{b} \in Q$ or $\sqrt{d/b} \in Q$.

Now let N be a particular normal extension of degree 8 over Q with Galois group D_4, say $N = Q(\sqrt{\alpha}, \sqrt{\bar{\alpha}})$ for some roots $\sqrt{\alpha}, \sqrt{\bar{\alpha}}$ of a biquadratic polynomial $f(x) = x^4 + ax^2 + b$. What are its quartic subfields? Recall the permutation representation of the Galois group is

$$\{e, (1234), (13)(24), (1432), (13), (14)(32), (24), (12)(34)\},$$

where $\sqrt{\alpha}, \sqrt{\bar{\alpha}}, -\sqrt{\alpha}, -\sqrt{\bar{\alpha}}$ are indexed 1, 2, 3, 4 respectively. The quartic subfields

of N are fixed by the subgroups of D_4 of order 2, of which there are exactly 5:

Generator of subgroup	Fixed field
(13)	$(Q(\sqrt{\bar{\alpha}})$
(24)	$Q(\sqrt{\alpha})$
(12)(34)	$Q(\sqrt{\alpha} + \sqrt{\bar{\alpha}})$
(14)(32)	$Q(\sqrt{\alpha} - \sqrt{\bar{\alpha}})$
(13)(24)	$Q(\alpha + \sqrt{\alpha\bar{\alpha}} + \bar{\alpha})$

$Q(\alpha + \sqrt{\alpha\bar{\alpha}} + \bar{\alpha})$ is the normal quartic subfield of N: for $Q(\alpha + \sqrt{\alpha\bar{\alpha}} + \bar{\alpha}) = Q(\pm\sqrt{d} \pm \sqrt{b})$ is generated by roots of $x^4 - 2(d+b)x^2 + (d-b)^2$, where $d = a^2 - 4b$. This is irreducible and therefore normal by the lemma. $Q(\sqrt{\alpha})$ and $Q(\sqrt{\bar{\alpha}})$ are the conjugate pair of nonnormal quartic fields generated by the roots of $f(x)$. How are they related to the other pair of nonnormal quartics $Q(\sqrt{\alpha} + \sqrt{\bar{\alpha}})$ and $Q(\sqrt{\alpha} - \sqrt{\bar{\alpha}})$? Let us put

$$\begin{aligned} \beta &= (\sqrt{\alpha} + \sqrt{\bar{\alpha}})^2 = -a + 2\sqrt{b}, \\ \tilde{\beta} &= (\sqrt{\alpha} - \sqrt{\bar{\alpha}})^2 = -a - 2\sqrt{b}. \end{aligned} \tag{1}$$

Then β and $\tilde{\beta}$ are the nonzero roots of the cubic resolvent of $f(x)$, $x^3 - 2ax + dx$ (the cubic resolvent of a general quartic equation with roots $\theta_1, \dots, \theta_4$ is the cubic equation with roots $(\theta_1 + \theta_2)(\theta_3 + \theta_4)$, $(\theta_1 + \theta_3)(\theta_2 + \theta_4)$, and $(\theta_1 + \theta_4)(\theta_2 + \theta_3)$).

This construction may be used to calculate the roots of a quartic equation [**4**]: Cardano's formulae give the roots of $f(x)$ in the form $\frac{1}{2}(\pm\sqrt{\beta} \pm \sqrt{\tilde{\beta}})$ instead of $\pm\sqrt{\alpha}$, $\pm\sqrt{\bar{\alpha}}$. But of course the roots are the same whichever way they are written, as can be seen from

$$\begin{aligned} \sqrt{\alpha} + \sqrt{\alpha} &= \sqrt{\beta}, \\ \sqrt{\beta} + \sqrt{\tilde{\beta}} &= 2\sqrt{\alpha}. \end{aligned} \tag{2}$$

We now have a complete picture of the chains of subfields of N:

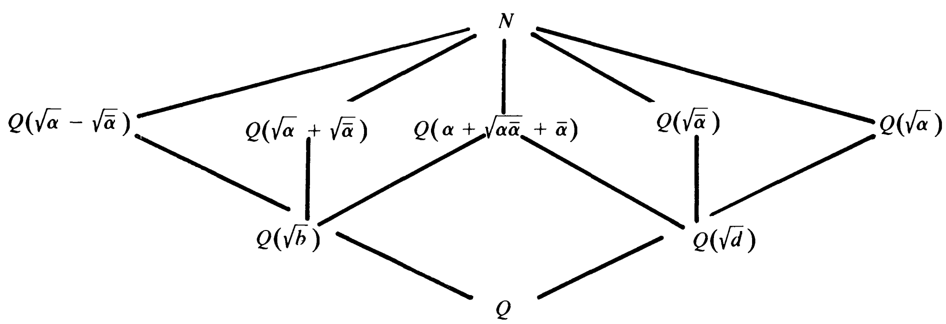

FIG. 1

Of course the quartic fields could have been written

$$Q(\sqrt{\beta}),\quad Q\left(\sqrt{\tilde{\beta}}\right),\quad Q\left(\beta+\sqrt{\beta\tilde{\beta}}+\tilde{\beta}\right),\quad Q\left(\sqrt{\beta}+\sqrt{\tilde{\beta}}\right),\quad Q\left(\sqrt{\beta}-\sqrt{\tilde{\beta}}\right)$$

which illustrates the duality of the two nonnormal towers leading to N: each may be generated by surds of the nonzero roots of the cubic resolvent of a biquadratic minimal polynomial of the other! Further, if $K = Q(\sqrt{\alpha})$ then $K^* = Q(\sqrt{\beta})$ and by (2) $K^{**} = K$. The two minimal polynomials may be written

$$f(x) = x^4 + ax^2 + b$$

and

$$f^*(x) = x^4 + 2ax^2 + d$$

which are not dual because $f^{**} \neq f$ although their roots generate the same extensions.

Well, what has all this to do with the incredible identities? The two polynomials which Shanks uses to construct his identities are equivalent to nonnormal biquadratic polynomials whose roots generate dual quartic fields. When the polynomials are written in biquadratic form the 'incredible identities' are immediate consequences of (2).

Now we can go through the examples in Shanks' article: Let

$$A = \sqrt{5} + \sqrt{22+2\sqrt{5}} \quad\text{and}\quad B = \sqrt{11+2\sqrt{29}} + \sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}$$

so we wish to explain the identity $A = B$. Put $\alpha = (11+\sqrt{5})/2$ so $\sqrt{\alpha}$ has minimal polynomial $f(x) = x^4 - 11x^2 + 29$. Since $\sqrt{\alpha} = \frac{1}{2}(A - \sqrt{5})$ and $\sqrt{5} \in Q(A)$, $Q(\sqrt{\alpha})$ is a subfield of $Q(A)$. But both are extensions of degree 4 over Q, so $Q(\sqrt{\alpha}) = Q(A)$ and $f(x)$ is equivalent to the minimal polynomial of A, $x^4 - x^3 - 3x^2 + x + 1$, the first of the quartics in Shanks' article. Now put $\beta = 11 + 2\sqrt{29}$ as in (1), so that by (2)

$$22 + 2\sqrt{5} = \left(\sqrt{11+2\sqrt{29}} + \sqrt{11-2\sqrt{29}}\right)^2,$$

so

$$A = \sqrt{5} + \sqrt{11+2\sqrt{29}} + \sqrt{11-2\sqrt{29}}.$$

Now it is easy to see that $A = B$ because

$$\left(5+\sqrt{11-2\sqrt{29}}\right)^2 = \sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}.$$

The Hasse Diagram for the dual field is

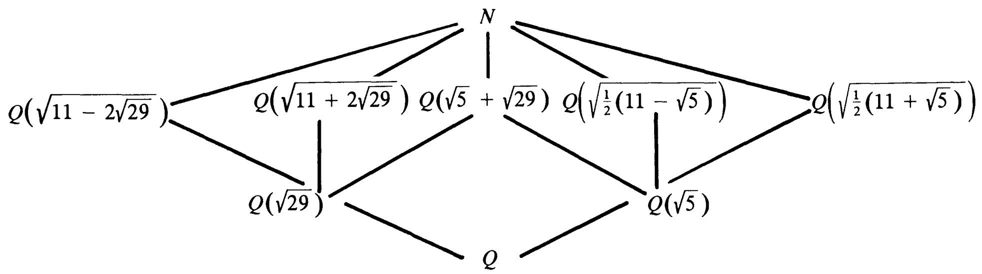

FIG. 2

Finally we show that the second of the quartics in the article, $x^4 - x^3 - 5x^2 - x + 1$, is equivalent to the minimal polynomial of the dual fields, that is $f^*(x) = x^4 - 22x^2 + 5$. Using Cardano's formulae we find that the largest root of Shanks' quartic is $\frac{1}{4}(C+1)$, where

$$C = \sqrt{29} + \sqrt{7+2\sqrt{5}} + \sqrt{7-2\sqrt{5}}\,.$$

But (2) yields $C = \sqrt{29} + \sqrt{14+2\sqrt{29}}$, so that $Q(C) = Q(\sqrt{14+2\sqrt{29}})$. Finally, since

$$14 + 2\sqrt{29} = (5+\sqrt{29})^2(11-2\sqrt{29}),$$

this field is conjugate to $Q(\sqrt{11} + 2\sqrt{29})$.

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Romberg Integration by Taylor Series

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The Trapezoidal Rule provides a simple method for approximating the value of the definite integral $I = \int_a^b f(x)\,dx$ whenever f is a continuous function on the interval $[a, b]$. The main point of this article is to prove, using nothing more

advanced than Taylor Series, that the error in the Trapezoidal Rule can be expressed as a series in even powers of the step size h provided that the Taylor Series for f converges for every x_0 in $[a, b]$. (This result is not new but the proof is simpler than others with which I am familiar.) Specifically, if we define the Trapezoidal Rule $T(h)$ by

$$T(h) = \big(f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big)\frac{h}{2}$$

(where n is an integer greater than one, $h = (b-a)/(n-1)$ and $x_i = a + (i-1)h$ for $i = 1, 2, \ldots, n$), then we will show that

$$I - T(h) = a_2h^2 + a_4h^4 + a_6h^6 + \cdots .$$

(We actually prove a bit more; see equation (1) below.) This result is essential for producing more accurate approximations from $T(h)$: for example, if we cut the step size h in half, we obtain

$$I - T(h/2) = a_2h^2/4 + a_4h^4/4^2 + a_6h^6/4^3 + \cdots .$$

A little algebra shows that

$$I - \frac{4T(h/2) - T(h)}{3} = -\frac{1}{4}a_4h^4 - \frac{5}{16}a_6h^6 - \cdots,$$

and we have obtained a more accurate fourth order approximation from a second order approximation. (This technique of obtaining higher order approximations from lower order ones is dubbed *Richardson's Extrapolation*. In this case, it's not hard to see that we have just obtained Simpson's Rule!)

It is clear that this procedure can be continued indefinitely; the resulting method is called *Romberg Integration*: Start with $h = b - a$ and define a lower triangular array,

$$\begin{array}{llll} R_{1,1} & & & \\ R_{2,1} & R_{2,2} & & \\ R_{3,1} & R_{3,2} & R_{3,3} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

by the formulas

$$R_{n,1} = T\big(h/2^{n-1}\big) \quad \text{for } n = 1, 2, 3, \ldots$$

and

$$R_{n,m+1} = \frac{4^m R_{n,m} - R_{n-1,m}}{4^m - 1} \quad \text{for } m = 1, 2, \ldots, n-1.$$

Then a relatively simple induction argument may be given to show that each entry in column m is an order h^{2m} approximation to I and that the last entry in row n has order h^{2n}.

All the steps in this derivation of Romberg integration are standard. The only real difficulty is to prove (1) below. This is usually done by an appeal to the "Euler-Maclaurin Formula" which gives the explicit form of the constants involved in terms of Bernoulli numbers. The proofs of this result with which I am familiar are not accessible to most undergraduates: they either use many special properties of the Bernoulli polynomials [**1**, pp. 136–140] or are based on the Peano kernel formulation of the error in the Trapezoidal Rule and integration by parts [**2**, pp. 222, 250–252]. The derivation given here, however, uses Taylor series in a manner analogous to that often used in a first course in numerical analysis to derive the expansion for the central difference approximation to the derivative in even powers of h.

Before giving the proof, we need a little more notation: For simplicity, assume that f has a convergent Taylor series on $[a, b]$ for every x_0 in $[a, b]$. The Trapezoidal Rule applied to $f^{(k)}$ may be written as

$$T_k := \left(\sum_{i=1}^{n-1} f^{(k)}(x_i) + \sum_{i=1}^{n-1} f^{(k)}(x_{i+1}) \right) \frac{h}{2}.$$

Furthermore, define

$$I_k := \int_a^b f^{(k)}(x)\, dx.$$

Then, for $k > 0$, $I_k = f^{(k-1)}(b) - f^{(k-1)}(a)$. Also let

$$E_k := I_k - T_k.$$

We wish to show that

$$E_0 = \sum_{k=1}^{m} c_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] h^{2k} + O(h^{2(m+1)}), \tag{1}$$

where c_{2k} are constants. (The Euler-Maclaurin Formula asserts that

$$c_{2k} = -\frac{B_{2k}}{(2k)!},$$

where the B_{2k} are the Bernoulli constants. For completeness, we will give a (nontrivial) method for generating the values of c_{2k} without reference to the Bernoulli constants, although these are not needed for Romberg integration.)

Now to the proof: The Taylor Series for f is

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!}.$$

Replacing x_0 by x_i, integrating from x_i to x_{i+1} and using $h = x_{i+1} - x_i$ yields:

$$\int_{x_i}^{x_{i+1}} f(x)\, dx = \sum_{k=0}^{\infty} f^{(k)}(x_i) \frac{h^{k+1}}{(k+1)!}.$$

Next, replace x_0 by x_{i+1} and integrate from x_i to x_{i+1} again:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \sum_{k=0}^{\infty} f^{(k)}(x_{i+1})(-1)^k \frac{h^{k+1}}{(k+1)!}.$$

If we take the average of the last two integrals and sum from $i = 1$ to $n - 1$, the even values of k will yield T_k and the odd values of k will all cancel out except at the endpoints, a and b. Specifically,

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty}\left[\left(\sum_{i=1}^{n-1} f^{(2k)}(x_i) + \sum_{i=1}^{n-1} f^{(2k)}(x_{i+1})\right)\frac{h}{2}\right]\frac{h^{2k}}{(2k+1)!}$$
$$- \sum_{k=1}^{\infty}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)\frac{h^{2k}}{2(2k)!}.$$

Using the notation introduced above and rearranging terms, we obtain

$$I_0 - T_0 = \sum_{k=1}^{\infty}\left(-\left(\frac{2k+1}{2}\right)I_{2k} + T_{2k}\right)\frac{h^{2k}}{(2k+1)!}.$$

Adding and subtracting I_{2k} yields

$$I_0 - T_0 = \sum_{k=1}^{\infty}\left(-\left(\frac{2k-1}{2}\right)I_{2k} - (I_{2k} - T_{2k})\right)\frac{h^{2k}}{(2k+1)!}. \tag{2}$$

It is now necessary to eliminate the differences $I_{2k} - T_{2k}$ from (2). But (2) can be applied in turn to $I_2 - T_2$, $I_4 - T_4$, etc. Thus we can see that we may repeatedly substitute a series expansion for $I_2 - T_2$, $I_4 - T_4, \ldots, I_{2m} - T_{2m}$ in higher even powers of h and obtain a series of the form

$$E_0 = \sum_{k=1}^{m} c_{2k} I_{2k} h^{2k} + O(h^{2(m+1)})$$
$$= \sum_{k=1}^{m} c_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]h^{2k} + O(h^{2(m+1)})$$

as desired.

Actually doing the substitutions for $I_{2k} - T_{2k}$ and generating a formula for c_{2k} requires further notation: Define

$$a_{2k} := -\frac{2k-1}{2(2k+1)!},$$
$$b_{2k} := -\frac{1}{(2k+1)!}.$$

Then (2) takes the form:

$$E_0 = \sum_{k=1}^{\infty}(a_{2k} I_{2k} + b_{2k} E_{2k})h^{2k}.$$

Applying this result to $f^{(2j)}$ yields

$$E_{2j} = \sum_{k=1}^{\infty} \left(a_{2k} I_{2(k+j)} + b_{2k} E_{2(k+j)}\right) h^{2k}.$$

Next define a sequence p_{2j} recursively by

$$p_0 := 1;$$

$$p_{2j} := \sum_{i=0}^{j-1} b_{2(j-i)} p_{2i}.$$

(Note: p_{2j} is the sum of all products of the b's where the subscripts in the product add up to $2j$.) It can then be shown by induction that, after substituting for $E_2, E_4, \ldots, E_{2m}$, we get

$$E_0 = \sum_{k=1}^{m} c_{2k} I_{2k} h^{2k} + \sum_{k=m+1}^{\infty} \left(c_{2k}^{(m)} I_{2k} + d_{2k}^{(m)} E_{2k}\right) h^{2k},$$

where

$$c_{2k}^{(m)} := \sum_{j=0}^{m} a_{2(k-j)} p_{2j} \quad \text{for } k > m,$$

$$d_{2k}^{(m)} := \sum_{j=0}^{m} b_{2(k-j)} p_{2j} \quad \text{for } k > m,$$

and, finally,

$$c_{2k} := \sum_{j=0}^{k-1} a_{2(k-j)} p_{2j} \quad \text{for } k \leqslant m.$$

The first few terms are:

$$c_2 = a_2 = -\frac{1}{12},$$

$$p_2 = b_2 = -\frac{1}{6},$$

$$c_4 = a_4 + a_2 p_2 = -\frac{3}{240} + \frac{1}{12}\frac{1}{6} = \frac{1}{720},$$

$$p_4 = b_4 + b_2 p_2 = -\frac{1}{120} + \frac{1}{6}\frac{1}{6} = \frac{7}{360},$$

$$c_6 = a_6 + a_4 p_2 + a_2 p_4 = -\frac{5}{10080} + \frac{3}{240}\frac{1}{6} - \frac{1}{12}\frac{7}{360} = -\frac{1}{30240}.$$

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Note on Characteristic Polynomials

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It is sometimes useful to know the norm of a 2×2 complex matrix A as an operator on 2-dimensional complex Euclidean space. The standard method for computing this is to take the square root of the largest root of the polynomial $\det(A^*A + \lambda I)$ ([3], p. 17). The method is cumbersome for higher order matrices, but there is a striking general identity for $\det(A^*A + \lambda I)$ that is evidently not well known.

For any $n \times n$ complex matrix A and any $j = 1, \ldots, n$, let $N_j(A)$ be the sum of the squares of the moduli of all minors of A of order j; set $N_0(A) = 1$.

THEOREM. *For any $n \times n$ complex matrix A and any complex number λ,*

$$\det(A^*A + \lambda I) = \sum_{j=0}^{n} N_{n-j}(A)\lambda^j. \tag{1}$$

In particular, for each $j = 1, \ldots, n$, $N_j(A) = e_j(\lambda_1, \ldots, \lambda_n)$ where e_j is the jth elementary symmetric function and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A*A.

The quantities $N_1(A), \ldots, N_n(A)$ are therefore unitary invariants. The norm of A as an operator on n-dimensional complex Euclidean space is a function of these invariants.

We outline two proofs of the identity (1). The first is very computational but straightforward. The second has greater elegance.

First proof. We write subscripts of the form $jk \ldots | pq \ldots$ on a determinant or matrix to indicate that rows $j, k, \ldots$ and columns $p, q, \ldots$ are deleted; a dash, —, means that nothing is deleted. Let

$$\det(A^*A + \lambda I) = P(\lambda) = \sum_{j=0}^{n} P_j \lambda^j.$$

Clearly $P_0 = |\det A|^2 = N_n(A)$ and $P_n = 1 = N_0(A)$. Fix $r = 1, \ldots, n-1$. Using the standard rule for differentiating determinants, we obtain, by an inductive argument,

$$P^{(r)}(\lambda) = r! \textstyle\sum_{1 \le k_1 < \cdots < k_r \le n} \det_{k_1 \ldots k_r | k_1 \ldots k_r}(A^*A + \lambda I),$$

and so

$$P_r = \textstyle\sum_{1 \le k_1 < \cdots < k_r \le n} \det_{k_1 \ldots k_r | k_1 \ldots k_r}(A^*A).$$

[1]Supported by NSF Grant DMS-84-06384.

For any term in the last sum, we have

$$\det_{k_1 \dots k_r | k_1 \dots k_r}(A^*A) = \det\left[\left(A_{-|k_1 \dots k_r}\right)^* A_{-|k_1 \dots k_r}\right]$$
$$= \sum\nolimits_{1 \le j_1 < \dots < j_r \le n} |\det_{j_1 \dots j_r | k_1 \dots k_r}(A)|^2,$$

where the last equality follows by the general product formula for determinants. Therefore $P_r = N_{n-r}(A)$, and (1) follows.

Second proof (due to P. M. Cohn). For any $r = 1, \dots, n$, let $A^{(r)}$ be the rth compound of A. This is defined as the square matrix of order $\binom{n}{r} \times \binom{n}{r}$ formed from all r-order minors of A relative to, say, the lexicographic ordering of the row and column positions. The Binet-Cauchy identity (see [**1**], p. 78, or [**2**], p. 21) asserts that

$$(AB)^{(r)} = A^{(r)}B^{(r)}$$

for any two $n \times n$ matrices A and B. Since also $(A^*)^{(r)} = (A^{(r)})^*$, the rth compound of a unitary matrix is unitary. One easily sees that

$$N_r(A) = \operatorname{tr}\left[(A^{(r)})^* A^{(r)}\right].$$

Therefore $N_r(A)$ is unchanged if A is replaced by UA or AV for any unitary matrices U and V. Thus (1) can be reduced to the case in which A is diagonal, and in this case the identity is proved in a straightforward way.

Acknowledgements. The identity (1) was first proved in the case $n = 3$ by Jenny Young. The author also thanks P. R. Halmos and P. M. Cohn for discussions of this material.

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THE TEACHING OF MATHEMATICS

EDITED BY JOAN P. HUTCHENSON AND STAN WAGON

For instructions about submitting material for publication in this department see the inside front cover.

A Remark on Finite Fields

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Recently, in the course of writing an elementary book on abstract algebra, I wanted to show the existence of a finite field having p^n elements, where p is any prime and n is any positive integer. Of course the usual proof consists of construct-

ing a field K over Z_p, the integers mod p, in which the polynomial $x^m - x$, with $m = p^n$, has a full complement of roots. (m will be p^n throughout). Then one shows that the roots themselves form a field F. The one point that remains is to prove that F is the sought-after field having exactly p^n distinct elements. This amounts to showing that the polynomial $x^m - x$ has $m = p^n$ distinct roots in K or, in other words, that the polynomial $x^m - x$ has no multiple roots.

The usual proof of this latter fact is to introduce the formal derivative $f'(x)$ of the polynomial $f(x)$, to develop the properties of this derivative, and to show that if $f(x)$ has a multiple root in some extension field, then $f(x)$ and $f'(x)$ are not relatively prime. Then, of course, since $(x^m - x)' = -1$ we see at a glance that $(x^m - x)'$ and $x^m - x$ are relatively prime, hence the roots of $x^m - x$ are all simple roots.

But, as I noticed in writing the book, all this machinery of the derivative is not needed, for there exists a trivial, one-line proof of the fact that $x^m - x$ has no multiple roots. Since this proof is so trivial it surely must have been noticed by many people, and should exist somewhere in print. However I have never seen it in print. I would be grateful to hear from any reader a specific source where it does exist.

Let α be a root of $x^m - x$ in some extension field K of Z_p. Thus

$$(x-\alpha)^m - (x-\alpha) = x^m - \alpha^m - x + \alpha = x^m - x,$$

since the characteristic of K is p and since $\alpha^m = \alpha$. Thus

$$x^m - x = (x-\alpha)^m - (x-\alpha) = (x-\alpha)\left[(x-\alpha)^{m-1} - 1\right].$$

Since $(x-\alpha)^{m-1} - 1$ is clearly not divisible by $x - \alpha$, we see that α is a simple root of $x^m - x$.

A Modern Fairy Tale?

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Far away from the hustle and bustle, tucked away in a rural corner of North America lies a phenomenally successful undergraduate mathematics program. Don't imagine some tiny private institute devoting its funds and attention to mathematics. Just picture a typical, publicly-funded, Arts and Science undergraduate institute of about 5000 students, with separate Departments of Mathematics and Computer Science. But while the total number of undergraduates has remained relatively fixed over the past 15 years, the number of mathematics majors here has doubled and doubled again, and again to over 400 now in third and fourth year. They don't offer a special curriculum: no mathematics for computer science, no visits by industry to applied mathematics seminars. This institution is seemingly unaware of the success of its mathematics department and there are no special favours granted. There is no

emphasis upon the general Arts and Science student taking mathematics: only the usual programs—in physics, chemistry, computer science and mathematics—require the introductory calculus course. It is just a standard, traditional pure mathematics department.

Nevertheless, more than half the freshman class elect calculus, because of the reputation of the mathematics department carried back to local high schools by satisfied mathematics students. And of the less than 1000 Bachelor degrees awarded, now almost 200 of these 1000 (20%) are in mathematics. In case you are unaware, 1% of Bachelor degrees granted in North America are in mathematics. These students graduate with a confidence in their ability and in the undergraduate mathematics training they received that convinces their prospective employers to hire them, at I.B.M., General Dynamics, Bell Laboratories, and so on. Until very recently, so few of them went into teaching that the local school board had difficulty hiring mathematics teachers. The department has so many students that they seek the cooperation of faculty from other departments in order to offer their applied mathematics courses—pure mathematics takes up all their resources. How do they do it? Do you imagine they must be the only college or university in a large city attracting all those students? No, picture them set in a small town, in an area served by many colleges and universities and almost a hundred miles from the nearest large city. They sit across the street from a nationally renowned college (now a university) specializing in science and technology.

Do they just lower their standards? Mathematics teachers in the university across the street say no; they open some of their applied mathematics courses to the overflow of these hundreds of mathematics students and see no significant difference between their performance and that of their own students.

A typical mathematics undergraduate studies four years (usually a combined degree) and might take:

Calculus I + Calculus II
(Set Theory and Logic) + Calculus III + Linear Algebra I + Theory of Sets
Modern Algebra I + Advanced Calculus I + Modern Algebra II
Topology + Advanced Calculus II + Problem Seminar

Would you believe (of a population of 5000 undergraduates, remember) a class of 200 in Modern Algebra I? This is a compulsory course—but 100 take Modern Algebra II which is elective. (The electives in the course pattern above are: Theory of Sets, Topology, Modern Algebra II and Advanced Calculus II.) The curriculum is very standard, even old-fashioned. The textbooks chosen (Pinter for Modern Algebra, Hu for Topology) are not exciting but comprehensive; recent, but traditional in tone. By the way, the first two chapters of Dugundji's book on Topology have been used in the Theory of Sets course in second year, where the emphasis is on creative proof-generation. One of the purposes was to show the students that they could learn to handle material written for an advanced mathematics level.

What is the basis of their success? The students say they feel that the faculty members really care about them, care that each one of them can develop to the maximum possible level. The faculty can be quite explicit about it: for example, the chairman tells his classes that he believes in each of them, and cares about them, that he is there to help them achieve their potential. He stresses that a mind is a terrible thing to waste. The faculty win the students over to enjoy and do mathematics. And it is simply the transforming power of love, love through encouragement, caring, and the fostering of a supportive environment.

Listen to some of the points the department makes explicit to all students majoring in mathematics, through a small booklet distributed to each of them:

> "We wish to help provide the most favourable learning environment needed for you to reach the level of academic excellence consistent with your high promise... members of the mathematics faculty will use many methods of teaching (for good teaching is dependent upon time and place). A given teacher may use several methods in a given class or different methods in different classes. Some of the aims of the faculty are to help you
> (a) learn the basic materials of the course and its applications,
> (b) give correct mathematical proofs,
> (c) write correct proofs in clear and neat form,
> (d) read mathematical literature with understanding and enjoyment,
> (e) develop the ability to work independently to enjoy that life of the mind in which pursuit of knowledge and understanding is its own justification and its own reward as a life-long goal, and
> (f) progress through the different levels of mathematical maturity and recognize the different levels of abstractions in mathematics as are explained below."

It is clear that pure mathematics is not a handicap in this department but in fact an asset, because it emphasizes learning to read and create mathematical proofs. They *tell* their students that this is a valuable human asset for life, study, understanding, and work; it makes them valuable to their future employer, to their community, and to themselves. By the time they enter the senior year, many can read and learn from mathematics texts and articles on their own. Teaching is organized to gently but steadily bring the students closer to these aims, increasing their confidence in their ability to reach these aims and in their progress and training along the way. Students are encouraged increasingly to work on individual projects and present them or report on them in class. Could this be why they graduate more women in mathematics than men, that they address and redress a lack of confidence many women may feel about their ability to be mathematicians or to do mathematics competently? Don't imagine any measurable differences between the sexes in mathematical preparation or numbers of entering students. The female mathematics students attribute their success to the supportive atmosphere. In the past ten years, almost every year the top graduating student at this institution, across all programs, has been a woman in mathematics.

Where are they? In northeastern New York State: SUNY College at Potsdam. The university across the street is Clarkson. SUNY College Potsdam's mathematics program was praised as one of North America's most successful mathematics programs, in the C.U.P.M. Report: *Recommendation for a General Mathematical Sciences Program* published by the M.A.A. in 1981.

What must a mathematics department do to attain this level of success? It seems to this author, from talking with students and faculty in visits to Potsdam over the past two years, that the faculty of such a mathematics department must love to teach, with all that this means about true communication, caring for students, and caring for their development. It is even a love that surpasses the love of mathematics. They would select their textbooks for their courses more on the basis of the book being the best possible aid to help them develop all the students in their classes, rather than primarily on the beauty of the mathematical treatment of the topic. They would teach at a pace which allows hardworking students the time to struggle with the problems and resolve them, rather than primarily to cover a certain amount of material. Most important, they would recognize that students need time to build the necessary skills, understanding and self-confidence to handle more advanced mathematics. The faculty and assisting sympathetic senior students would be easily available for supplementary support outside the classroom, because problems and goals would be there for each student, to help them blossom through reaching and attaining. The faculty would encourage and reward the successes of their students, even to the point of bringing all or most of them to a high level of achievement (and high grades) in their courses, rather than using the grading to filter only the brightest and quickest students into further mathematics studies. As the chairman at Potsdam expresses it: a proper (rather than excessive) concern about the curriculum and standards of the program is important.

What can individual teachers do in departments where the majority of mathematicians do not wish to devote their own efforts to such a program? Perhaps with the moral support of the department a group of caring teachers could form an effective relay team across the core mathematics courses in the early years (their legacy could be substantial). They must ensure that their team is sufficiently large and well motivated. To plan their next steps, can they meet in the same sort of atmosphere that they desire to foster in their own classrooms? They will need to openly lay detailed plans before the department for a five year experiment. Imagine if the department supported the team in the same way that the team hopes to support the majority of incoming majors, until they find their feet and know the confidence of accomplishment.

The recipe for success at Potsdam is very simple. In teaching as in parenting it is to instill self-confidence and a sense of achievement through the creation of an open, caring environment.

Background information: I first heard of the phenomenal success of the Potsdam mathematics program when I attended an invited talk at the 1982 joint A.M.S.-M.A.A. summer meeting, given by their chairman, Dr. C. F. Stephens, on "The pending death of the mathematics major." That Fall I invited him

to give a colloquium talk on the same theme at Carleton. By then I had begun to eliminate from my mind many "teaching-techniques" explanations of their success, but I had no clear idea of why they succeeded. The praise of their program in the CUPM Recommendations for a Mathematical Sciences Program caught my attention but left me no wiser. (Echoes of Tom Peter's experiences *In Search of Excellence* now ring in my ears.) In the Fall of 1983 I requested to visit Potsdam (70 miles from Ottawa) and was warmly received. I spoke to students, faculty and administrators, and sat in on classes, in my attempt to understand the source of their success. I then wrote to several alumni and spoke with faculty from Clarkson College (as it was then) in Potsdam, to ask their opinions. The next summer I gave invited talks at the Annual Meetings of the Canadian Mathematics Education Study Group and the Canadian Mathematics Society on my observations about Potsdam. Positive reaction encouraged me to visit Potsdam a second time, to test my theories. That round of discussions with students and faculty and attending classes confirmed my ideas. This article is one of the fruits of that learning experience.

Sub-Gaussian Techniques in Proving Strong Laws of Large Numbers

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1. Introduction. Borel's Strong Law of Large Numbers (SLLN) asserts that if S_n, $n \geq 1$, is the number of successes in the first n independent repetitions of a Bernoulli trial with success probability p, $0 < p < 1$, then

$$P\left[\lim_{n \to \infty} S_n/n = p\right] = 1.$$

Tomkins (1984) provided a clever and simpler proof than the proofs common to most textbooks. Typically, measure theoretic techniques are used in developing SLLN's and are usually accessible at the graduate level. Tomkins' proof used basic probabilistic results, the analytic tools of Maclaurin expansions and geometric series, and a lemma on monotone random variables. A crucial part of his proof used an exponential moment inequality typical of sub-gaussian random variables. In this note sub-gaussian techniques are introduced and used to provide a considerably shorter proof and to greatly expand SLLN's for bounded random variables which are independent but need not be identically distributed. These techniques are very accessible and appropriate for a calculus-based undergraduate probability and statistics course. In addition, it is indicated how truncation and sub-gaussian techniques can be used in combination to provide a simple proof of the SLLN for unbounded random variables with slightly stronger moment conditions than usual.

2. Background and motivation discussion. Let $\{X_n\}$ denote independent and identically distributed Bernoulli random variables with

$$P[X_n = 1] = p \quad \text{and} \quad P[X_n = 0] = 1 - p.$$

Typically, p is referred to as the probability of success for each trial. The sum $S_n = X_1 + \cdots + X_n$ counts the number of successes in n trials. An essential concept in probability and statistics is that the relative frequency of successes should

approach p as the number of trials increases. This convergence takes two standard modes technically as

$$\lim_{n\to\infty} P\left[\left|\frac{S_n}{n} - p\right| > \varepsilon\right] = 0 \quad \text{for each } \varepsilon > 0,$$

which is known as the weak law of large numbers (WLLN), and

$$P\left[\lim_{n\to\infty} \frac{S_n}{n} = p\right] = 1, \tag{1}$$

which is known as the strong law of large numbers (SLLN).

For independent random variables $\{X_n\}$ which are bounded by a constant M and have the same mean $\mu = EX_n$ for all n, it easily follows from Chebyshev's inequality that

$$P\left[\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right] \le \frac{\Sigma_{k=1}^n \operatorname{Var} X_k}{n^2\varepsilon^2} \le \frac{4M^2}{n\varepsilon^2}. \tag{2}$$

Thus, the WLLN quickly follows from (2). However, the SLLN is considerably harder to establish even in this restrictive setting (refer to standard textbooks).

In establishing the basic limit and monotone properties of probability distributions, the following two results are generally developed and used:

$$\lim_{n\to\infty} P(A_n) = P\left[\bigcup_{n=1}^{\infty} A_n\right] \quad \text{when } A_1 \subset A_2 \subset \cdots, \tag{3}$$

and

$$\lim_{n\to\infty} P(A_n) = P\left[\bigcap_{n=1}^{\infty} A_n\right] \quad \text{when } A_1 \supset A_2 \supset \cdots. \tag{4}$$

Using (3) and (4), we find that

$$\begin{aligned} P\left[\lim_{n\to\infty} X_n \ne X\right] &= P\left[\bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left[|X_n - X| > \frac{1}{k}\right]\right] \\ &= \lim_{k\to\infty} \lim_{m\to\infty} P\left[\bigcup_{n=m}^{\infty} \left[|X_n - X| > \frac{1}{k}\right]\right]. \end{aligned}$$

Thus, the SLLN would follow if

$$\lim_{m\to\infty} P\left[\bigcup_{n=m}^{\infty} \left[\left|\frac{S_n}{n} - p\right| > \varepsilon\right]\right] = 0 \quad \text{for each } \varepsilon > 0,$$

or (more restrictively) if

$$P\left[\bigcup_{n=m}^{\infty} \left[\left|\frac{S_n}{n} - p\right| > \varepsilon\right]\right] \le \sum_{n=m}^{\infty} P\left[\left|\frac{S_n}{n} - p\right| > \varepsilon\right] < \infty. \tag{5}$$

for each n. Thus, by the integral test,

$$\sum_{n=1}^{\infty} P[|S_n| > n\varepsilon] \le 2 \sum_{n=1}^{\infty} \exp(-\varepsilon^2 n^d/2C) < \infty,$$

and (5) is established. □

The following corollary contains the Borel SLLN and establishes (1) in the Bernoulli random variables setting. Moreover, the corollary shows that the divisor of the sum, S_n, can be n^b (instead of n) for any $b > \frac{1}{2}$. The requirement of $b > \frac{1}{2}$ is sharp since $n^{-1/2} \sum_{k=1}^{n}(X_k - p)$ would converge to a normal distribution (not zero) by the central limit theorem for the independent Bernoulli random variables.

COROLLARY. *Let $\{X_n\}$ be independent random variables which are bounded by a constant M. Then for $b > \frac{1}{2}$*

$$P\left[\lim_{n\to\infty} \frac{1}{n^b} \sum_{k=1}^{n} (X_k - EX_k) = 0\right] = 1. \tag{9}$$

If $|X_n| \le M$ for all n, then $|X_n - EX_n| \le 2M$, and Fact 3 provides the sub-gaussian property. Then, when dividing $S_n = \sum_{k=1}^{n} (X_k - EX_k)$ by n^b, the crucial inequality (8) becomes

$$P\left[|S_n| > n^b\varepsilon\right] \le 2\exp(-\varepsilon^2 n^{2b-1}/16M^2), \tag{10}$$

and the proof of the corollary easily follows.

The desired Borel SLLN of Tomkins (1984) follows from (9) by letting $b = 1$ and $M = 1$. The generality of the divisor n^b is credited to Marcinkiewicz and Zygmund (1937) for independent, identically distributed random variables where their techniques included truncation and measure theory. Truncation and sub-gaussian techniques are used in the next section to show how moment conditions can replace the assumption of bounded random variables which need not be identically distributed.

4. Additional results. Chung's (1947) SLLN provides that

$$P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = 0\right] = 1, \tag{11}$$

where $\{X_n\}$ are independent random variables such that $EX_n = 0$ for all n and

$$\sum_{n=1}^{\infty} E|X_n|^p/n^p < \infty$$

for some p, $1 \le p \le 2$. Chung's result and its measure-theoretic proof are not accessible to the typical calculus-based probability and statistics course. However, an easy proof of (11), results from using sub-gaussian techniques and the slightly

more restrictive moment condition

$$\sum_{n=1}^{\infty} E|X_n|^{2pq}/n^q < \infty, \tag{12}$$

where q is some positive constant and p is some constant, $1 < p \leq 2$.

In outlining the proof of (11) using (12), first define $Y_k = X_k I[|X_k| \leq k^{1/2p}]$. Then

$$P\left[\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Y_k\right] = 1, \tag{13}$$

since

$$\begin{aligned}\sum_{n=1}^{\infty} P[X_n \neq Y_n] &= \sum_{n=1}^{\infty} P\left[|X_n| > n^{1/2p}\right] \\ &\leq \sum_{n=1}^{\infty} \frac{E|X_n|^{2pq}}{n^q} < \infty\end{aligned}$$

by (12). Also, using (12) and $EX_n = 0$ for all n, one can show that,

$$\frac{1}{n}\sum_{k=1}^{n} EY_k \to 0. \tag{14}$$

Thus,

$$\sum_{n=1}^{\infty} P\left[\left|\frac{1}{n}\sum_{k=1}^{n}(Y_k - EY_k)\right| > \varepsilon\right] < \infty \quad \text{for each } \varepsilon > 0.$$

The major theorem then applies since $Y_k - EY_k$ is sub-gaussian with $\alpha_k = 2\sqrt{2}\,k^{1/2p}$ and $p > 1$. Thus, the proof of (11) is completed.

A special case which satisfies (12) is when $\sup_n E|X_n|^{2+\delta} < \infty$, for some $\delta > 0$ (in this case, let $q = \sqrt{1 + \delta/2} = p$). Thus, very general SLLN's for independent, non-identically distributed random variables can be obtained using sub-gaussian techniques.

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E 3198. *Proposed by I. J. Schoenberg, Madison, Wisconsin.*

Let $\Pi_x = (x_0, x_1, x_2, x_3, x_4, x_5) := (x_k)$, $0 \leqslant k \leqslant 5 \pmod 6$ be an arbitrary closed skew hexagon in R^3. Form a new hexagon $\Pi_y = (y_k)$ by

$$y_k = x_k + \frac{1}{2}(x_k - x_{k-3}) = \frac{3}{2}x_k - \frac{1}{2}x_{k-3}.$$

Finally, define a third hexagon $\Pi_w = (w_k)$ by

$$w_k = \frac{1}{3}(y_k + x_{k-1} + x_{k+1}).$$

Show that the hexagon Π_w lies in a plane π and is in π an affine image of a regular hexagon.

E 3199. *Proposed by H. Guelicher, Muenster, West Germany.*

In the triangle ABC, point Q is on the ray $\overrightarrow{BA}$, point R is on the ray $\overrightarrow{CB}$, and $BQ = CR = AC$. A line parallel to $\overleftrightarrow{AC}$ through R intersects $\overleftrightarrow{CQ}$ in a point T. A line parallel to $\overleftrightarrow{BC}$ through T intersects $\overleftrightarrow{AC}$ in a point S. Show that:

$$(AC)^3 = AQ \cdot BC \cdot CS.$$

E 3200. *Proposed by Paul Erdös, Hungarian Academy of Science.*

Suppose $a_1, a_2, \ldots$ are real numbers such that $0 < a_1 < a_2 < a_3 < \cdots$, $a_n \to \infty$, and $(a_1 + a_2 + \cdots + a_n)/a_n \to +\infty$. If $N(x) = \sum_{a_n < x} 1$, prove that $N(x)/\log x \to +\infty$ as $x \to +\infty$.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Polynomial First-Order System of ODE's

E 2939 [1982, 273]. *Proposed by Tian Jinghuang, Szechwan University, China.*

Consider the autonomous quadratic first-order system

$$dx/dt = X(x, y), \qquad dy/dt = Y(x, y), \tag{1}$$

where

$$X(x, y) = \sum a_{ik} x^i y^k, \qquad Y(x, y) = \sum b_{ik} x^i y^k$$

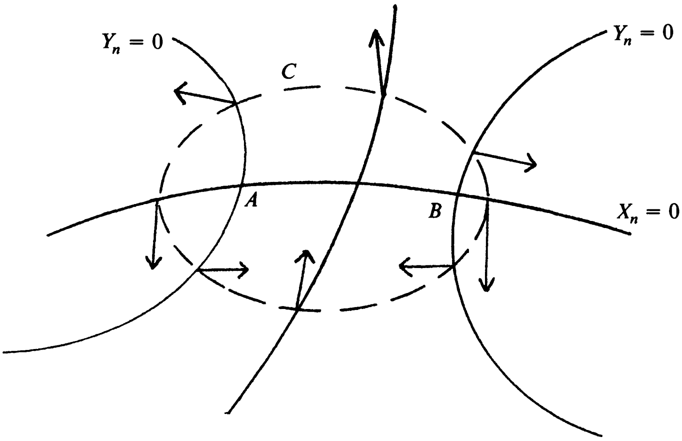

FIG. 2

obtained. For the case shown in Figure 2 (where a possible distribution of the field directions on C is again indicated), the change in the angle mentioned above is equal to $8 \cdot (\pi/2) = 4\pi$. For all the other possible distributions, the change in the angle is either 4π or -4π, and $\operatorname{ind} C = \pm 2$. Since the index of any elementary critical point is $+1$ or -1, the relation $\operatorname{ind} C = \operatorname{ind} A + \operatorname{ind} B$ yields the required result: $\operatorname{ind} A = \operatorname{ind} B$.

ADVANCED PROBLEMS

6539. *Proposed by C. A. Spiro, SUNY at Buffalo.*

Let $b_1 < b_2 < b_3 < \cdots$ be the distinct positive integers expressible as sums of two squares of integers. Prove that for any given positive integer d the equality $b_{n+1} - b_n = d$ holds for infinitely many n.

6540. *Proposed by Mo Song-Qing, Institute of Applied Physics and Computational Mathematics, Beijing, China.*

Suppose x is a given real number greater than 1. Let $a_n = [x^n]$ for $n = 1, 2, \ldots$, where $[x]$ is the integral part of x. Let S be the infinite decimal $S = 0.a_1a_2a_3\ldots$, where the notation indicates the expansion formed by writing down the decimal digits of a_1, a_2, $a_3, \ldots$ in turn. (For example, if $x = \pi$, then $S = 0.393197\ldots$.) Is it possible for S to be rational?

6541. *Proposed by Carl W. Helstrom, University of California, San Diego.*

Derive the continued fraction

$$\frac{K_0(x)}{K_1(x)} = \frac{1}{1+}\frac{2u}{1+}\frac{u}{1+}\frac{3u}{1+}\frac{3u}{1+}\frac{5u}{1+}\frac{5u}{1+}\cdots\frac{(2k+1)u}{1+}\frac{(2k+1)u}{1+}\cdots,$$

where $K_0(x)$ and $K_1(x)$ are modified Bessel functions of the second kind and $u = 1/(4x)$.

SOLUTIONS OF ADVANCED PROBLEMS

Distribution of Sums of Exponentials of Random Variables

6499 [1985, 511]. *Proposed by Doug Hensley, Texas A&M University.*

Suppose $X_1, X_2, \dots$ are independent random variables all uniformly distributed on [0, 1]. Show that

$$\lim_{n\to\infty} \operatorname{Prob}\left(\sum_{k=1}^{n} e^{-nX_k} \le 1\right) = e^{-\gamma}$$

where γ is Euler's constant.

Solution by the proposer. We establish a more general result. Let $\rho(u)$ be the Dickman-de Bruijn function determined by $\rho(u) = 1$ for $0 < u \le 1$, and for $u > 1$ by the delay differential equation

$$u\rho'(u) = -\rho(u-1).$$

Then, for each $U > 0$,

$$\lim_{n\to\infty} \operatorname{Prob}\left(\sum_{k=1}^{n} e^{-nX_k} \le U\right) = e^{-\gamma}\int_0^U \rho(u)\,du.$$

For the proof, set

$$F(u) := \operatorname{Prob}(e^{-nX} \le u) = \operatorname{Prob}\left(1 - X \le 1 + \frac{\log u}{n}\right).$$

If X is uniformly distributed on [0, 1] so is $1 - X$, and the p.d.f. (probability density function) for $F(u)$ is

$$f_n(u) = \begin{cases} 1/(nu) & e^{-n} \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the p.d.f. of the sum is the convolution of the p.d.f.'s it suffices to show that

$$g_n(u) \to e^{-\gamma}\rho(u),$$

6541. *Proposed by Carl W. Helstrom, University of California, San Diego.*

Derive the continued fraction

$$\frac{K_0(x)}{K_1(x)} = \frac{1}{1+}\,\frac{2u}{1+}\,\frac{u}{1+}\,\frac{3u}{1+}\,\frac{3u}{1+}\,\frac{5u}{1+}\,\frac{5u}{1+}\cdots\frac{(2k+1)u}{1+}\,\frac{(2k+1)u}{1+}\cdots,$$

where $K_0(x)$ and $K_1(x)$ are modified Bessel functions of the second kind and $u = 1/(4x)$.

SOLUTIONS OF ADVANCED PROBLEMS

Distribution of Sums of Exponentials of Random Variables

6499 [1985, 511]. *Proposed by Doug Hensley, Texas A&M University.*

Suppose $X_1, X_2, \ldots$ are independent random variables all uniformly distributed on [0, 1]. Show that

$$\lim_{n\to\infty} \operatorname{Prob}\left(\sum_{k=1}^{n} e^{-nX_k} \le 1\right) = e^{-\gamma}$$

where γ is Euler's constant.

Solution by the proposer. We establish a more general result. Let $\rho(u)$ be the Dickman-de Bruijn function determined by $\rho(u) = 1$ for $0 < u \le 1$, and for $u > 1$ by the delay differential equation

$$u\rho'(u) = -\rho(u-1).$$

Then, for each $U > 0$,

$$\lim_{n\to\infty} \operatorname{Prob}\left(\sum_{k=1}^{n} e^{-nX_k} \le U\right) = e^{-\gamma}\int_0^U \rho(u)\,du.$$

For the proof, set

$$F(u) := \operatorname{Prob}(e^{-nX} \le u) = \operatorname{Prob}\left(1 - X \le 1 + \frac{\log u}{n}\right).$$

If X is uniformly distributed on [0, 1] so is $1 - X$, and the p.d.f. (probability density function) for $F(u)$ is

$$f_n(u) = \begin{cases} 1/(nu) & e^{-n} \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the p.d.f. of the sum is the convolution of the p.d.f.'s it suffices to show that

$$g_n(u) \to e^{-\gamma}\rho(u),$$

this reduces to $e^{-\gamma}$ when $\alpha = 1$. J. M. F. Chamayou (France) observed that

$$\mathrm{Prob}[e^{-2X_1} + e^{-2X_2} \leq 1] = 1 - \frac{1}{4}\big(L_2(1) - L_2(e^{-2})\big) = 0.624\ldots,$$

where L_2 is the dilogarithm function, and that Monte Carlo simulations for $n \geq 4$ show that the convergence to $e^{-\gamma} = 0.561\ldots$ is fast. Chamayou then obtained various generalizations of the result, and referred to the paper, J. M. F. Chamayou, A Probabilistic Approach to a Differential Difference Equation Arising in Analytic Number Theory, Math. Comp., 27–121(1973), 197–203 and the manuscript, J. M. F. Chamayou, Random Difference Equations, Iterated Integrals, and the Golomb Constant.

A similar problem is treated in L. Takács, On stochastic processes connected with certain physical recording apparatuses, Acta Math. Acad. Sci. Hungar., 6(1955), 363–380.

Also solved by Kee-wai Lau (Hong Kong) and Eric Willekens (Belgium).

Insufficient Conditions for the Existence of Square Roots in Groups

6500 [1985, 512]. *Proposed by Eugene M. Luks, University of Oregon, and Michael B. Ward, Bucknell University.*

In the first printing of an algebra text, the following problem appeared:

Let G be a group, and H a normal subgroup of G. Prove: If every element of G/H has a square root, and every element of H has a square root, then every element of G has a square root.

The statement is true if G is abelian or finite. Is it true if G/H and H are abelian? Is it true if G is torsion? Is it true in general?

Solution by David E. Manes, SUNY at Oneonta, Oneonta, New York. The answer to all three questions is negative. Indeed, the standard wreath product W of two Prüfer groups of type 2^∞ is a counterexample. The base group B of W is a divisible abelian 2-group, while W/B is of type 2^∞. However, not every element of W has a square root. The details may be found in D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, vol. 2, Springer-Verlag, Berlin (1972). (All solutions received gave essentially the same counterexample.)

Manes points out that the result is true for hypercentral groups, i.e., groups G whose upper central series reaches G when continued transfinitely. This follows from the theory of semi π-radicable groups developed in §9 of the above cited reference, and is essentially due to Černikov. Here π is a set of primes; the problem under discussion is the special case $\pi = \{2\}$.

Also solved by Jeffrey M. Cohen, Alberto Espuelas, and the second proposer.

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REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN EWING

Differential Equations. By A. N. Tikhonov, A. B. Vasil'eva and A. G. Sveshnikov. Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

WARREN S. LOUD
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Differential equations are of wide mathematical interest: they are used extensively as mathematical models in applications from the physical sciences, engineering, economics, and the biological sciences, and at the same time provide significant concrete examples for abstract mathematical theories, giving indications of profitable directions for research in many mathematical areas.

One or two generations ago courses in differential equations treated them as an application of integral calculus. The object was to develop techniques for finding explicit solutions. "Cookbook" courses that gave a large number of rather disjointed "recipes" for finding explicit solutions were commonplace.

The currently prevailing approach is to find ways of deducing properties of solutions even when explicit formulas are not available. Existence and uniqueness theorems for solutions of initial value problems are easily proved, so that solutions

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One or two generations ago courses in differential equations treated them as an application of integral calculus. The object was to develop techniques for finding explicit solutions. "Cookbook" courses that gave a large number of rather disjointed "recipes" for finding explicit solutions were commonplace.

The currently prevailing approach is to find ways of deducing properties of solutions even when explicit formulas are not available. Existence and uniqueness theorems for solutions of initial value problems are easily proved, so that solutions

of initial value problems can be discussed with the knowledge that they make mathematical sense even though there is no explicit formula. Existence and uniqueness proofs that use successive approximations give the possibility of constructing useful approximations to solutions. Linear algebra allows one to describe the structure of the family of all solutions of linear differential equations. One may analyze continuity and differentiability of solutions with respect to initial conditions and parameters, which is important for applications since it is desired that small changes in initial conditions or parameters arising, say, through measurement errors, will result in only small changes in solutions.

The advent of high-speed computers has brought the ability to obtain rapidly more detailed knowledge of solutions than was formerly possible and has stimulated the development of more effective numerical methods of approximating solutions.

With all this current activity and interest in differential equations there arises a vast selection of directions for further study, all accessible to intermediate-level students and all of importance in applications. There are many suitable topics in the area of linear differential equations alone. The subject of linear systems with constant coefficients may be easily reduced to the properties of matrices. This is an excellent application of linear algebra, and is one of the principal reasons why linear algebra belongs in the standard curriculum. Power series solutions about ordinary points and regular singular points can be studied using elementary complex-variable theory. Expansions of solutions about irregular singular points give a natural introduction to asymptotic series, which although possibly divergent, still have partial sums that give useful information about behavior of solutions. Another area associated with linear equations is that of linear boundary-value problems. Here, one applies further notions from linear algebra such as the Fredholm alternative. The subject of boundary-value problems introduces Green's functions in a natural way. Also for problems not having unique solutions the idea of a generalized Green's function can be introduced. Boundary-value problems also lead to eigenvalue problems and to eigenfunction expansions, which open the door to Fourier series and other orthogonal expansions. Going in still another direction boundary-value problems lead to the theory of oscillation of solutions.

There are also many nonlinear problems that are appropriate for intermediate-level courses. Perhaps the most accessible is that of stability of an equilibrium of an autonomous system or of a periodic solution. One can proceed directly from the definitions of stability and asymptotic stability, or one can use the method of Lyapunov functions. In many cases, but by no means in all, stability is equivalent to the stability of the linear system formed by linearizing about the solution being studied. Another subject with many applications is the phase plane study of nonlinear autonomous systems. Although the Poincaré-Bendixson theory requires some topology, it is accessible at this level. There is then the topic of perturbations, particularly for periodic solutions. It is also possible to introduce various aspects of bifurcation theory in this connection. Another topic is the theory of singular perturbations with their application to boundary-layer phenomena.

The availability of computers has made the construction and analysis of numerical approximation techniques an important topic, and most texts at this level include something in this area.

What background is desirable for profitable study of these topics in differential equations at an intermediate level? A firm background in calculus with understanding and appreciation of continuity, differentiability, and convergence of series is certainly needed. Linear algebra is also needed at least through canonical forms of matrices, eigenvalues and eigenvectors, and the language of subspaces. Advanced calculus is helpful, particularly the implicit function theorem. Also helpful are beginning topology at least through compactness and some functional analysis, so that the ideas of function spaces and contraction mappings are accessible.

The book by Tikhonov, Vasil'eva and Sveshnikov is in the Springer Series in Soviet Mathematics, and is a translation of the 1980 Russian edition. The translation is excellent, with very little awkward phraseology. They include most of the topics I have mentioned above. They discuss asymptotic approximations in the case of solutions of perturbed systems, but not for expansions of solutions near an irregular singular point. They do not consider oscillation theory and Poincaré-Bendixson theory. An unusual feature is the inclusion of singular perturbations, a topic not usually included in texts, but included here because of the fundamental contributions of the first author to the subject. The authors have chosen in their discussions to present simple cases with mention of possible generalizations rather than to discuss the most general situations. There is also a chapter on first-order partial differential equations, added because their theory depends so strongly on ordinary differential equations. There are numerous examples throughout but no formal exercises.

In closing let me mention several other texts on differential equations at about this level. They all reflect the fact that American students, particularly in engineering, cannot be expected to bring as extensive a mathematical background to this subject as do their Soviet counterparts. The book by Boyce and DiPrima [**1**] covers to some extent all the topics mentioned above except for irregular singular points and the various perturbation problems. The book by Hirsch and Smale [**3**] covers much less of the subject of linear equations, but goes more deeply into the nonlinear topics. The recent book by Brauer and Nohel [**2**] covers most of the linear topics mentioned above, but has a limited coverage of nonlinear problems. In contrast with the book by Tikhonov, Vasil'eva, and Sveshnikov, all the American books have extensive sets of exercises that illustrate numerous important applications.

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Blaschke Products: Bounded Analytic Functions. By Peter Colwell. University of Michigan Press, Ann Arbor, MI, 1985. viii + 140 pp., $15.00.

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A Blaschke product (called a BP in this review) is a special kind of bounded analytic function, defined in the open unit disc U of the complex plane, and essentially determined by its zeros and their multiplicities in the following manner.

Suppose that (a_n) is a (possibly finite or even empty) sequence of not necessarily distinct points of U, $a_n \neq 0$, which satisfies the so-called Blaschke condition

$$(*) \qquad \sum_n (1 - |a_n|) < \infty,$$

that m is a nonnegative integer, and that c is a constant, $|c| = 1$. The BP associated to these data is the function B defined for z in U by the formula

$$B(z) = cz^m \prod_n \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \bar{a}_n z}.$$

Note that each factor $b_n(z)$ in this product is a Möbius transformation that maps U onto U, and has $b_n(0) > 0$.

The finite BP's are easily seen to be precisely those functions f that are analytic in U and satisfy $|f(z_i)| \to 1$ for every sequence $\{z_i\}$ in U such that $|z_i| \to 1$ as $i \to \infty$. This follows from the reflection principle.

Quite often one restricts c to be 1 in the definition of BP's.

In the presence of $(*)$, the factors $|a_n|/a_n$ ensure that the product defines an analytic function B in U whose zeros are located at precisely the prescribed points a_n, plus, of course, at 0 if $m > 0$. This follows from the convergence of the series $\Sigma|1 - b_n(z)|$. Of course, $|B| \leqslant 1$.

It should be noted that, contrary to what seems to be a widespread misconception (see, for example, page 1 of the book under review), B is analytic even when $(*)$ is violated, because then

$$B(z) = \lim_{N \to \infty} z^m \prod_{n=1}^{N} b_n(z) = 0$$

for every z in U—an admittedly uninteresting case.

Let us now recall some standard terminology: H^∞ is the space of all bounded analytic functions in U, with norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in U\}.$$

For $0 < p < \infty$, the Hardy space H^p consists of all analytic functions f in U for which

$$\|f\|_p = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

The radial limits of an f defined in U will be denoted by

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

for those $e^{i\theta}$ on the unit circle T where this limit exists.

The condition $(*)$ seems to have occurred for the first time in the theorem of Blaschke (1915) which asserts that a set of distinct points $a_n \in U$ violates $(*)$ if and only if the following is true: If $f_i \in H^\infty$, $\|f_i\|_\infty \leqslant 1$ for $i = 1, 2, 3, \ldots,$ and $\lim_{i\to\infty} f_i(a_n)$ exists for every n, then the sequence $\{f_i(z)\}$ must converge uniformly on every compact subset of U.

As a consequence, one sees that H^∞-functions are determined by their values on $\{a_n\}$ if and only if $(*)$ fails.

In 1923, F. Riesz discovered the canonical factorization theorem which explains the importance of BP's in the function theory of the unit disc: If $0 < p \leqslant \infty$, $f \in H^p$, $f \not\equiv 0$, then the zeros of f, counted according to multiplicity, must satisfy $*$, and if B is the corresponding BP, then $f = Bg$, where g has no zeros in U, and $g \in H^p$; in fact, $\|g\|_p = \|f\|_p$.

He proved also that $|B^*(e^{i\theta})| = 1$ a.e. on T.

This factorization was extremely helpful in the early investigations of the H^p-spaces, since it made it possible to reduce many problems to their analogues in the Hilbert space H^2. For example, if $f \in H^1$ then $f = Bg^{1/2} \cdot g^{1/2}$ represents f as a product of two H^2-functions; also, every $f \in H^1$ is a sum of two zero-free H^1-functions. It is an interesting fact, discovered about 50 years later, that these decompositions are not available in several complex variables, i.e., when U is replaced by a polydisc or by the unit ball of $\mathbb{C}^n$.

The functions $f \in H^\infty$ that have $|f^*| = 1$ a.e. on T are called inner functions. Every BP is thus an inner function. There are others, the so-called singular ones, given by

$$S(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right\},$$

where μ is any finite positive Borel measure that is concentrated on a set of Lebesgue measure zero. Moreover, every inner function is a product BS. Inner functions attracted the attention of functional analysts when Beurling showed, in 1949, how they can be used to describe the closed subspaces of a separable Hilbert space that are carried into themselves by a unilateral shift operator, and they continue to play an important role in that area of analysis where operator theory interacts with classical function theory.

In his 1935 thesis, Frostman characterized the BP's as precisely those analytic functions in U that satisfy

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \big|\log|f(re^{i\theta})|\big|\, d\theta = 0,$$

and he proved that if f is any inner function, then $(f - \alpha)/(1 - \bar{\alpha} f)$ is a BP for

Schwarz lemma implies that $|f'(t)| \leqslant \|f\|_\infty/(1-t^2)$. Thus $\varepsilon_f(r) \leqslant \frac{1}{2}\|f\|_\infty$. If $\lim_{t\to 1} f(t)$ exists, one can say more, namely, $\lim_{r\to 1}\varepsilon_f(r) = 0$.

QUESTION: Is this last limit relation true for every $f \in H^\infty$? The answer is no, and there is a BP that shows it.

Put $a_n = (e^n - 1)/(e^n + 1)$ for $n = 0, \pm 1, \pm 2, \ldots$. Let φ be the Moebius transformation that fixes 1 and -1 and moves $0 = a_0$ to a_1. Then $\{a_n\}$ satisfies $(*)$, and a simple calculation shows that $\varphi(a_n) = a_{n+1}$ for all n. Therefore,

$$B(z) = z\prod_{n=1}^{\infty}\frac{a_n^2 - z^2}{1 - a_n^2 z^2},$$

the BP that has a simple zero at each a_n, satisfies the functional equation

$$B(\varphi(z)) = -B(z).$$

As x moves from a_n to a_{n+1} along the real axis, $B(x)$ covers a path Λ_n from 0 to 0. Note that $\Lambda_n = \Lambda_{n+2}$ for all n, and that every Λ_n has the same length, say L. Hence

$$\int_0^{a_n}|B'(t)|\,dt = Ln = L\log\frac{1+a_n}{1-a_n}.$$

This answers our question, because $\lim_{r\to 1}\varepsilon_B(r) = L > 0$.

Mathematics and the Search for Knowledge. By Morris Kline. Oxford University Press, New York, 1985, 245 pp.

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An old conundrum, many times resurrected, is this: Why do mathematics and physics fit together so surprisingly well? There is a famous article by Eugene Wigner, or at least an article with a famous title: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." After all, pure mathematics is created by lone fanatics sitting at their desks or scribbling on their blackboards. These wild men go where they please, led only by some notion of "beauty," "elegance," or "depth," which nobody can really explain.

Often, as in Lobachewsky's non-Euclidean geometry, or in Cayley's matrix theory, or in Riemann's n-dimensional manifold theory, or in the algebraic topology of the mid-twentieth century, pure mathematics seemingly had left far behind any physical interpretation or utility. And yet, over and over—certainly in the cases mentioned—physicists later found in these "useless" mathematical abstractions just the tools they needed.

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An old conundrum, many times resurrected, is this: Why do mathematics and physics fit together so surprisingly well? There is a famous article by Eugene Wigner, or at least an article with a famous title: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." After all, pure mathematics is created by lone fanatics sitting at their desks or scribbling on their blackboards. These wild men go where they please, led only by some notion of "beauty," "elegance," or "depth," which nobody can really explain.

Often, as in Lobachewsky's non-Euclidean geometry, or in Cayley's matrix theory, or in Riemann's n-dimensional manifold theory, or in the algebraic topology of the mid-twentieth century, pure mathematics seemingly had left far behind any physical interpretation or utility. And yet, over and over—certainly in the cases mentioned—physicists later found in these "useless" mathematical abstractions just the tools they needed.

Why does this happen?

Is there some arcane psychological principle by which the most original and creative mathematicians choose as "interesting" or "attractive" just those directions in which Nature herself wants to go? Such an answer might only be explaining one mystery by means of a greater mystery.

On the other hand, perhaps the "miracle" is an illusion. Perhaps for every bit of abstract purity that finds application, there are a dozen others that find no such application, that eventually die, disappear and are forgotten. This second explanation could in principle be checked out, perhaps by a doctoral candidate in the history of mathematics. I have not checked it myself, but my gut feeling is that it is false, that most of the mainstream research in pure mathematics does connect somehow with applications.

A third explanation, a more profound one that goes to the very nature of the mathematical and physical universes, is this: Mathematics has evolved from the study of number and shape, and these two studies, arithmetic and visual geometry, were obtained by abstraction from physical reality. Evolving from this root, no matter how far it goes, mathematics can never escape from its inner identity with physical reality. Every so often, this inner identity pops out spectacularly when, for example, the geometry of vector bundles is identified as the mathematics of the gauge field theory of elementary-particle physics. This third explanation has a satisfying feel of philosophical depth. It recalls Leibniz's windowless monads, the body and soul, which at the dawn of time God set to be in tune with each other forever. But this explanation has an unsatisfactory aspect too. It supposes that all mathematical growth is inevitable. We know that is not so. Bad mathematics is possible—whether pointless, ugly, or trivial. This fact forces us to acknowledge that in the evolution of mathematics there is an element of human choice, or human taste if you prefer. Thereby we return to the mystery we started with. What enables certain humans to choose better than they have any way of knowing?

The new book by Morris Kline consists largely of an account, in very accessible, nontechnical terms, of the principal accomplishments of mathematical physics, from the Greeks through 1932 (Dirac). (There is a cursory reference to more recent physics in one sentence on page 243.) There are substantial chapters on planetary theory, on gravity, on electromagnetism, on special and general relativity, and on quantum mechanics up to the positron. As always in Kline's books, there is a tremendous abundance of well-chosen quotations from mathematicians, physicists, and philosophers, from Copernicus and Kepler on up to Heisenberg, Born, and Dirac.

There is not a great deal of explanation of mathematical ideas; in fact, a reader can start and finish this book without knowing any calculus at all. The introductory and concluding chapters are of a more philosophical turn. They do not claim to offer new insights on why mathematics and physics discover each other as predestined mates. But certainly they will be informative and enlightening for a wide class of readers, especially those with a strong curiosity about the world of physics, and little or no mathematical preparation.

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
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S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatabilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, S*(13-16), P, L*. *Descartes' Dream: The World According to Mathematics.* Philip J. Davis, Reuben Hersh. Harcourt Brace Jovanovich, 1986, xviii + 321 pp, $19.95. [ISBN: 0-15-125260-2] A potpourri of independent essays derived from the authors' articles and talks about the impact on society of mathematics itself and of its stepchild the computer. It purports to be a philosophy of applied mathematics, an inquiry into the contemporary impact of Descartes' dream of a unifying method—reason—for approaching all knowledge. The writing is personal and casual, not tight and philosophical, punctuated by occasional rhetorical excess ("Cartesianism calls for the primacy of world mathematization") and uncontrolled meandering (e.g., into "The Computerization of Love" and "Mathematics and the End of the World"). Not a coherent exposition, it is nevertheless a compelling collage woven around a central theme of caution that the limitless application of mathematics—the fulfillment of Descartes' dream—too often quenches human spirit and human life. LAS

Elementary, T(13: 1). *Trigonometry with Applications.* Terry H. Wesner, Harry L. Nustad, Philip H. Mahler. Wm C Brown, 1986, xiii + 459 pp. [ISBN: 0-697-00236-5] Trigonometry for college students assuming intermediate algebra and graphing skills. Covers standard material; trigonometry of right triangles and unit circle, functional properties, inverses, and identities. Well-organized chapters with student-directed notes (e.g., "Note: Read A' as A-Prime") and mastery points. Exercises include many applications. MR

Precalculus, T*(13: 1). *Algebra and Trigonometry.* Ronald J. Harshbarger, James J. Reynolds. Brooks/Cole, 1987, xi + 506 pp, $30.75. [ISBN: 0-534-06288-1] Attractive text for pre-calculus students. Exercises contain more applied problems than most. Answers to odd exercises run over 50 pages and contain numerous impressively rendered figures—a very strong point in the text's favor. "Preparing for Calculus" exercises are featured throughout. Some calculator problems. Tables. Have a look. JK

Education, P*, L***.** *The Underachieving Curriculum: Assessing U.S. School Mathematics From An International Perspective.* Curtis C. McKnight, et al. Stipes Pub (10-12 Chester St., Champaign, IL 61820), 1987, xiv + 127 pp, $8 (P). [ISBN: 0-87563-298-X] "In school mathematics the United States is an underachieving nation and our curriculum is helping to create a nation of underachievers. We are not what we ought to be; we are not even close to what we can be." A stark, forthright analysis for the general public (e.g., for local school boards) of the results of the Second International Assessment of Mathematics, in which the United States was outranked in most measures by almost every other industrialized country. Covers achievement, attitudes, curriculum, opportunity to learn, and "deceptive explanations." Must reading for every American. LAS

History, S(15-17), L*. *A.M. Turing's ACE Report of 1946 and Other Papers.* Ed: B.E. Carpenter, R.W. Doran. MIT Pr, 1986, vii + 140 pp, $20. [ISBN: 0-262-03114-0] Volume 10 in the Babbage Reprint Series for the History of Computing: Tur-

ing's original proposal for an Automatic Computing Engine (ACE), his 1947 London Mathematical Society lecture amplifying details of this proposal, and Michael Woodger's 1958 report on the first practical implementation of ACE. Begins with a brief introduction to Turing and the ACE project. LAS

History, P, L. *Methods, Concepts and Ideas in Mathematics: Aspects of an Evolution.* A.F. Monna. CWI Tract V. 23. Math Centrum, 1986, iv + 170 pp, Dfl. 26.40 (P). [ISBN: 90-6196-299-4] A personal historical essay on the development of selected themes in nineteenth and twentieth century mathematics, especially algebraization and existence theorems. Intended as "theory of history," as a look at some of the main trends in mathematics from a point outside. LAS

History, S*(15-17), L*. *A Source Book in Mathematics, 1200-1800.* Ed: D.J. Struik. Princeton U Pr, 1986, xiv + 427 pp, $12.50 (P). [ISBN: 0-691-02397-2] Paperback edition of 1969 Harvard University Press original edition (TR, June-July 1969; Extended Review, February 1970). A compendium of mathematical classics from the Latin world from the thirteenth through the eighteenth century. A very valuable resource at a modest price. LAS

Foundations, P, L. *The Identification of Progress in Learning.* Ed: T. Hägerstrand. Cambridge U Pr, 1985, xi + 204 pp, $39.50. [ISBN: 0-521-30087-8] Keynote papers from a 1983 multidisciplinary colloquium sponsored by the European Science Foundation providing expert commentary on the criteria of progress in several scientific fields. The mathematics contribution is by Michael Atiyah, with commentary by René Thom. LAS

Foundations, T(14: 1), L.** *A Transition to Advanced Mathematics, Second Edition.* Douglas Smith, Maurice Eggen, Richard St. Andre. Brooks-Cole, 1985, xi + 276 pp, $30.50. [ISBN: 0-534-05796-9] A revision of a popular text (TR, April 1984) to include more topics from discrete mathematics—counting, graphs, and trees. The book remains an attractive text for a useful course. TAV

Combinatorics, T(16-17: 1), S**, P**, L*.** *Enumerative Combinatorics, Volume I.* Richard P. Stanley. Math. Ser. Wadsworth, 1986, xi + 306 pp, $42.95. [ISBN: 0-534-06546-5] An engaging text which looks at enumeration, sieve methods, posets and generating functions. This is a well-written book which has a particularly rich collection of examples and graded exercises. Also includes lots of references. CEC

Combinatorics, T(17-18: 2). *Combinatorial Theory, Second Edition.* Marshall Hall, Jr. Ser. in Disc. Math. Wiley, 1986, xv + 440 pp, $39.95. [ISBN: 0-471-09138-3] Revision of the 1968 text (TR, January 1968; Extended Review, August-September 1969) includes recent work of Egorychov on the minimum value of the permanent for a doubly stochastic matrix, Wilson's work on asymptotics for block designs, and the relationship between design theory and the theory of error-correcting codes. A "...conscious effort...to bring it up to date... ." LC

Combinatorics, P. *Combinatorics and Ordered Sets.* Ed: Ivan Rival. Contemp. Math., V. 57. AMS, 1986, xvi + 285 pp, $29 (P). [ISBN: 0-8218-5051-2] Nine expository papers from a conference at Humboldt State University, August 1985. LC

Combinatorics, S, P, L**.** *Mathematical Gems III.* Ross Honsberger. Dolciani Math. Expos., No. 9. MAA, 1985, 250 pp, $28. [ISBN: 0-88385-313-2] Eighteen vignettes of number theory, combinatorics and geometry, frequently inspired by problems in *Crux Mathematicorum.* The third collection of Honsberger "gems" designed to delight, entertain, and challenge anyone who has studied freshman college mathematics. Ideal source for an undergraduate mathematics seminar. LAS

Discrete Mathematics, P. *Discrete Iterations: A Metric Study.* Francois Robert. Transl: Jon Rokne. Ser. in Comput. Math., V. 6. Springer-Verlag, 1986, xvi + 195 pp, $59. [ISBN: 0-387-13623-1] Discrete iterative models (iteration graphs, automata networks) are associated with iterations of a function $F: X \rightarrow X, X$ a product of n copies of $\{0,1\}$ (usually). Notions of convergence from normed spaces (contractions, fixed points, Gauss-Seidel serial-parallel iterations, Newton's method) for the iterative schema $x_{r+1} = F(x_r)$ are transformed into the discrete setting where, because of the exact arithmetic, round-off is not an issue. RM

Topological Groups, S(18), P*. *Representation Theory of Semisimple Groups: An Overview Based on Examples.* Anthony W. Knapp. Math. Ser., V. 36. Princeton U Pr, 1986, xvii + 773 pp, $75. [ISBN: 0-691-08401-7] Self-contained survey of results and techniques, omitting some proofs and sketching others. Emphasizes explicit formulas and examples, especially $SL(2,R)$. Numerous exercises end each chapter. Some good expository sections outline main ideas. BC

Topological Groups, P. *Representations of Rank One Lie Groups.* David H. Collingwood. Res. Notes in Math., V. 137. Pitman, 1985, 244 pp, $39.95 (P). [ISBN: 0-273-08697-9] Research monograph on principal series representations and the Kazhdan-Lusztig conjectures for connected semisimple real rank one matrix groups. Applications to Harish-Chandra, Verma, and Jacquet modules. Leans to-

ward "proof by example," but the examples are very detailed. BC

Algebra, T(16-17: 1), S. *Categories.* T.S. Blyth. Longman, 1986, 152 pp, $34.95. [ISBN: 0-582-98804-7] A slim introductory text for beginners that aims at "a couple of dozen lectures with enough interest to provide a good idea of what categories are about." Numerous examples, exercises, solutions, index. JS

Calculus, P*, L*. *Toward A Lean and Lively Calculus.* Ed: Ronald G. Douglas. Notes, No. 6. MAA, 1986, xxvii + 249 pp, $12.50 (P). [ISBN: 0-88385-056-7] Long-awaited report of the Calculus Conference held at Tulane University in January 1986. Opening papers by conference organizer Ron Douglas identify numerous problems with calculus as currently taught. Reports from three conference working groups outline alternative curricula, teaching strategies, and implementation options. 18 papers by conferees provide analysis of courses, texts, applications and the new symbolic algebra packages. An important stimulus for needed reform. LAS

Real Analysis, T(17: 1). *Lebesgue Measure and Integration.* P.K. Jain, V.P. Gupta. Wiley, 1986, viii + 260 pp, $17.95. [ISBN: 0-470-20296-3] Contains all the usual material on Lebesgue measure and integration in a formal definition-theorem-proof-example format. A good source of problems at all levels. TAV

Real Analysis, T(17-18: 1), S, P. *Differential Calculus.* A. Avez. Transl: D. Edmunds. Wiley, 1986, xii + 179 pp, $32.95. [ISBN: 0-471-90873-8] A crisp, enthusiastic introduction to the basics of differential analysis from an advanced point of view. Covers the derivative, Taylor's formula, implicit and inverse function theorems, differential equations, and calculus of variations, mostly in Banach spaces. Eight brief appendices fill in background. There are frequent references to further reading, but no exercises. PZ

Complex Analysis, P. *Lecture Notes in Mathematics-1188: Fonctions de Plusieurs Variables Complexes V.* Ed: Francois Norguet. Springer-Verlag, 1986, 301 pp, $25 (P). [ISBN: 0-387-16460-X] Eight long papers on topics in several complex variables, dating 1979-1985, from the Séminaire Norguet at the University of Paris VII. An extended expository-research monograph, by G. Roos, studies integral formulae in several complex variables. PZ

Complex Analysis, P. *The Bieberbach Conjecture: Proceedings of the Symposium on the Occasion of the Proof.* Ed: Albert Baernstein, II, et al. Math. Surveys & Mono., No. 21. AMS, 1986, xvi + 218 pp, $45. [ISBN: 0-8218-1521-0] 15 mathematical papers, 4 personal accounts—by L. de Branges and others, and a poem from the March 1985 Purdue conference celebrating de Branges' proof of the 69-year old Bieberbach conjecture in geometric function theory of a complex variable. The first paper, by L. Ahlfors, assesses the proof and its significance for classical analysis. Another paper lists open problems. PZ

Differential Equations, T(14-15). *Elementary Differential Equations with Linear Algebra.* David L. Powers. Prindle, Weber & Schmidt, 1986, x + 573 pp. [ISBN: 0-87150-957-1] Written with engineering students in mind; applications motivate, methods precede theory; proofs are largely passed over in favor of explanations and illustrations of their meaning. Assumes one year of calculus and power series; develops linear algebra with the use of eigenvalues in systems of differential equations as a goal. Concludes with chapters on the Laplace Transform and on numerical methods. Appears to be an attractive text for the intended audience. AWR

Differential Equations, T(14-15: 1), S, L. *Elementary Differential Equations.* William Ted Martin, Eric Reissner. Dover, 1986, xiii + 331 pp, $7.95 (P). [ISBN: 0-486-65024-3] Unabridged republication of the *Second Edition* published in 1961 by Addison-Wesley. Solid presentation in the old style which treats the one-time standard applications (mechanics and electric circuits, for example) thoroughly and clearly. Not as pretentious as current textbooks, not as encyclopedic, a bit on the stodgy side, but an excellent presentation. Highly recommended for those seeking to review differential equations. Answers to both even *and* odd exercises. JK

Differential Equations, P. *Monotone Iterative Techniques for Nonlinear Differential Equations.* G.S. Ladde, V. Lakshmikantham, A.S. Vatsala. Mono., Adv. Texts & Surveys in Pure & Appl. Math., V. 27. Pitman, 1985, x + 236 pp, $110. [ISBN: 0-273-08707-X] An introduction to use of the method of upper and lower solutions together with monotone iterative techniques in the study of nonlinear ordinary and partial differential equations. AO

Differential Equations, P. *One-dimensional Inverse Problems of Mathematical Physics.* M.M. Lavrent'ev, K.G. Reznitskaya, V.G. Yakhno. AMS Transl, Ser. 2, V. 130. AMS, 1986, vi + 70 pp, $92. [ISBN: 0-8218-3099-6] An investigation of one-dimensional inverse problems such as those associated with certain types of seismic analysis. AO

Numerical Analysis, P. *Foundations of the Numerical Analysis of Plasticity.* Tetsuhiko Miyoshi. Math. Stud., V. 107. Elsevier Science, 1985, xi + 249 pp, $55.50 (P). [ISBN: 0-444-87671-5] Presents a framework for the development and analysis of numerical methods for the solution of dynamic and quasi-static plasticity problems. AO

Analysis, P. *Some Random Series of Functions, Second Edition.* Jean-Pierre Kahane. Stud. in Adv.

Math., V. 5. Cambridge U Pr, 1985, xiii + 305 pp, $47.50. [ISBN: 0-521-24966-X] A minor revision of the 1968 *First Edition.* The author presents the theory of series with coefficients being random variables, then applies the theory to a variety of geometric and analytic problems. Contains an extensive bibliography. TAV

Analysis, P. *Methods in Approximation: Techniques for Mathematical Modelling.* Richard E. Bellman, Robert S. Roth. Math. & Its Applic. D Reidel, 1986, xv + 224 pp, $49. [ISBN: 90-277-2188-2] A collection of techniques for mathematical approximation. Includes techniques such as one to find a linear differential equation whose exact solution is a good approximation to the solution of a given nonlinear differential equation. Also discusses polynomial, spline, and exponential approximation as well as the finite element method. AO

Analysis, P. *Strong Approximation by Fourier Series.* László Leindler. Akademiai Kiado, 1985, 210 pp. [ISBN: 963-05-4044-4] The concept of strong approximation is due to Hardy and Littlewood (1913), but its connection to Fourier series is much newer (Alexits, 1963). This monograph attempts to bring together the most important results on the subject in the last 20 years. A formidable task, well executed. TAV

Analysis, P. *Theory of Holors: A Generalization of Tensors.* Perry Moon, Domina Eberle Spencer. Cambridge U Pr, 1986, xix + 392 pp, $69.50. [ISBN: 0-521-24585-0] The word *holor* was coined by the authors to describe mathematical entities made up of one or more independent quantities (e.g., complex numbers, matrices, tensors). Describes a unified theory of holors (tensor or nontensor) using a single notation that applies to all holors. AO

Analysis, T(17-18: 1, 2), S*. *Asymptotic Expansions of Integrals.* Norman Bleistein, Richard A. Handelsman. Dover, 1986, xiii + 425 pp, $10.95 (P). [ISBN: 0-486-65082-0] Unabridged, corrected republication of the 1975 edition by Holt, Rinehart and Winston. New preface. Introductory material presented in a well-organized manner, sufficient for a one-year course. Readers will need a good background in advanced calculus, differential equations, and complex variables. Applications. Exercises. A valuable and inexpensive reference. JK

Differential Geometry, P. *Nonlinear Problems in Geometry.* Ed: Dennis M. DeTurck. Contemp. Math., V. 51. AMS, 1985, ix + 130 pp, (P). [ISBN: 0-8218-5053-9] Proceedings of an AMS Special Session held May 3-4, 1985, in Mobile, Alabama. Most of the lectures are represented here. Some work that was invited but not presented also appears. Topics include elliptic problems on complete open manifolds, the long-time behavior of solutions to geometric evolution equations, symmetrization, spectral theory, and several others. JK

Differential Geometry. *The Mathematics of Surfaces.* Ed: J.A. Gregory. Inst. of Math. & Its Applic. Conf. Ser., No. 6. Clarendon Pr, 1986, xi + 282 pp, $49. [ISBN: 0-19-853609-7] A series of papers, mostly by engineers and computer scientists, on aspects of classical differential geometry and applications to areas such as computer-aided design. Topics covered include differential forms, spline algorithms, surface intersections, and parametric representations. The mathematics discussed is very concrete and is presented with an eye toward applications. SG

Geometry, S(18), P. *Finite Projective Spaces of Three Dimensions.* J.W.P. Hirschfeld. Math. Mono. Clarendon Pr, 1985, x + 316 pp, $65. [ISBN: 0-19-853536-8] This is the second volume in a planned set of three. The first volume gave introductory material on finite fields and projective spaces, elementary properties of n-dimensional spaces over finite fields and the line and plane. This volume is devoted to finite projective spaces in three dimensions. Includes an excellent bibliography. CEC

Operations Research, T(14-15). *Extremal Methods of Operations Research.* Paul R. Gribik, Kenneth O. Kortanek. Pure & Appl. Math., V. 97. Dekker, 1985, viii + 312 pp, $37.50. [ISBN: 0-8247-7474-4] An introductory level book of three chapters: transportation problems, network flow, linear programming. An unusually extensive set of solutions to exercises occupies the last 95 pages. Written with college sophomores in mind, the typed-for-camera format may work against acceptance by the intended audience—which would be too bad since this is a nice introduction to the topic. AWR

Operations Research, T*(15: 1, 2), L. *Quantitative Methods for Management Decisions.* William P. Cooke. McGraw-Hill, 1985, xv + 686 pp, $33.95. [ISBN: 0-07-012518-X] Intended for a survey course in OR/MS—the mathematical prerequisite is college algebra. Covers topics in linear programming, dynamic programming, networks, queues, simulation. Many very nice applications and exercises. While the level of sophistication is not high, the level of presentation is. TAV

Optimization, T(17: 1), P. *Optimal Sequential Block Search.* Li Weixuan. Res. & Expos. in Math., V. 5. Heldermann Verlag, 1984, vii + 209 pp, (P). [ISBN: 3-88538-205-9] Presents an abstract treatment of sequential block searches, also called line searches, for use in optimizing unimodal functions of one real variable over a given interval. Prerequisite: elementary matrix theory. SM

Control Theory, T*(16-17: 3), S, L. *Introduction to Differential Games and Control Theory.* V.N. Lagunov. Sigma Ser. in Appl. Math., V. 1. Heldermann Verlag, 1985, viii + 285 pp, $88 (P). [ISBN: 3-88538-401-9] Text for a three semester course for graduate students or advanced undergraduates. Covers the general theory of games, optimal control, and the fundamentals of differential games. Well written with intuitive examples as well as mathematical rigor. SM

Systems Theory, P. *Metamodeling: A Study of Approximations in Queueing Models.* Subhash Chandra Agrawal. MIT Pr, 1985, xvii + 262 pp, $22.50 (P). [ISBN: 0-262-01080-1] Queueing network models are the most widely used analytical tool for estimating performance of a computer system. Many technical questions arise—the author provides a model for the modelling process to give a framework for answering those questions. Well conceived and clearly written. TAV

Probability, P. *Brownian Motion and Stochastic Flow Systems.* J. Michael Harrison. Wiley, 1985, xxi + 140 pp, $31.95. [ISBN: 0-471-81939-5] A compact treatment of Brownian motion and the Ito calculus. Special emphasis is given to the fundamental Ito formula and its generalizations. A systematic account of regulated Brownian motion leads to analysis of stochastic flows. TAV

Probability, T*(18: 2), P*. *Markov Processes: Characterization and Convergence.* Stewart N. Ethier, Thomas G. Kurtz. Wiley, 1986, x + 534 pp, $47.50. [ISBN: 0-471-08186-8] Beginning with the idea that weak convergence is tied to the characterization of a limit process, the authors lay the foundation for their work, then provide a comprehensive treatment. Many exercises, a very good bibliography. TAV

Computer Literacy, S, P, L*. *Using Computers: The Human Factors of Information Systems.* Raymond S. Nickerson. MIT Pr, 1986, xiv + 434 pp, $22.50. [ISBN: 0-262-14040-3] A substantial systematic survey of trends and research in the interface between people and computers: physical and cognitive interface, artificial intelligence, information services, quality of life. Well referenced to a massive 60-page bibliography. Name and subject indices. LAS

Computer Programming, T(13-14: 1). *Reasoning With a Computer in Pascal.* Daniel Solow. Addison-Wesley, 1986, xiii + 481 pp, $25.95 (P). [ISBN: 0-201-12060-7] Introduction to programming in Pascal, with emphasis on problem-solving process, techniques, program tracing and error correction. Uses nice program development style, Pascal-like pseudo-code, general programming concepts with less emphasis on Pascal specifics. RM

Computer Programming, S(14-15). *Practical PL/I.* Gordon R. Clarke, Sue Green, Peter Teague. Comput. Soc. Mono. in Informatics. Cambridge U Pr, 1985, x + 217 pp, $24.95 (P). [ISBN: 0-521-31768-1] A guide to the effective use of the programming language PL/1. Not a language reference manual, but a review of major language features together with examples of their use. AO

Computer Programming, T(13: 1). *BASIC Programming: A Problem-Solving Approach.* Elton Earl Beougher. Gorsuch Scarisbrick, 1986, x + 222 pp, $17 (P). [ISBN: 0-89787-412-9] Standard BASIC for the beginning computer user. About ten programming exercises per chapter. Discusses doing graphics and handling disk files on the Apple II and IBM PC microcomputers. An appendix compares BASIC dialects on those, Commodores, Zenith Z-150's, and TRS-80's. DFA

Computer Programming, T(13: 1), S, P. *Essentials of BASIC with Structure.* Coleman Barnett. Gorsuch Scarisbrick, 1986, vii + 199 pp, $15 (P). [ISBN: 0-89787-414-5] An introduction to BASIC which includes chapters on sequence logic, selection logic, loop logic, structured design and organization, arrays, sequence checking, control break, formating printed output, and debugging. The development of good programming techniques through flowcharts is described and illustrated. Includes a reasonable number of programming exercises. CEC

Computer Science, P. *On the Design of ALEPH.* D. Grune. CWI Tract, V. 13. Math Centrum, 1986, v + 191 pp, Dfl. 28.80 (P). [ISBN: 90-6196-284-6] ALEPH, A Language Encouraging Program Hierarchy, exploits the analogy between grammars and programs, parsing and problem solving. Programming methods (selection of alternatives, decomposition into sub-problems, packaging actions into named sub-procedure) are reflected in the structure of a context-free grammar (left hand side of production corresponding to procedure header, right hand side to body). The programming language procedures are top-down descriptions of actions similar to the rules of an affix grammar, whose compiler can deal with uninitialized variable and side-effect problems. Describes the design of ALEPH, a compiler, and the manual. RM

Computer Science, T(16-17: 1), P, L. *Data Structures of Pascal, Algol 68, PL/1 and Ada.* Johan Lewi, Jan Paredaens. Springer-Verlag, 1986, xii + 395 pp, $42 (P). [ISBN: 0-387-15121-4] A textbook for a course in programming language concepts. Each chapter is devoted to a single language concept. Compares and evaluates Pascal and the Pascal-like subsets of Algol 68, PL/1, and Ada. AO

Computer Science, P. *Proceedings of the Fourth*

British National Conference on Databases (BNCOD 4). Ed: A.F. Grundy. Comput. Soc. Workshop Ser. Cambridge U Pr, 1985, x + 229 pp, $44.50. [ISBN: 0-521-32020-8] Twelve papers presented at a conference held July 10-12, 1985 at the University of Keele. The papers cover topics in access and concurrency control, models and mapping, and database management system design and use. AO

Applications (Biology), P*. *Lecture Notes in Biomathematics-68: The Dynamics of Physiologically Structured Populations.* Ed: J.A.J. Metz, O. Diekmann. Springer-Verlag, 1986, xii + 511 pp, $47.40 (P). [ISBN: 0-387-16786-2] Based on notes from a colloquium in Amsterdam in 1983. Participants were mainly biologists. One objective of structured population dynamics is the use of biological knowledge to describe the interaction of a population and its environment (including other populations) by specifying how birth, death, and growth processes depend on environmental quantities. Notes contain a systematic exposition of the present state of the mathematical theory, a collection of papers on special topics, techniques and results, together with biological interpretations in terms of special models. JK

Applications (Communication Theory), T(14-16: 1), S, L. *A First Course in Coding Theory.* Raymond Hill. Appl. Math. & Comput. Sci. Ser. Clarendon Pr, 1986, xii + 251 pp, $35. [ISBN: 0-19-853804-9] Probably the most elementary text available. The author is aiming at second-year undergraduates, assuming no more than some familiarity with matrix arithmetic. Concerned almost exclusively with block codes but not restricted to binary codes. Exercises, solutions, bibliography, index. JS

Applications (Economics), S(17), P. *Transport System Optimization and Pricing.* Jan Owen Jansson. Wiley, 1984, v + 280 pp, $54.95. [ISBN: 0-471-10264-4] The author presents a new theory for pricing transportation services with emphasis placed on the effects of user cost. Majority of book deals with case studies involving different modes of transportation services. Discussion of monopoly effects. SM

Applications (Management), T(17), P. *Multiple-Criteria Decision Making: Concepts, Techniques, and Extensions.* Po-Lung Yu. Math. Conc. & Methods. in Sci. & Eng., V. 30. Plenum Pr, 1985, xiv + 388 pp, $49.50. [ISBN: 0-306-41965-3] A graduate-level introduction to MCDM. The stated pre-requisites are courses in optimization and operations research. Such preparation had best be mathematically oriented. Besides game theoretic notions and multiple-criteria simplex methods, there is extensive coverage of differential game theory. AWR

Applications (Medicine), T(16-17: 1), S. *Mathematics and Physics of Neutron Radiography.* A.A. Harms, D.R. Wyman. Texts in Math. Sci. D Reidel, 1986, xiii + 163 pp, $39.50. [ISBN: 90-277-2191-2] An introduction to the mathematics and physics of neutron radiography. AO

Applications (Physics), P. *Mathematics + Physics: Lectures on Recent Results, Volume 2.* Ed: L. Streit. World Scientific, 1986, vii + 344 pp, $33 (P). [ISBN: 9971-978-60-1] Eight lectures on mathematical aspects of physics from Center for Interdisciplinary Research at Bielefeld University. All articles are generally readable and provide brief introductions to each subject. Topics covered include topology and anomalies, tunneling in one-dimension, and computer quantum field theory. Excellent article on nonstandard analysis giving applications to differential equations and probability theory (hyperfinite random walks). MR

Applications (Physics), T(17-18, 1, 2), S, P. *Mathematical Models in Applied Mechanics.* A.B. Tayler. Appl. Math. & Comput. Sci. Ser. Clarendon Pr, 1986, viii + 280 pp, $29.95; $14.95 (P). [ISBN: 0-19-853533-3; 0-19-853541-1] Designed to be a coherent text, this book develops the theory and applications of partial differential equations through a series of problems which have arisen from industrial research: furnace reaction analysis, submarine detection, coal steam exploration, the hot rolling of steel, percolation in a sand dune, welding, the shape of laser melt-pools, the spreading of oil films, and others. However, all discussion of numerical methods have been omitted. MU

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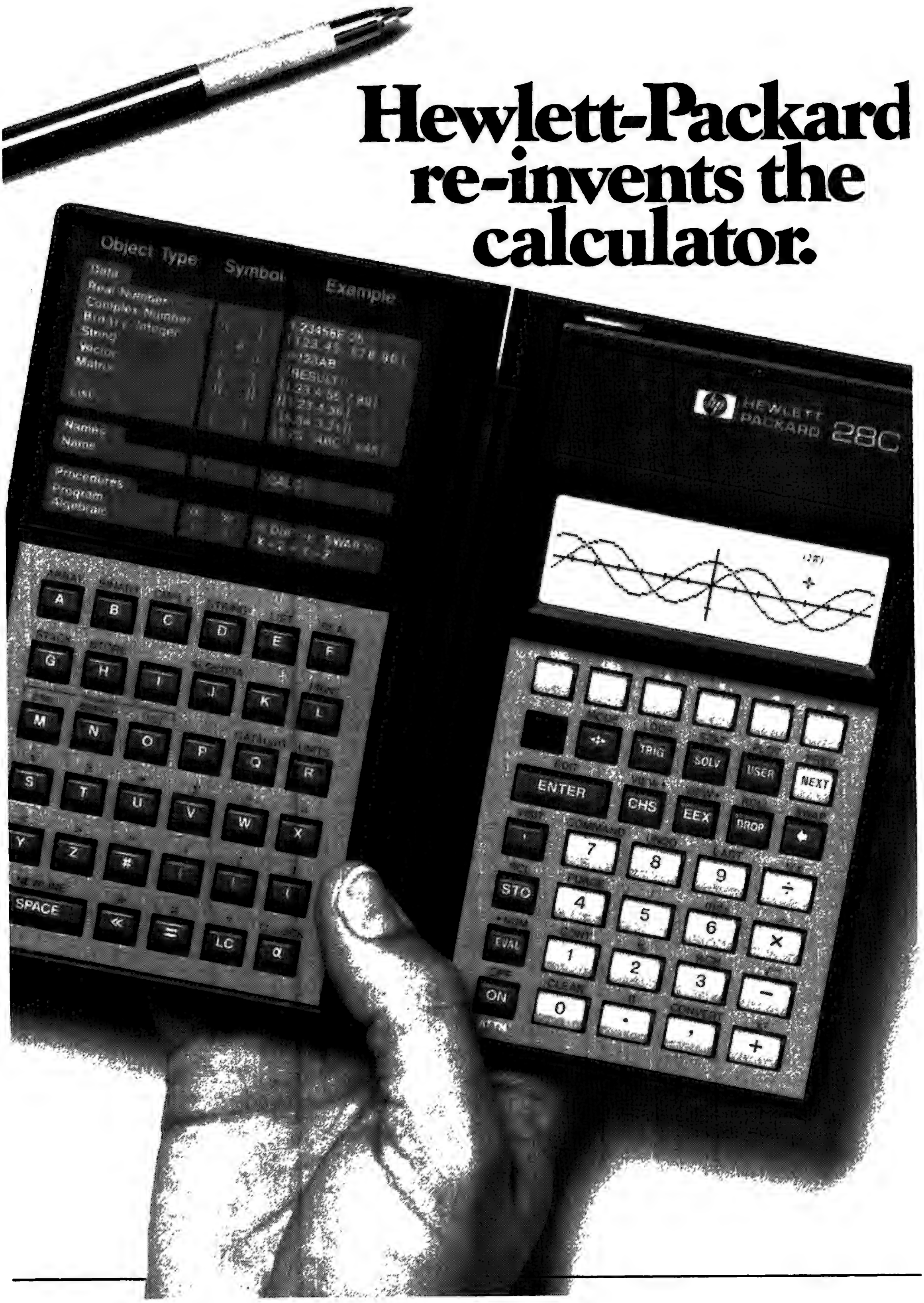
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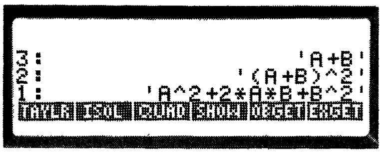

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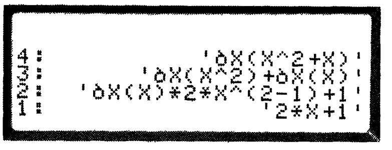

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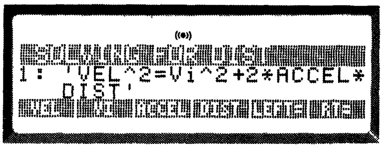

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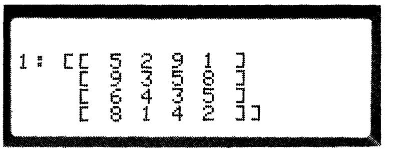

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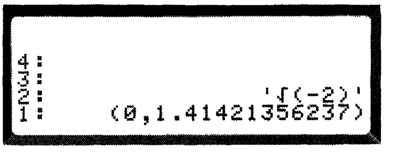

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THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 4 April 1987

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(ISSN 0002-9890)

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Subscription correspondence, changes of address, and inquiries about nondelivered or defective issues: Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department. Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

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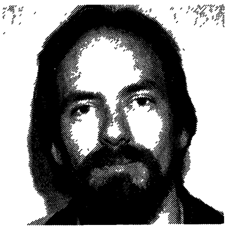
On the Runge Example

JAMES F. EPPERSON, *University of Georgia, Athens*



The author received his Ph.D.—on the numerical solution of the Stefan Problem—from Carnegie-Mellon University (1980), under George Fix.

1. Introduction. The "Runge phenomenon," or Runge example, is the classic illustration of polynomial interpolation *non*convergence. Stated briefly, if $p_n(x)$ is the polynomial interpolating

$$f(x) = (1 + x^2)^{-1}, \qquad x \in [-5,5] \tag{1}$$

at the equidistant nodes $x_j^{(n)} = -5 + 10(j/n)$ $(j = 0, 1, \ldots, n)$, then $p_n \to f$ uniformly only if $|x| < x_c \approx 3.63$. If $x_c < |x| < 5$, then divergence occurs.

The example is given at the end of Runge's paper **[9]**, in which he discusses the general theory of convergence for interpolation at equidistant nodes. Essentially, the distribution of nodes defines (in the limit, as $n \to \infty$), a family of curves $C(\rho)$—Runge called them U-curves—which are centered about the origin. For a fixed $\rho > 0$, convergence occurs if f is analytic inside $C(\rho)$. In the case of Runge's example, the contour $C(\rho^*)$ which passes through the singularities of f at $\pm i$ also crosses the axis at $x_c = \pm 3.63\ldots$, which explains the result given above.

Unfortunately a complete and rigorous development of these results (see below) requires some subtle analysis as well as the evaluation of complex integrals via the theory of residues. This yields some very elegant and aesthetically pleasing mathematics, but it also places the material beyond the comprehension of most students in undergraduate numerical analysis courses. On the other hand, the *fact* of the Runge example is important enough that one should present it to such a class. But to present it without any justification is difficult, at best, since the Runge example is not the least bit intuitive—and few texts provide any help.

The present paper attempts to fill this gap by bringing together several explanations/developments of the Runge example which do not require extensive complex analysis. Not surprisingly, these are not as sharp as the complete (complex) analysis but they do provide some insight to the essential points.

For completeness, we also outline the complex error analysis, and provide some discussion of the role played by the Chebyshev nodes. Our style is informal, and, hence, many of the proofs are done by reference or merely in outline. The primary goal is to provide an adequate basis for explaining polynomial interpolation nonconvergence. (We might also note in passing that the complex remainder theory is an excellent application of residue theory, and as such could well be presented to a beginning class in complex variables.)

2. A Selected Review of Interpolation Remainder Theory. Let $n > 0$ and $f \in C^{n+1}(I)$, $I = [a, b]$, be given. Denote the interpolation nodes by $\{x_j^{(n)}\}$, for $0 \leqslant j \leqslant n$. Then, if p_n is the polynomial of degree n which interpolates f at those nodes, the usual error estimate is [1, p. 56]:

$$f(x) - p_n(x) = \frac{1}{(n+1)!}\left\{\prod_{j=0}^{n}\left(x - x_j^{(n)}\right)\right\} f^{(n+1)}(\xi), \tag{2}$$

where ξ is a point on the interval containing x and the nodes. Convergence proofs based on (2) require some estimate of $|f^{(k)}(x)|$ as $k \to \infty$; for example, if $|f^{(k)}(x)| \leqslant M$ for all k and all $x \in I$, then it is easy to show $p_n \to f$ uniformly on I. (It is worth noting that this holds for an arbitrary distribution of the nodes $\{x_j^{(n)}\}$.) Because it is so difficult to estimate the derivative term, (2) has been of little use in explaining the Runge example. There is a related estimate [4]

$$f(x) - p_n(x) = \left\{\prod_{j=0}^{n}\left(x - x_j^{(n)}\right)\right\} f[x_0, \dots, x_n, x], \tag{3}$$

where $f[x_0, \dots, x_n, x]$ is the $(n+1)$st Newton divided difference of f [4]. This form of the remainder has been used by Isaacson and Keller [4] to treat the Runge example, and we expand upon this approach in §3d, below.

From now on, the polynomial $w_n(x)$ will be defined as

$$w_n(x) = \prod_{j=0}^{n}\left(x - x_j^{(n)}\right). \tag{4}$$

The complex analogue of (2) and (3) is a contour integral [1, p. 67]:

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_T} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{5}$$

Here C_T is the boundary of a domain T, f is analytic in T, and z and all the nodes are contained in T. Note that T is allowed to contain "holes."

A key point in the analysis of the error is the behavior of $|w_n(z)|$ as $n \to \infty$. From [1, p. 84] we have the following:

LEMMA 1. *Assume the $\{x_j^{(n)}\}$ are equidistant nodes on $[a, b]$, and define*

$$\sigma_n(z) = |w_n(z)|^{1/(n+1)}.$$

Then

$$\lim_{n \to \infty} \sigma_n(z) = \sigma(z) \tag{6}$$

exists for all z. In particular,

$$\sigma(z) = \exp\left\{\frac{1}{b-a}\int_a^b \log|z - s|\, ds\right\}. \tag{7}$$

Note that (7) says that $\sigma(z)$ is the geometric mean of $|z - s|$, which is not surprising, since $|w_n(z)|^{1/(n+1)}$ is the geometric mean of the $|z - x_j^{(n)}|$.

For $\rho > 0$, consider now the family of curves

$$C(\rho) = \{z \in \mathbb{C} | \sigma(z) = \rho\}.$$

These are smooth concentric curves about the midpoint of $[a, b]$; in fact, their level curves in terms of (x, y) coordinates can be computed by integrating in (7). These curves and the placement of z relative to them are the key to convergence: if $\zeta \in C(\rho)$ and $z \in C(\rho')$, $\rho' < \rho$, we can show

$$\lim_{n \to \infty} \left| \frac{w_n(z)}{w_n(\zeta)} \right| = 0,$$

which can be used to prove convergence of p_n to f.

PROPOSITION. *Let the interpolation nodes $\{x_j^{(n)}\}$ be contained in a contour $C(\rho)$ and suppose f is analytic inside $C(\rho)$. Then $p_n \to f$ uniformly on $C(\rho')$, $\rho' < \rho$.*

Proof. Let z be in $C(\rho')$; then (5) implies

$$|f(z) - p_n(z)| \leqslant C \max_{\zeta \in C(\rho)} \left| \frac{w_n(z)}{w_n(\zeta)} \right|.$$

This follows from the maximum modulus theorem and the fact that $|z - \zeta| > 0$ (since the two curves $C(\rho)$ and $C(\rho')$ do not intersect). But, for ε arbitrarily small and n sufficiently large,

$$\left| \frac{w_n(z)}{w_n(\zeta)} \right|^{1/(n+1)} = \left(\frac{\sigma_n(z)}{\sigma_n(\zeta)} \right) \leqslant \left(\frac{\rho' + \varepsilon}{\rho - \varepsilon} \right) < 1;$$

thus

$$|f(z) - p_n(z)| \leqslant C\theta^{n+1}$$

for $0 < \theta < 1$, and convergence follows.

Suppose now that f is not analytic inside $C(\rho)$; suppose in fact, that there is a single simple pole at z^*, and, in order to enclose z in a contour $C(\rho)$, we must also enclose z^*. Then, for the error representation (5) to be valid, the contour C_T must consist of the union of $C(\rho)$ and a small path around z^*, say C^*. Then

$$\frac{1}{2\pi i} \int_{C(\rho)} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{C_T} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{C^*} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Since z is inside $C(\rho)$, the integral on the left goes to zero as $n \to \infty$, exactly as in the previous case; the first integral on the right is exactly the error $f(z) - p_n(z)$; and the last integral can be quickly evaluated by a residue. Thus we have

$$f(z) - p_n(z) = \left(\frac{w_n(z)}{w_n(z^*)} \right) \left(\frac{f(z^*)}{z^* - z} \right) + \delta_n, \tag{8}$$

where $|\delta_n| \to 0$ as $n \to \infty$. Now, since $z \in C(\rho)$ and $z^* \in C(\rho^*)$, $\rho^* < \rho'$, it

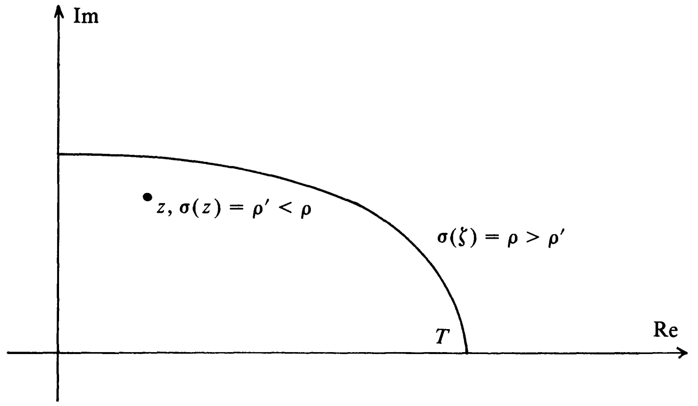


FIG. 1. Convergence occurs in the region T so long as f is analytic there.

follows that, as before,

$$\left|\frac{w_n(z)}{w_n(z^*)}\right|^{1/(n+1)} = \left(\frac{\sigma_n(z)}{\sigma_n(z^*)}\right) \geqslant \frac{\rho' - \varepsilon}{\rho^* + \varepsilon} > 1.$$

Thus the error grows without bound. (See Figures 1 and 2.) If we enclose more than one pole, then the *-terms in (8) are replaced by an appropriate summation over all the poles; divergence still occurs.

In the case of Runge's example (1), the contour $C(\rho_c)$ which passes through the singularities $\pm i$ crosses the x-axis at $x_c = 3.6333843024$. Thus, for $|x| < x_c$ we can draw C_T such that each $\zeta \in C_T$ is on a contour $C(\rho_\zeta)$, $\rho_\zeta > \rho_c - \varepsilon$, and C_T encloses

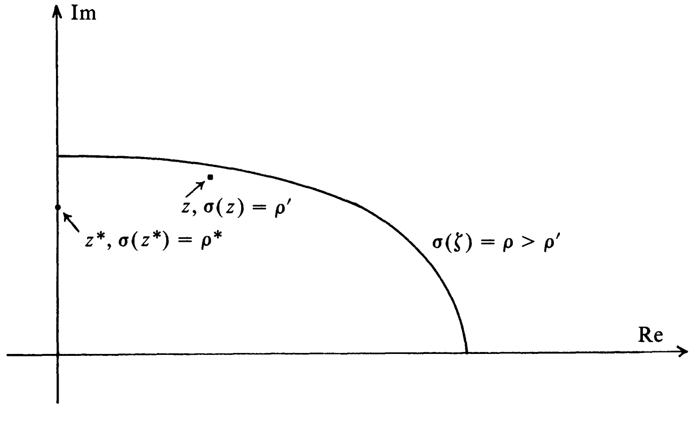


FIG. 2. Divergence occurs at z because we must enclose the pole z^* in order to also enclose z.

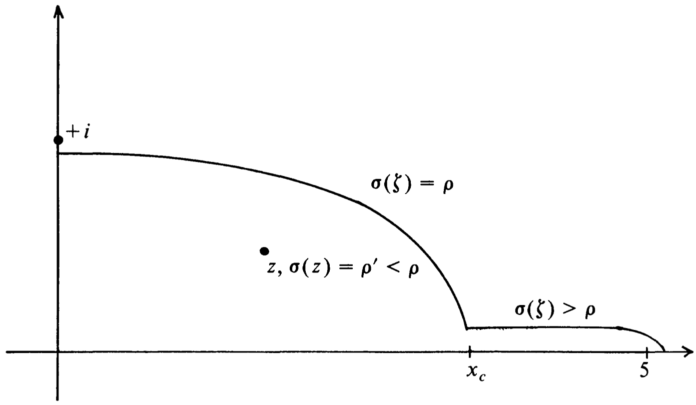


FIG. 3. The Runge Example: convergence occurs for z because $\sigma(z) < \rho$ and the contour C_T (solid curve) consists of points ζ satisfying $\sigma(\zeta) \geq \rho$.

all the interpolation nodes as well as x, but not the poles $\pm i$. (See Figure 3.) Since $|x| < x_c$ it follows that $x \in C(\rho)$, $\rho < \rho_c - \varepsilon$; hence

$$\min_{\zeta \in C_T} \left| \frac{w_n(x)}{w_n(\zeta)} \right|^{1/(n+1)} \to \theta < 1$$

and convergence follows.

If x is outside x_c then any contour enclosing x must also enclose the poles and divergence occurs.

We summarize the foregoing discussion in the following theorem:

THEOREM 1. *Let $\{p_n\}$ be a sequence of polynomials interpolating f at the equidistant points $\{x_k^{(n)}\}$, $0 \leqslant k < n$, with each $x_k^{(n)} \in [a, b]$. Assume f is analytic on $[a, b]$. Let $\sigma(z)$ be as defined in* (6). *Then*:

i) *If f is analytic for all z such that $\sigma(z) < \rho$, then $p_n \to f$ for each z^* such that $\sigma(z^*) < \rho$. The convergence is uniform for all z^* such that $\sigma(z^*) \leqslant \rho^* < \rho$.*

ii) *If f has a pole z^*, and z is such that $\sigma(z) > \sigma(z^*)$, then $p_n(z) \nrightarrow f(z)$.*

If we switch from uniform nodes to the Chebyshev nodes (these are the roots of the $(n+1)$st Chebyshev polynomial—see §4), then the contours $C(\rho)$ become true ellipses having foci at $x = \pm 5$ [**1**, p. 83]. Thus, it is easy to choose a contour which completely encloses the interval without hitting the singularities at $\pm i$. In fact, the following theorem holds.

THEOREM 2. *If f is analytic in an open domain containing $I = [a, b]$, and p_n interpolates f at the Chebyshev nodes on $[a, b]$, then*

$$p_n \to f$$

uniformly on $[a, b]$. (*Note that if* $[a, b] \neq [-1, 1]$ *the Chebyshev nodes must be transformed*.)

3. The Runge Example Without Residues. In this section we present several discussions aimed at justifying (if only partially) the Runge example without having to use residues or the complex remainder (5).

3a. *Interpolation is Ill-Conditioned* [**3**]. Here we show that small errors in computing the $\{f(x_k^{(n)})\}$—round-off error, or experimental "noise"—can be greatly magnified by interpolation at equidistant nodes, especially for x "near the edge of the interval."

Consider the Lagrange form of the interpolating polynomial [**1**, p. 33]:

$$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x),$$

where

$$l_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{x - x_k^{(n)}}{x_j^{(n)} - x_k^{(n)}} \right).$$

If the nodes are equally spaced then

$$x_j^{(n)} = a + jh, \qquad h = \left(\frac{b-a}{n} \right).$$

Suppose now that a small error is made in computing each $f(x_j)$. That is, we actually form

$$\bar{p}_n(x) = \sum_{j=0}^{n} \bar{f}(x_j) l_j(x),$$

where $\bar{f}(x_j) = f(x_j) + \varepsilon_j$. Then the error due to roundoff is

$$E_n = \sum_{j=0}^{n} l_j(x) \varepsilon_j.$$

Let $x = a + h/2$ so that x is near the edge of the interval. Then

$$l_j\left(a + \frac{h}{2}\right) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{\frac{1}{2} - k}{j - k} \right),$$

and, after a page or so of calculation, this becomes

$$l_j\left(a + \frac{h}{2}\right) = \left(\frac{-1}{2j - 1} \right) \binom{n}{j} \left[\frac{(2n)!}{2^{2n} (n!)^2} \right].$$

Using Stirling's formula the bracketed term can be estimated as

$$\left[\frac{(2n)!}{2^{2n}(n!)^2}\right] \sim \frac{1}{\sqrt{n}}.$$

Thus,

$$l_j\left(a + \frac{h}{2}\right) \sim \frac{1}{\sqrt{n}}\left(\frac{-1}{2j-1}\right)\binom{n}{j}.$$

If $n = 2m$ and $j = m$, i.e., even for j near the middle of the interval, we have

$$\binom{n}{j} = \binom{2m}{m} = \frac{(2m)!}{(m!)^2} \sim \frac{2^n}{\sqrt{m}},$$

so that

$$l_m\left(a + \frac{1}{2}\right) \sim \left(\frac{-1}{n-1}\right)m^{-1}(2^n).$$

Hence the small error $\varepsilon_{n/2}$ is multiplied by a factor which grows exponentially. Unless we are extremely lucky with cancellation, the amplified error will eventually dominate the calculation. Unfortunately, this approach to the Runge phenomenon ignores the role of f and leaves the impression that the divergence is perhaps due to machine error, which is not at all the case. On the other hand, it is a valuable demonstration of why polynomial interpolation (with equidistant nodes) at high degree is not, in general, a good approximation technique, even for analytic functions.

3b. *Growth of the Interpolate* [**2**], [**7**]. Obviously if $\|p_n\| \to \infty$ then p_n isn't a good approximation to "nice" functions f. If $\|\quad\|$ is the sup-norm, then we have

$$\begin{aligned}\|p_n\| &= \left\|\sum_{j=0}^{n} l_j f(x_j)\right\| \\ &\leqslant \|f\| \sum_{j=0}^{n} \|l_j\| \\ &= \|f\|\Lambda_n,\end{aligned}$$

and there are functions f for which this bound is sharp [**2**], i.e., equality holds. For equally spaced nodes $x_j^{(n)}$ it can be shown [**8**, pp. 87–99] that

$$\Lambda_n \geqslant Cn^{-3/2}(2^{n-1}), \tag{9}$$

thus the norm of the interpolate is unbounded. (The argument leading to (9) is very similar to what we used in the preceding subsection.) Again, we have a result showing that the l_j's can grow rapidly, but it is not clear how this worst case estimate would apply to a very smooth function like $(1 + x^2)^{-1}$. The functions for which $\|p_n\| = \|f\|\Lambda_n$ are not, in general, very smooth.

3c. *Derivative Bounds*. The previous two discussions made no hypotheses on the function being interpolated. Here we consider what sort of behavior for $f(x)$ would lead to convergence or divergence.

The simplest estimate (2) says

$$f(x) - p(x) = \frac{1}{(n+1)!} w_n(x) f^{(n+1)}(\xi_x).$$

Let $x = \bar{x}_j = x_j + \frac{1}{2}h$, i.e., x is halfway between two nodes. Then

$$\begin{aligned} w_n(\bar{x}_j) &= \prod_{k=0}^{n} \left(x_j + \frac{1}{2}h - x_k \right), \\ &= h^{n+1} \prod_{k=0}^{n} \left(j - k + \frac{1}{2} \right). \end{aligned}$$

For $j = 0$, $\bar{x}_j$ is a point near the edge of $[a, b]$, and in this case

$$\begin{aligned} |w(\bar{x}_0)| &= h^{n+1} \prod_{k=0}^{n} \left(k - \frac{1}{2} \right), \\ &= h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \frac{(2n)!}{n!}, \end{aligned}$$

so that

$$\begin{aligned} |f(\bar{x}_0) - p(\bar{x}_0)| &= h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \frac{(2n)!}{n!(n+1)!} |f^{(n+1)}(\xi_x)| \\ &= \frac{1}{n+1} h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \left[\frac{(2n)!}{(n!)^2} \right] |f^{(n+1)}(\xi_x)| \end{aligned}$$

and Stirling's formula yields

$$|f(\bar{x}_0) - p_n(\bar{x}_0)| \sim h^{n+1} n^{-3/2} |f^{(n+1)}(\xi_x)|. \tag{10}$$

Now computational experience as well as the preceding sections lead us to believe that the error at the ends of the interval should go to infinity as n does, i.e., (10) can be considered a worst case estimate. Yet the leading factors in (10) behave like $n^{-3/2} h^{n+1}$ so that the derivative term must grow quite rapidly to force divergence. A quasi-uniform estimate of this type can be had by noting (from Stirling's formula)

$$\frac{1}{(n+1)!} |w_n(x)| \leqslant \frac{(b-a)^{n+1}}{(n+1)!} \leqslant \frac{C}{\sqrt{n+1}} \left(\frac{e(b-a)}{n+1} \right)^{n+1}.$$

Thus, for any x,

$$|f(x) - p_n(x)| \leqslant \frac{C}{\sqrt{n+1}} \left(\frac{e(b-a)}{n+1} \right)^{n+1} |f^{(n+1)}(\xi_x)|.$$

Unfortunately this approach does not yield a lower bound, and so divergence does not necessarily follow for $|f^{(n+1)}(\xi_x)|$ large. But it does show that $\|f^{(n+1)}\| \sim \sqrt{n+1}\,(n+1)/e(b-a)^{n+1}$ is necessary for there to be any chance of divergence.

In summary, then, we have:

THEOREM 3. *Let* $\{p_n\}$ *interpolate* f *at the equidistant nodes* $\{x_j^{(n)}\}$, $0 \leqslant j \leqslant n$. *If*

$$\lim_{n\to\infty}\left\{\frac{\|f^{(n+1)}\|}{\sqrt{n+1}}\left(\frac{e(b-a)}{n+1}\right)^{n+1}\right\} = 0,$$

then

$$p_n \to f,$$

uniformly on $[a, b]$.

The above discussion (and theorem) also appears to contradict the oft-read dictum "divergence of interpolation polynomials is due to the growth of the $l_k(x)$ functions." (See [**2**, p. 25] and [**7**, p. 35] for statements to this effect.) While the importance of the growth of the l_k should not be underestimated, (10) clearly indicates that the interpolation will converge for functions whose derivatives behave "moderately well" as $n \to \infty$. (Note, however, that the results of §3a still show that polynomial interpolation is *ill-conditioned* as $n \to \infty$.)

3d. *A Real Variable Estimate*. Consider a generalization of Runge's example

$$f(x) = (x^2 + s^2)^{-1}, \qquad x \in [-a, a].$$

In this subsection we will develop a series of estimates which show that polynomial interpolation (with equidistant nodes) to f converges only for $|x|$ sufficiently small and s sufficiently large. We do this with the estimate (3), for which we need an expression for the divided difference.

LEMMA 2.

$$f[x_0, \dots, x_n, x] = \left(\frac{1}{iw_n(si)}\right)\left(\frac{r_n}{x^2+s^2}\right), \qquad r_n = \begin{cases} x, & n \text{ even} \\ si, & n \text{ odd} \end{cases}.$$

Proof. We write $f(x)$ as

$$f(x) = \frac{1}{2si}\left\{\frac{1}{x - si} - \frac{1}{x + si}\right\}$$

and note that the divided difference operator is linear. Further, letting

$$g_1(x) = \frac{1}{x - si}, \qquad g_2(x) = \frac{1}{x + si},$$

we have (by induction on m)

$$g_1[\xi_0, \dots, \xi_m] = (-1)^m \frac{1}{\prod_{j=0}^{m} (\xi_j - si)},$$

$$g_2[\xi_0, \dots, \xi_m] = (-1)^m \frac{1}{\prod_{j=0}^{m} (\xi_j + si)}.$$

Thus, using (4),

$$f\left[x_0^{(n)}, \dots, x_n^{(n)}, x\right] = \frac{1}{2si}\left\{\frac{1}{(x - si)w_n(si)} - \frac{1}{(x + si)w_n(-si)}\right\}$$

But the symmetric distribution of the nodes implies that $x_j^{(n)} = -x_{n-j}^{(n)}$, so $w_n(-si) = (-1)^{n+1}w_n(si)$. Hence

$$f\left[x_0^{(n)}, \dots, x_n^{(n)}, x\right] = \frac{1}{2si}\left\{\frac{1}{(x - si)w_n(si)} - \frac{(-1)^{n+1}}{(x + si)w_n(si)}\right\}$$
$$= \left\{\frac{1}{iw_n(si)}\right\}\left\{\frac{r_n}{x^2 + s^2}\right\},$$

which completes the proof.

It follows immediately from the lemma and (3) that

$$|f(x) - p_n(x)| = \frac{|r_n|}{s^2 + x^2}\left|\frac{w_n(x)}{w_n(si)}\right|. \tag{11}$$

Thus (just as in the complex analysis) convergence depends entirely on the limiting value of $|w_n(x)/w_n(si)|$. In particular, $p_n(x) \to f(x)$ if and only if $\sigma(x)$ (as defined in (6)) is less than $\sigma(si)$. Further, $p_n \to f$ uniformly whenever

$$\max_{x \in [-a, a]} \sigma(x) < \sigma(si).$$

Thus uniform convergence is a function of the two parameters a (half-length of interval of interpolation) and s (distance from poles to real-axis). An ordinary calculation shows that

$$\max_{x \in [-a, a]} \sigma(x) = \sigma(a) = 2a/e.$$

Further, we find that $\sigma(si) \leqslant \sigma(a)$ if and only if

$$\ln(1 + \xi^2) + 2\xi \arctan(1/\xi) - \ln 4 > 0,$$

where $\xi = s/a$. This function is monotone increasing for $\xi \geqslant 0$ and has a unique root $\xi^* \approx .5255$. Thus we get uniform convergence for $s > \xi^* a$. Note that for

Runge's original function this is not achieved until the interval shrinks to (approximately) $[-1.9, 1.9]$, whereas on $[-5,5]$ we must move the singularities to (approximately) ± 2.63. This leads to the following result.

THEOREM 4. *Let f be of the form*

$$f(x) = (x^2 + s^2)^{-1}$$

for $x \in [-a, a]$. If p_n interpolates f at the equidistant nodes $\{x_j^{(n)}\}$, $0 \leqslant j \leqslant n$, then $p_n \to f$ uniformly on $[-a, a]$ if and only if

$$s > \xi^* a,$$

for $\xi^* \approx .5255$.

This last theorem is the crux of the matter, for it clearly indicates the separate roles played by the singularities and by the interval length.

4. Chebyshev Nodes. Consider now what happens if the Chebyshev nodes are used instead of equidistant nodes. Define

$$t_j^{(n)} = \cos\left[\left(j + \frac{1}{2}\right)\frac{\pi}{n+1}\right], \qquad 0 \leqslant j \leqslant n$$

as the Chebyshev nodes, i.e., the roots of the $(n+1)$st Chebyshev polynomial $T_{n+1}(x)$. (In what follows we assume the interval is now $[-1,1]$.) Then we have [**1**, p. 61]

$$\prod_{j=0}^{n}\left(x - t_j^{(n)}\right) = \tilde{T}_{n+1}(x) = \left(\frac{1}{2}\right)^n T_{n+1}(x),$$

and $|T_{n+1}(x)| \leqslant 1$ for all $x \in [-1,1]$. It is a well-known (but by no means trivial) result that, for this choice of nodes [**8**, p. 94],

$$\Lambda_n = \sum_{j=0}^{n} \|l_j\| \leqslant \frac{2}{\pi}\log n + 4.$$

Thus, the growth of the l_k functions is much less than for equidistant nodes. Moreover, we see that the effects of round off error are much less:

$$\|E_n\| \leqslant \varepsilon_{\max}\Lambda_n \leqslant \left(\frac{2}{\pi}\log n + 4\right)\varepsilon_{\max}.$$

In terms of the derivative bounds of §3c, we have

$$\begin{aligned}|f(x) - p_n(x)| &\leqslant \frac{1}{(n+1)!}|\tilde{T}_{n+1}(x)|\,|f^{(n+1)}(\xi_x)|,\\ &\leqslant \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n |f^{(n+1)}(\xi_x)|.\end{aligned}$$

Since the Chebyshev polynomials are, in an appropriate sense, minimal over $[-1,1]$,

we can in fact assert that this choice of nodes minimizes the error by minimizing the upper bound for $|w_n(x)|/(n+1)!$. Divergence is still possible if $|f^{(n+1)}(\xi_x)|$ grows too rapidly.

Finally, note that the analysis of §3d carries through for any set of (symmetric) nodes as far as (11). Thus, for the Chebyshev nodes in $[-1,1]$, we have that

$$|f(x) - p_n(x)| \sim \left|\frac{w_n(x)}{w_n(si)}\right|.$$

But $w_n(x) = \tilde{T}_{n+1}(x)$ for any x, hence

$$\left|\frac{w_n(x)}{w_n(si)}\right| = \left|\frac{T_{n+1}(x)}{T_{n+1}(si)}\right| \leqslant \frac{1}{|T_{n+1}(si)|} = \frac{1}{|\cos(n+1)z|},$$

where $z = \arccos(si)$. Standard properties of the elementary functions on $\mathbb{C}$ then imply that

$$z = \frac{\pi}{2} - i\log\left(s + \sqrt{s^2+1}\right).$$

Thus

$$\begin{aligned}|\cos(n+1)z| &= \left|\cos\left((n+1)\frac{\pi}{2} - i(n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right| \\ &= \begin{cases}\left|\cosh\left((n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right|, & n \text{ odd}, \\ \left|\sinh\left((n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right|, & n \text{ even},\end{cases} \\ &\geqslant C(1+s)^{n+1},\end{aligned}$$

and so

$$|f(x) - p_n(x)| \leqslant \frac{C}{(1+s)^{n+1}}, \quad \text{for all } x,$$

which implies uniform convergence for all $s > 0$.

Historical Comment. Although most references use only Runge's name it appears that Meray ([5], [6]) also contributed to understanding this phenomenon.

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The Remarkable Theorem of Lévy and Steinitz

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1. Introduction. Everyone knows Riemann's theorem that a conditionally convergent series of real numbers can be rearranged to sum to any real number. An alternate formulation is the following: the set of all sums of rearrangements of a given series of real numbers is the empty set, a single point, or the entire real line.

What is the corresponding theorem for series of complex numbers?

Our informal survey has shown that surprisingly few mathematicians know the answer to this question, although the answer is a very natural one that was published more than eighty years ago.

The theorem is the following: *the set of all sums of rearrangements of a given series of complex numbers is the empty set, a single point, a line in the complex plane, or the whole complex plane.*

The analogue holds in n dimensions:

THE LÉVY-STEINITZ THEOREM. *The set of all sums of rearrangments of a given series of vectors in a finite-dimensional real Euclidean space is either the empty set or a translate of a subspace (i.e., a set of the form $v + M$, where v is a given vector and M is a linear subspace).*

Since a finite-dimensional complex vector space is a real vector space of twice the dimension, the Lévy-Steinitz Theorem implies that the set of rearrangements of a series in a complex Euclidean space is the empty set or a translate of a real subspace.

The theorem was first proven by P. Lévy [4] in 1905. In 1913, Steinitz [6] pointed out that Lévy's proof was incomplete, especially in the higher-dimensional cases. Steinitz [6] filled the gap in Lévy's proof and also found an entirely different approach.

The purpose of this article is to make this beautiful result more widely known.

We present Steinitz' approach, as modified by Gross [1]. The main reason that this theorem is not better known is that the difficulty of the proof seems to be out of

proportion to the result. We have endeavored to divide the proof into easily-digested pieces with the hope of making it both accessible and interesting.

We begin with the "Polygonal Confinement Theorem" as proven by Gross [**1**]; this says that an arbitrarily large but finite set of vectors, each of length less than one, which sums to 0, can be rearranged so that none of the partial sums is more than a certain constant which depends only on the dimension of the vector space.

In section 3 we discuss "The Rearrangement Theorem," which states that some rearrangement of a series converges to S if a subsequence of the sequence of partial sums of the series converges to S and the sequence of terms converges to 0. This theorem, which is a consequence of the Polygonal Confinement Theorem, is surely of interest in its own right.

We present the Lévy-Steinitz Theorem in section 4. In section 5 we briefly discuss certain related results and references.

I was told of the Lévy-Steinitz Theorem by Israel Halperin. The first few times that he started to explain the proof to me, I didn't listen; I assumed that I could prove the theorem in some easier way. Finally, after I realized I couldn't prove it, I let him describe the proof. The exposition that follows is mainly based on these private lectures, for which I am extremely grateful.

2. The Polygonal Confinement Theorem.

In the Steinitz-Gross proof of the Lévy-Steinitz Theorem the basic technical lemma is the following.

THE POLYGONAL CONFINEMENT THEOREM ([**6**], [**1**]). *For each dimension n there is a constant C_n such that whenever $\{v_i: i = 1, \dots, m\}$ is a finite family of vectors in $\mathbf{R}^n$ which sums to 0 and satisfies $\|v_i\| \leqslant 1$ for all i, there is a permutation P of $(2, \dots, m)$ with the property that*

$$\left\| v_1 + \sum_{i=2}^{j} v_{P(i)} \right\| \leqslant C_n$$

for every j. Moreover, we can take $C_1 = 1$ and $C_n \leqslant \sqrt{4C_{n-1}^2 + 1}$ for every n.

Proof. The case $n = 1$ is easy. If, for example, $v_1 > 0$, we can choose $P(2)$ so that $v_{P(2)} < 0$, and keep choosing negative v's until the sum of all the chosen vectors becomes negative. Then choose the next v to be positive, and keep choosing positive v's until the sum of all the chosen vectors becomes positive. Continue in this manner until all the v's are used. Since $|v_i| \leqslant 1$ for all i, it is clear that each partial sum in this arrangement is within 1 of 0. Hence, $C_1 = 1$.

The general case is proven by induction. Assume that $n > 1$ and that C_{n-1} is known to be finite, and consider a collection $\{v_i\}$ of vectors satisfying the hypotheses.

Since $\{v_i\}$ is finite there are a finite number of possible partial sums of the v's that begin with v_1; let L be such a partial sum with maximal length among all such

partial sums. Then $L = v_1 + u_1 + \cdots + u_s$, where $\{u_1, \ldots, u_s\} \subset \{v_i\}$. Let $\{w_1, \ldots, w_t\}$ denote the other v's, so that $L + w_1 + \cdots + w_t = 0$.

We use the notation $(u|v)$ to denote the Euclidean inner product of u and v.

We begin with a proof that the $\{u_i\}$ point in the same general direction as L, while the $\{w_i\}$ point in the opposite direction; (a diagram makes this very plausible).

Claim (a): $(u_i|L) \geqslant 0$ for all i.

To see this, suppose that $(u_i|L) < 0$, for some i. Then

$$\left((L - u_i)|\frac{L}{\|L\|}\right) = \|L\| - \frac{1}{\|L\|}(u_i|L) > \|L\|,$$

so $\|L - u_i\| > \|L\|$, which contradicts the assumption that L is a longest such partial sum.

Claim (b): $(v_1|L) \geqslant 0$.

For if $(v_1|L) < 0$, then

$$\left(\frac{-L}{\|L\|}\middle|(v_1 + w_1 + \cdots + w_t)\right) = \left(\frac{-L}{\|L\|}\middle|(v_1 - L)\right) = \|L\| - \frac{1}{\|L\|}(L|v_1) > \|L\|,$$

so $v_1 + w_1 + \cdots + w_t$ would be a longer partial sum than L.

Claim (c): $(w_i|L) \leqslant 0$ for all i. For if there was an i with $(w_i|L) > 0$, then

$$\left((L + w_i)|\frac{L}{\|L\|}\right) = \|L\| + \frac{(w_i|L)}{\|L\|} > \|L\|,$$

and, therefore, $\|L + w_i\| > \|L\|$. But $\|L + w_i\|$ is the length of a partial sum of the required kind. Thus this would also contradict $\|L\|$ being the longest length of such a partial sum.

We use the inductive hypothesis in the $(n - 1)$-dimensional space

$$L^\perp = \{v \in \mathbf{R}^n : (v|L) = 0\}.$$

We let v' denote the component of a vector v in $L^\perp$; i.e.,

$$v' = v - \frac{(v|L)}{\|L\|^2}L.$$

Then $L = v_1 + u_1 + \cdots + u_s$ implies $v_1' + u_1' + \cdots + u_s' = 0$. For a similar reason, $w_1' + \cdots + w_t' = 0$. By the inductive hypothesis, there exists a permutation Q of $(1, \ldots, s)$ such that

$$\left\|v_1' + \sum_{i=1}^{j} u'_{Q(i)}\right\| \leqslant C_{n-1} \quad \text{for } j = 1, \ldots, s,$$

and there exists a permutation R of $(2, \dots, t)$ such that

$$\left\| w_1' + \sum_{i=2}^{j} w_{R(i)}' \right\| \leqslant C_{n-1} \quad \text{for } j = 2, \dots, t.$$

Define $R(1) = 1$.

Now the idea is to keep the above orders within the u's and the w's (which will keep the components in $L^{\perp}$ of partial sums from being too large) and alternately "feed in" u's and w's to keep the components along L of length at most 1 (as in the proof of the case $n = 1$).

More precisely, since $(v_1|L) \geqslant 0$ and $(w_i|L) \leqslant 0$, we can choose a smallest r, say r_1, such that

$$(v_1|L) + \sum_{i=1}^{r_1} (w_{R(i)}|L) \leqslant 0.$$

Then we choose a smallest s_1 such that

$$(v_1|L) + \sum_{i=1}^{r_1} (w_{R(i)}|L) + \sum_{i=1}^{s_1} (u_{Q(i)}|L) \geqslant 0.$$

Then a smallest r_2 such that

$$(v_1|L) + \sum_{i=1}^{r_1} (w_{R(i)}|L) + \sum_{i=1}^{s_1} (u_{Q(i)}|L) + \sum_{i=r_1+1}^{r_2} (w_{R(i)}|L) \leqslant 0.$$

And so on. Arrange the vectors $\{v_i\}$ in the order

$$(v_1, w_{R(1)}, \dots, w_{R(r_1)}, u_{Q(1)}, \dots, u_{Q(s_1)}, w_{R(r_1+1)}, \dots, w_{R(r_2)}, \dots).$$

In this arrangement, clearly the components along L of each partial sum have norm at most 1. The choice of the arrangements Q and R by the inductive hypothesis insures that the components orthogonal to L of the partial sums have norms at most $C_{n-1} + C_{n-1}$. Hence, the norm of each partial sum is at most $\sqrt{(2C_{n-1})^2 + 1}$. This completes the proof.

3. The Rearrangement Theorem.

The Rearrangement Theorem is a crucial ingredient of Steinitz' proof of the Lévy-Steinitz Theorem and is also of independent interest.

For the proof of the Rearrangement Theorem it is convenient to isolate the following consequence of the Polygonal Confinement Theorem.

LEMMA 1. *If* $\{v_i\colon i = 1, \dots, m\} \subset \mathbf{R}^n$ *and* $\|\Sigma_{i=1}^m v_i\| \leqslant \varepsilon$, $\|v_i\| \leqslant \varepsilon$ *for all* i, *then there is a permutation* P *of* $(1, \dots, m)$ *such that*

$$\|v_{P(1)} + v_{P(2)} + \cdots + v_{P(r)}\| \leqslant \varepsilon(C_n + 1)$$

for $1 \leqslant r \leqslant m$.

Proof. Define $v_{m+1} = -v_1 - \cdots - v_m$ so that $\Sigma_{i=1}^{m+1} v_i = 0$. By the Polygonal Confinement Theorem, there is a permutation P of $(2, \dots, m+1)$ such that

$$\left\| \frac{1}{\varepsilon} v_1 + \sum_{i=2}^{r} \frac{1}{\varepsilon} v_{P(i)} \right\| \leqslant C_n$$

for all r.

Then $\|v_1 + \Sigma_{i=2}^{r} v_{P(i)}\| \leqslant \varepsilon C_n$ for all r. Let $P(1) = 1$.

Now order the $\{v_i\}$ according to P, but omit v_{m+1}; since $\|v_{m+1}\| \leqslant \varepsilon$ this changes the norms of the partial sums by at most ε. Hence in this arrangement all the partial sums have norm at most $\varepsilon C_n + \varepsilon$. This proves the Lemma.

THE REARRANGEMENT THEOREM. *In $\mathbf{R}^n$, if a subsequence of the sequence of partial sums of a series of vectors converges to S, and if the sequence of terms of the series converges to 0, then there is a rearrangement of the series that sums to S.*

Proof. Let $\{v_i\}_{i=1}^{\infty}$ be a sequence of vectors in $\mathbf{R}^n$. For each m let $S_m = \Sigma_{i=1}^{m} v_i$. We assume that $\{S_{m_k}\} \to S$ for some subsequence $\{S_{m_k}\}$, and we must show how to rearrange the $\{v_i\}$ so that the entire sequence of partial sums converges to S. The idea is to use Lemma 1 to obtain rearrangements of each of the families $(v_{m_k+1}, \dots, v_{m_{k+1}-1})$ so that all the partial sums of these families are small. Then S_m is close to S_{m_k} if m is between m_k and m_{k+1}.

This can be stated as follows. Let $\delta_k = \|S_{m_k} - S\|$; then $\{\delta_k\} \to 0$. Now

$$\left\| \sum_{i=m_k+1}^{m_{k+1}-1} v_i \right\| = \left\| \sum_{i=1}^{m_{k+1}} v_i - \sum_{i=1}^{m_k} v_i - v_{m_{k+1}} \right\| < \delta_{k+1} + \delta_k + \|v_{m_{k+1}}\|.$$

For each k let

$$\varepsilon_k = \max\{\delta_{k+1} + \delta_k, \sup\{\|v_i\| : i \geqslant m_k\}\}.$$

Then $\{\varepsilon_k\} \to 0$, and

$$\left\| \sum_{i=m_k+1}^{m_{k+1}-1} v_i \right\| < 2\varepsilon_k.$$

By Lemma 1, for each k there is a permutation P_k of $(m_k + 1, \dots, m_{k+1} - 1)$ such that

$$\left\| \sum_{i=m_k+1}^{r} v_{P_k(i)} \right\| \leqslant 2\varepsilon_k (C_n + 1)$$

for $r = m_k + 1, \dots, m_{k+1} - 1$.

Now arrange the $\{v_i\}$ as follows. Keep v_{m_k} in position m_k for each k. Then order the v_i for $(m_k + 1) \leqslant i \leqslant (m_{k+1} - 1)$ according to P_k. In this arrangement, if $m_k + 1 \leqslant m \leqslant m_{k+1} - 1$ then $S_m - S_{m_k}$ is a sum of the form $\Sigma_{i=m_k+1}^{m} v_{P_k(i)}$ with $m < m_{k+1}$, and hence has norm at most $2\varepsilon_k(C_n + 1)$. Since $\{S_{m_k}\} \to S$ and $\{\varepsilon_k\} \to 0$, it follows that $\{S_m\} \to S$.

4. The Lévy-Steinitz Theorem.

To prove the Lévy-Steinitz Theorem we will need another consequence of the Polygonal Confinement Theorem in addition to the Rearrangement Theorem.

LEMMA 2. *If* $\{v_i\}_{i=1}^m \subset \mathbf{R}^n$, $w = \sum_{i=1}^m v_i$, $0 < t < 1$, *and* $\|v_i\| \leqslant \varepsilon$ *for all* i, *then either* $\|v_1 - tw\| \leqslant \varepsilon\sqrt{C_{n-1}^2 + 1}$ *or there is a permutation* P *of* $(2, \dots, m)$ *and an* r *between* 2 *and* m *such that* $\|v_1 + \sum_{i=2}^r v_{P(i)} - tw\| \leqslant \varepsilon\sqrt{C_{n-1}^2 + 1}$.

Proof. Suppose $w \neq 0$ (otherwise the result is trivial). Consider first the case $n = 1$. By multiplying through by -1 if necessary, we can assume that $w > 0$; let s denote the smallest i such that

$$v_1 + v_2 + \cdots + v_i > tw.$$

Since

$$v_1 + v_2 + \cdots + v_{s-1} \leqslant tw$$

and $|v_s| \leqslant \varepsilon$, it follows that

$$|v_1 + v_2 + \cdots + v_s - tw| \leqslant \varepsilon.$$

Thus in the case $n = 1$ the Lemma holds with $C_{n-1} = C_0$ being defined to be 0. Note also that, in the case $n = 1$, no rearranging is necessary to get an appropriate partial sum.

Now consider the general case of $\mathbf{R}^n$ for $n > 1$. Since $w = \sum_{i=1}^m v_i$, the projections $\{v_i'\}$ of the $\{v_i\}$ onto $\{w\}^\perp$ add up to 0. Since $\|v_i\| \leqslant \varepsilon$ for all i, the Polygonal Confinement Theorem yields a permutation P of $(2, \dots, m)$ such that

$$\left\| \frac{1}{\varepsilon}v_1' + \frac{1}{\varepsilon}v_{P(2)}' + \cdots + \frac{1}{\varepsilon}v_{P(j)}' \right\| \leqslant C_{n-1}$$

for $j = 2, \dots, m$.

Also,

$$\left(v_1 \middle| \frac{w}{\|w\|}\right) + \left(v_{P(2)} \middle| \frac{w}{\|w\|}\right) + \cdots + \left(v_{P(m)} \middle| \frac{w}{\|w\|}\right) = \|w\|,$$

and $|(v_i|w)/\|w\|| \leqslant \varepsilon$ for all i. Hence, the case $n = 1$ yields an r such that

$$\left| \left(v_1 \middle| \frac{w}{\|w\|}\right) + \left(v_{P(2)} \middle| \frac{w}{\|w\|}\right) + \cdots + \left(v_{P(r)} \middle| \frac{w}{\|w\|}\right) - t\|w\| \right| \leqslant \varepsilon.$$

The bounds on the components yield a bound on the vector, so

$$\|v_1 + v_{P(2)} + \cdots + v_{P(r)} - tw\|^2 \leqslant \varepsilon^2 C_{n-1}^2 + \varepsilon^2,$$

which is the Lemma.

Now we can finally prove the main theorem.

THE LÉVY-STEINITZ THEOREM ([4], [6]). *The set of all sums of rearrangements of a given series of vectors in* $\mathbf{R}^n$ *is either the empty set or a translate of a subspace.*

Proof. Let S denote the set of all sums of convergent rearrangements of the series $\Sigma_{i=1}^{\infty} v_i$. Suppose S is not empty. By replacing v_1 by $v_1 - v$, where v is any element of S, we can assume that $0 \in S$. We must show that S is a subspace.

The proof will require rearranging the series a number of times. The outline of the proof that 0, s_1 and s_2 in S implies $s_1 + s_2$ is in S is the following. We choose a sequence $\{\varepsilon_m\}$ of positive numbers that converges to 0. We form a partial sum of Σv_i, in some order, that is within ε_1 of s_1. Then we construct a partial sum that contains all the vectors we have already used and that is within ε_1 of 0, then a partial sum containing all the vectors already used that lies within ε_1 of s_2, then one within ε_2 of s_1, within ε_2 of 0, ε_2 of s_2, and so on. The vectors used between a sum close to 0 and the next sum close to s_2 approximately add up to s_2. Interchanging them with those between the preceeding sum close to s_1 and the sum close to 0 produces partial sums close to $s_1 + s_2$. The Rearrangement Theorem finishes the proof.

We now present the details. Let $\{\varepsilon_m\}$ be a sequence of positive numbers that converges to 0. Since an arrangement converges to s_1, there exists a finite set I_1 of positive integers such that $1 \in I_1$ and $\|\Sigma_{i \in I_1} v_i - s_1\| < \varepsilon_1$. Since an arrangement converges to 0, there is a finite set J_1 of positive integers such that $J_1 \supset I_1$ and $\|\Sigma_{i \in J_1} v_i - 0\| < \varepsilon_1$, and a set $K_1 \supset J_1$ such that $\|\Sigma_{i \in K_1} v_i - s_2\| < \varepsilon_1$. There is also a set I_2 containing both K_1 and $\{2\}$ such that $\|\Sigma_{i \in I_2} v_i - s_1\| < \varepsilon_2$. And so on. That is, we inductively construct sets I_m, J_m, and K_m of positive integers such that

$$\{1, \dots, m-1\} \subset K_{m-1} \subset I_m \subset J_m \subset K_m,$$

$$\left\| \sum_{i \in I_m} v_i - s_1 \right\| < \varepsilon_m, \qquad \left\| \sum_{i \in J_m} v_i - 0 \right\| < \varepsilon_m, \qquad \text{and} \left\| \sum_{i \in K_m} v_i - s_2 \right\| < \varepsilon_m.$$

For each m, starting at $m = 1$, arrange the indices in J_m so that those in I_m come at the beginning, and then arrange the indices in K_m so that those in J_m come at the beginning. Then arrange the indices of I_{m+1} so that those of K_m come at the beginning. Thus there is a permutation P of the set of positive integers and increasing sequences $\{i_m\}$, $\{j_m\}$, $\{k_m\}$ such that $i_m < j_m < k_m < i_{m+1}$, and

$$\left\| \sum_{i=1}^{i_m} v_{P(i)} - s_1 \right\| < \varepsilon_m, \qquad \left\| \sum_{j=1}^{j_m} v_{P(j)} \right\| < \varepsilon_m, \qquad \left\| \sum_{k=1}^{k_m} v_{P(k)} - s_2 \right\| < \varepsilon_m$$

for each m.

Note that

$$\left\| \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2 \right\| = \left\| \sum_{i=1}^{k_m} v_{P(i)} - \sum_{j=1}^{j_m} v_{P(j)} - s_2 \right\| < \varepsilon_m + \varepsilon_m.$$

It follows that

$$\left\| \sum_{i=1}^{i_m} v_{P(i)} + \sum_{i=j_m+1}^{k_m} v_{P(i)} - (s_1 + s_2) \right\| < 3\varepsilon_m.$$

For each m, rearrange the vectors in $\{v_{P(i)}: i = i_m, \dots, k_m\}$ by interchanging the vectors $\{v_{P(i)}: i = i_m + 1, \dots, j_m\}$ with the vectors $\{v_{P(i)}: i = j_m + 1, \dots, k_m\}$. In this new arrangement, the above shows that there is a subsequence of the sequence of partial sums that converges to $s_1 + s_2$. Since we are assuming $S \neq \varnothing$, $\{v_{P(i)}\} \to 0$, so the Rearrangement Theorem implies that there is another arrangement that converges to $s_1 + s_2$. Therefore, $(s_1 + s_2) \in S$.

It remains to be shown that $s \in S$ implies $ts \in S$ for all real t. The additivity of S implies this for t a positive integer, so it suffices to consider the cases $t \in (0,1)$ and $t = -1$.

We start with the arrangement P used above to show the additivity of S. Fix $t \in (0,1)$. We use Lemma 2. As shown above,

$$\left\| \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2 \right\| < 2\varepsilon_m$$

for each m. Let $\delta_m = \sup\{\|v_{P(i)}\|: i = j_m + 1, \dots, k_m\}$, and let

$$u_m = \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2.$$

By Lemma 2, there is a permutation Q_m of $\{P(j_m + 1), \dots, P(k_m)\}$ and an r_m so that

$$\left\| \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - t(s_2 + u_m) \right\| \leqslant M\delta_m, \quad \text{where } M = \sqrt{C_{n-1}^2 + 1}\,.$$

Then

$$\left\| \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - ts_2 \right\| < M\delta_m + 2\varepsilon_m.$$

Now

$$\left\| \sum_{i=1}^{j_m} v_{P(i)} + \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - ts_2 \right\| < M\delta_m + 3\varepsilon_m,$$

so in this arrangement a subsequence of the sequence of partial sums converges to ts_2. The Rearrangement Theorem yields $ts_2 \in S$.

It only remains to be shown that $-s_2 \in S$. But

$$\left\| \sum_{i=1}^{j_{m+1}} v_{P(i)} - \sum_{i=1}^{k_m} v_{P(i)} - (0 - s_2) \right\| < \varepsilon_{m+1} + \varepsilon_m,$$

so

$$\left\| \sum_{i=k_m+1}^{j_{m+1}} v_{P(i)} - (-s_2) \right\| < \varepsilon_{m+1} + \varepsilon_m.$$

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LETTERS TO THE EDITOR

Editor,

In a recent article in this MONTHLY [1], Stephen L. Campbell has given a proof, different from the historically well-known Cantor diagonalisation process, of the countability of the set of rational numbers.

His method is similar in spirit to that given in [2], where a Gödel index $2^n 3^p 5^q$ is constructed to correspond to the rational number $(-1)^n p/q$, where $n = 0$ or 1 depending on whether the rational number is positive or negative, respectively. This method is easily generalized to cover the case of the set of polynomials with rational coefficients as well. In my opinion, this method is more elementary.

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1. Stephen L. Campbell, Countability of Sets, this MONTHLY, 93 (1986) 480–481.
2. Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (revised edition), Holt, Rinehart and Winston, NY (1966), p. 338.

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NOTES

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The Monotonicity Theorem, Cauchy's Interlace Theorem, and the Courant-Fischer Theorem

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1. Introduction. In this note some important theorems on eigenvalues of Hermitian matrices are reworked from a unified viewpoint of exploiting a simple dimensional identity to obtain easier and quicker proofs. The usual procedure of invoking the minimax characterization or Sylvester's Law of Inertia to prove these results lead to longer proofs (see, for instance, [**1**, 186–192] or [**2**, 99–104]).

Our proofs depend on the following simple dimensional identity:

$$\dim(S_1 \cap S_2) = \dim S_1 + \dim S_2 - \dim(S_1 + S_2), \tag{1}$$

where S_1 and S_2 are subspaces of a finite-dimensional vector space. Thus, this note may also be viewed as a collection of good applications of the dimensional identity (1).

Before proceeding, we state the following basic facts used in the subsequent proofs without explicit reference: (a) the eigenvalues of a Hermitian matrix are real and the corresponding eigenvectors may be taken to be orthonormal; (b) letting $\alpha_1 \leqslant \cdots \leqslant \alpha_k$ denote a subset of eigenvalues of a Hermitian matrix $\mathscr{A}$ and letting $u_1, \ldots, u_k$ denote an orthonormal set of corresponding eigenvectors, we have $\alpha_1 \leqslant x^H \mathscr{A} x \leqslant \alpha_k$ for any x in the span of $u_1, \ldots, u_k$, where $x^H x = 1$. (The symbol "H" denotes conjugate transpose.)

2. The Monotonicity Theorem [**1**, p. 191].

Let $\mathscr{A}$ and $\mathscr{B}$ be Hermitian and let $\mathscr{A} + \mathscr{B} = \mathscr{C}$. Let the eigenvalues of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be $\alpha_1 \leqslant \cdots \leqslant \alpha_n$, $\beta_1 \leqslant \cdots \leqslant \beta_n$ and $\gamma_1 \leqslant \cdots \leqslant \gamma_n$, respectively. Then

$$(1)\ \alpha_j + \beta_{i-j+1} \leqslant \gamma_i, \qquad (i \geqslant j)$$

$$(2)\ \gamma_i \leqslant \alpha_j + \beta_{i-j+n}, \qquad (i \leqslant j)$$

$$(3)\ \alpha_i + \beta_1 \leqslant \gamma_i \leqslant \alpha_i + \beta_n.$$

Proof. Let

$$\mathscr{A} u_i = \alpha_i u_i, \qquad \mathscr{B} v_i = \beta_i v_i, \qquad \mathscr{C} w_i = \gamma_i w_i,$$

$$u_i^H u_j = v_i^H v_j = w_i^H w_j = \delta_{ij}, \qquad i, j = 1, \ldots, n.$$

Consider first the case $i \geqslant j$ and let

$$S_1 = \operatorname{span}\{u_j, \ldots, u_n\}, \qquad \dim S_1 = n - j + 1;$$

$$S_2 = \operatorname{span}\{v_{i-j+1}, \ldots, v_n\}, \quad \dim S_2 = n - i + j;$$

$$S_3 = \operatorname{span}\{w_1, \ldots, w_i\}, \qquad \dim S_3 = i.$$

Proof. Let

$$\mathscr{A} u_i = \alpha_i u_i, \quad u_i^H u_j = \delta_{ij}, \qquad j = 1, \ldots, n.$$

Let

$$S_1 = \operatorname{span}\{u_k, \ldots, u_n\} \quad \text{and} \quad S_2 = S^k, \text{(any k-dimensional subspace)}.$$

Then §1 (1) guarantees the existence of an $x \in S_1 \cap S^k$, $x^H x = 1$, giving $x^H \mathscr{A} x \geqslant \alpha_k$.

On the other hand, for any $u \in \operatorname{span}\{u_1, \ldots, u_k\}$, a k-dimensional subspace, we have $u^H \mathscr{A} u \leqslant \alpha_k$ and $u_k^H \mathscr{A} u_k = \alpha_k$, proving the first equality of the theorem.

To prove the second, choose

$$S_1 = \operatorname{span}\{u_1, \ldots, u_k\}, \qquad S_2 = (S^{k-1})^{\perp},$$

and proceed in a similar line of argument as above.

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The *n*th Derivative as a Limit

J. B. WILKER
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Several well-known texts in analysis (e.g. [1], p. 124 and [3], p. 100) ask for a proof that

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

whenever $f''(x_0)$ exists. This raises the question of determining similar formulae for $f^{(n)}(x_0)$ $(n \geq 2)$ whenever it exists. The binomial aspect of the given formula suggests successive differences, but it turns out that something quite a bit more general does work and is, after the fact, completely natural.

THEOREM. *Let $a_0, a_1, a_2, \ldots, a_n$ be $n+1$ mutually distinct real numbers and let $w_0, w_1, w_2, \ldots, w_n$ be corresponding weights determined by the Vandermonde system*

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & & & & \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n! \end{pmatrix}.$$

Proof. Let

$$\mathscr{A} u_i = \alpha_i u_i, \quad u_i^H u_j = \delta_{ij}, \qquad j = 1, \dots, n.$$

Let

$$S_1 = \operatorname{span}\{u_k, \dots, u_n\} \quad \text{and} \quad S_2 = S^k, \text{(any } k\text{-dimensional subspace)}.$$

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expression for g shows that the resulting sequence of functions begins

$$(x - \tfrac{1}{x})^n$$

$$n(x - \tfrac{1}{x})^{n-1}(x + \tfrac{1}{x})$$

$$n(n-1)(x - \tfrac{1}{x})^{n-2}(x + \tfrac{1}{x})^2 + n(x - \tfrac{1}{x})^n$$

$$\vdots$$

The lemma follows because these expressions retain a factor of $x - 1/x$ in every term until the nth differentiation and then the only term that survives evaluation at $x = 1$ is $n!(x + 1/x)^n$.

In connection with the theorem, the lemma shows that the amplitudes $a_k = (n - 2k)/2$ $(k = 0, 1, 2, \dots, n)$ determine the weights $w_k = (-1)^k\binom{n}{k}$ and hence

$$f^{(n)}(x_0) = \lim_{h \to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x_0 + \frac{n - 2k}{2} h\right)}{h^n}.$$

For sufficiently smooth functions there is a corresponding formula in numerical differentiation (cf. [2]: Theorem 2, p. 203, and Exercise 3, p. 198). It has an error term proportional to $h^2 f^{(n+2)}(\xi)$, where

$$\xi \in \left[x_0 - \frac{nh}{2}, x_0 + \frac{nh}{2}\right].$$

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Limiting Distribution for the Generalized Matching Problem

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The well-known "matching problem" appears in most textbooks on probability; for instance, see [1], [3]. One version of the matching problem is as follows. At a party m men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. It is known that the number of matches is asymptotically a standard Poisson random

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variable. The purpose of this note is to generalize the problem above as follows. At a party m men take off their hats. The hats are then mixed up. We divide the m men into two groups A and B. Only n ($n \leqslant m$) men in the group A will randomly select their hats, and we count the number X_m^n of matches. We will show that the limiting distribution of X_m^n is the Poisson distribution with parameter $1/\alpha$ when $m = [\alpha n]$ and $\alpha \geqslant 1$. When $\alpha = 1$, this model coincides with the classical one.

Let E denote the event that no match occurs, and to make explicit the dependence on n and m, write $P_m^n = P(E)$. To compute the probability P_m^n of no match, let M_1 be the event that the first man selects his own hat and M_2 be the event that the first man selects a hat belonging to one of the next $n - 1$ men in the group A, and M_3 be the event that the first man selects a hat belonging to a man in the group B. Then

$$P_m^n = P(E) = P(E|M_1)P(M_1) + P(E|M_2)P(M_2) + P(E|M_3)P(M_3).$$

Clearly,

$$P(E|M_1) = 0 \quad \text{and} \quad P(E|M_3) = P_{m-1}^{n-1}.$$

Now $P(E|M_2)$ is the probability of no matches when $n - 1$ men select from a set of $m - 1$ hats that does not contain the hat of the extra man of these $n - 1$ men. This can happen in either of two mutually exclusive ways. Either there are no matches and the extra man does not select the extra hat (this being the hat of the man that chose first), or there are no matches and the extra man does select the extra hat. The probability of the first of these events is just P_{m-1}^{n-1}, which is seen by regarding the extra hat as the extra man's hat. As the second event has probability $\dfrac{1}{m-1}P_{m-2}^{n-2}$, we have

$$P(E|M_2) = P_{m-1}^{n-1} + \frac{1}{m-1}P_{m-2}^{n-2}.$$

Thus we obtain

$$\begin{aligned} P_m^n &= \left(P_{m-1}^{n-1} + \frac{1}{m-1}P_{m-2}^{n-2}\right)\frac{n-1}{m} + P_{m-1}^{n-1}\frac{m-n}{m} \\ &= \frac{n-1}{m(m-1)}P_{m-2}^{n-2} + \frac{m-1}{m}P_{m-1}^{n-1}. \end{aligned} \tag{1}$$

By the same idea, we have

$$\begin{aligned} P_m^1 &= \frac{m-1}{m}, \\ P_m^2 &= \frac{1}{m} + \frac{(m-2)^2}{m(m-1)}. \end{aligned} \tag{2}$$

We can calculate the probability P_m^n by the recurrence relations (1) and (2).

To obtain the probability of exactly k matches, we consider any fixed set of k men in the group A. The probability that they, and only they, select their own hats is

$$\frac{1}{m}\frac{1}{m-1}\cdots\frac{1}{m-(k-1)}P_{m-k}^{n-k}=\frac{(m-k)!}{m!}P_{m-k}^{n-k}.$$

Where P_{m-k}^{n-k} is the probability that the other $n-k$ men, selecting among $m-k$ hats including their hats, have no matches. As there are $\binom{n}{k}$ choices of a set of k men among n men, the desired probability $P(X_m^n=k)$ of exactly k matches is

$$P(X_m^n=k)=\binom{n}{k}\frac{(m-k)!}{m!}P_{m-k}^{n-k}=\frac{1}{k!}\frac{n!}{(n-k)!}\frac{(m-k)!}{m!}P_{m-k}^{n-k}. \tag{3}$$

For large n and m, $\dfrac{n!}{(n-k)!}\dfrac{(m-k)!}{m!}$ is approximately equal to $\left(\frac{n}{m}\right)^k$. Hence we obtain

$$P(X_m^n=k)\sim\frac{1}{k!}\left(\frac{n}{m}\right)^k P_{m-k}^{n-k}. \tag{4}$$

Let both n and m be permitted to increase with $m=[\alpha n]$, for fixed $\alpha\geqslant 1$. Then we will show that the limiting distribution of the random variable X_m^n is the Poisson distribution with parameter $1/\alpha$. For the remaining of this note we will assume that $m=[\alpha n]$ and α is any real number greater than or equal to 1. When $\alpha=1$ i.e., $m=n$, it is known that (P_n^n) converges to e^{-1}. From this observation, we might guess that (P_m^n) converges to $e^{-1/\alpha}$. Indeed our guess turns out to be true. First we note by the repeated application of equation (1) that if (P_m^n) converges to $e^{-1/\alpha}$ then (P_{m-k}^{n-k}) converges to the same number $e^{-1/\alpha}$ for any fixed k. In order to prove that the sequence (P_m^n) converges to $e^{-1/\alpha}$, it is enough to show that any subsequence of (P_m^n) has a convergent sub-subsequence whose limit is $e^{-1/\alpha}$. Since the sequence (P_m^n) is bounded by 1, any subsequence of (P_m^n) has a convergent sub-subsequence. To avoid complex notation we denote a convergent sub-subsequence again by (P_m^n), and let L be the limit of the sequence (P_m^n). It remains to show that $L=e^{-1/\alpha}$. It is evident that the random variable X_m^n assumes values $0,1,\ldots,n$, and so we have

$$1=\sum_{k=0}^{n}P(X_m^n=k)=\sum_{k=0}^{n}\frac{1}{k!}\frac{n!}{(n-k)!}\frac{(m-k)!}{m!}P_{m-k}^{n-k}.$$

On the other hand we can regard the sum as the integral with respect to the counting measure. Since

$$\frac{1}{k!}\frac{n!}{(n-k)!}\frac{(m-k)!}{m!}P_{m-k}^{n-k}\leqslant\frac{1}{k!},$$

From (4), we have

$$P(X_m^n = k) \to \frac{1}{k!}\left(\frac{1}{\alpha}\right)^k e^{-1/\alpha}.$$

Thus we obtain our main result.

THEOREM. *Let X_m^n be the number of matches. When $m = [\alpha n]$ and α is any real number greater than or equal to 1, the limiting distribution of X_m^n is the Poisson distribution with parameter $1/\alpha$.*

It turns out that this approximation is remarkably good for moderate values of n. Table 1 gives the comparison between the Poisson approximation and the exact probability of random variable X_m^n when $\alpha = 2, 3, 5$, with $m = \alpha n$.

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2. N. L. Johnson and S. Kotz, Urn Models and Their Application, Wiley, New York, 1977.
3. S. M. Ross, Introduction to Probability Models, 2nd ed., Academic Press, 1980.

Computing Binomial Coefficients

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The purpose of this paper is to show how results of the last century about binomial coefficients have been rediscovered with a micro-computer and the way that they can be used to compute $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for large values of n.

1. Introduction. The usual means for computing binomial coefficients is based on the formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and tables such as [5] have been constructed using this relation. Unfortunately, to get for instance the value of $\binom{1000}{353}$ you have to compute $353 \times (1000 - 353) = 228391$ terms of Pascal's triangle.

Our aim is to give a fast process to get the factorization of $\binom{n}{k}$ into primes. The exact value of $\binom{n}{k}$ is easy to obtain from the factorization.

NOTATION. Let I and J be two positive integers and a_i's and b_i's defined by

$$I = a_r 2^r + a_{r-1}2^{r-1} + \cdots + a_1 2 + a_0,$$
$$J = b_r 2^r + b_{r-1}2^{r-1} + \cdots + b_1 2 + b_0, \quad 0 \leqslant a_i, b_i \leqslant 1 \quad (i = 0, \ldots, r).$$

From (4), we have

$$P(X_m^n = k) \to \frac{1}{k!}\left(\frac{1}{\alpha}\right)^k e^{-1/\alpha}.$$

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$$J = b_r 2^r + b_{r-1}2^{r-1} + \cdots + b_1 2 + b_0, \quad 0 \leqslant a_i, b_i \leqslant 1 \quad (i = 0, \dots, r).$$

Using Boolean notation we set

$$(I \text{ and } J) = c_r 2^r + \cdots + c_1 2 + c_0, \quad c_i = \operatorname{Min}(a_i, b_i) \quad (i = 0, \ldots, r),$$
$$(I \text{ or } J) = d_r 2^r + \cdots + d_1 2 + d_0, \quad d_i = \operatorname{Max}(a_i, b_i) \quad (i = 0, \ldots, r).$$

EXAMPLE:

$43 \rightleftarrows 101011, \quad 25 \rightleftarrows 011001, \quad (43 \text{ and } 25) \rightleftarrows 001001,$

$(43 \text{ or } 25) \rightleftarrows 111011$ then $(43 \text{ and } 25) = 9, \quad (43 \text{ or } 25) = 59.$

2. Factorization of binomial coefficients into primes. We start with the following remark. The pattern of Fig. 1 was obtained with a computer by two different ways:

(1) for every $(n, k) \in \mathbb{N}^2$ (n, k) is plotted if and only if $\binom{n}{k}$ is odd (an analogous pattern can be seen in [**8**]);

(2) for every $(I, J) \in \mathbb{N}^2$ $((I \text{ or } J), (I \text{ and } J))$ is plotted.

This observation suggests that

(R1) $\binom{n}{k}$ is divisible by 2 if and only if $\binom{n}{k}$ cannot be written as $((I \text{ or } J), (I \text{ and } J))$.

This is the same as:

(R2) Let n and k be two positive integers $(k \leqslant n)$, $n = a_r \ldots a_0$ and $k = b_r \ldots b_0$ their binary representation. $\binom{n}{k}$ is divisible by 2 if and only if there exists i such that $b_i > a_i$.

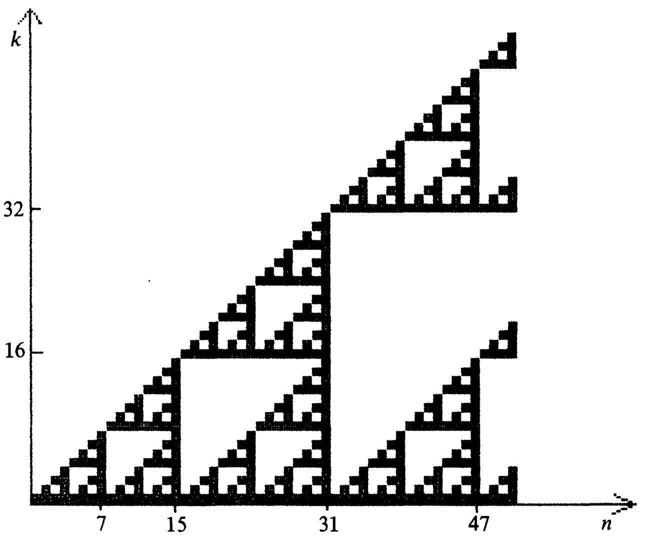

FIG. 1. Odd binomial coefficients.

The last result, discovered here by studying diagrams, is not only true but is an easy corollary of a theorem known as Lucas' Lemma ([**1**] and [**4**, p. 417]).

THEOREM. *Let n and k be two positive integers and p be a prime. Let $a_r \dots a_0$ and $b_r \dots b_0$ be the p-ary representation of n and k, respectively. Then*

$$\binom{n}{k} \equiv \binom{a_r}{b_r}\binom{a_{r-1}}{b_{r-1}} \cdots \binom{a_0}{b_0} \pmod{p}$$

(*with conventional value* $\binom{a}{b} = 0$ *if* $a < b$).

A natural question arises: What is the power of 2 in the factorization of $\binom{n}{k}$ into primes? The pattern of Fig. 2 shows binomial coefficients divisible by 2 but not by 4 and on Fig. 3 binomial coefficients divisible by 4 but not by 8. These drawings suggest there is an underlying mathematical reason for their regularity. The list of n and k such that (n, k) is plotted either in Fig. 2 or 3 is easily provided by the computer. A careful examination of the list shows that

(R3) The power of 2 in the factorization of $\binom{n}{k}$ into primes is equal to the number of borrow(s) in the subtraction $n - k$ in base 2.

Furthermore, computational experiments show that (R3) remains valid when replacing 2 by any prime:

(R4) Let p be a prime. The power of p in the factorization of $\binom{n}{k}$ into primes is equal to the number of borrow(s) in the subtraction $n - k$ in base p.

This result (re)discovered by the author on a micro-computer was already known though never formulated as (R4). The first version of (R4) by Kummer ([**3**] pp. 115–116), rephrased by Singmaster [**7**], says:

(R5) The power of p in the factorization of $\binom{n}{k}$ into primes is equal to the number of carry(ies) when summing $(n - k)$ and k.

A second version of (R4) is due to Kazantzidis [**2**].

Now we give a sketch for an elementary proof of (R4) providing an algorithm for computing the power of p in the factorization into primes (for extensive proofs see [**2**], [**3**], [**7**]).

(1) The power of p in the factorization of $n!$ into primes is $\sum_{i>0}[n/p^i]$. ($[x]$ denotes the greatest integer lower than or equal to x.)

(2) The power of p in the factorization of $\binom{n}{k}$ into primes is:

$$E(n,k) = \sum_{i>0}\left([n/p^i] - [k/p^i] - [(n-k)/p^i]\right). \tag{1}$$

(3) Writing $n = a_n p^2 + b_n p + c_n$, $k = a_k p^2 + b_k p + c_k$, with $0 \leqslant b_i,\ c_i < p$ $(i = k, n)$ we have:

(a) $E(n, 0) = 0$.

(b) If $c_n \geqslant c_k$, then $E(n, k) = E([n/p], [k/p])$.

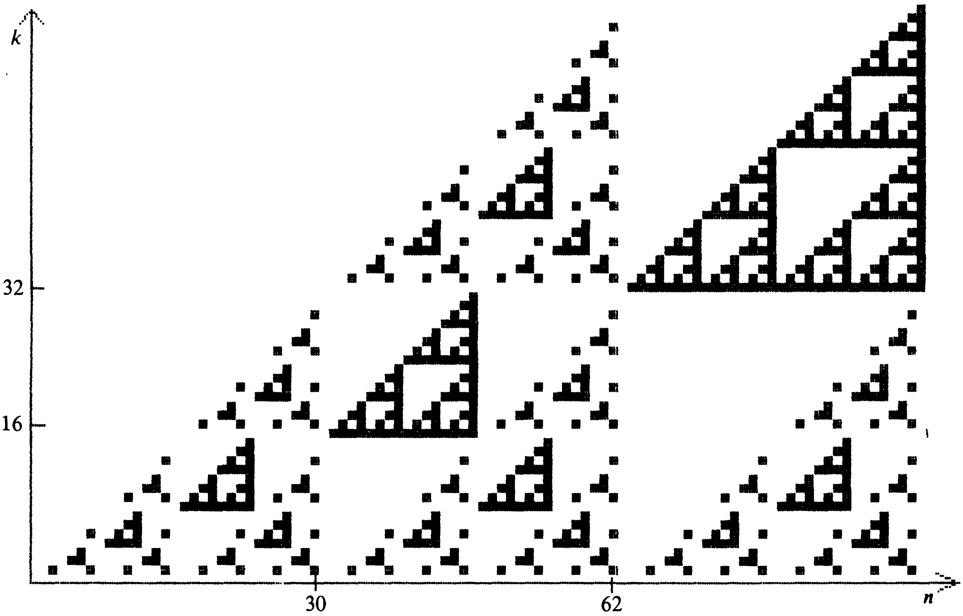

FIG. 2. Binomial coefficients divisible by 2 but not by 4.

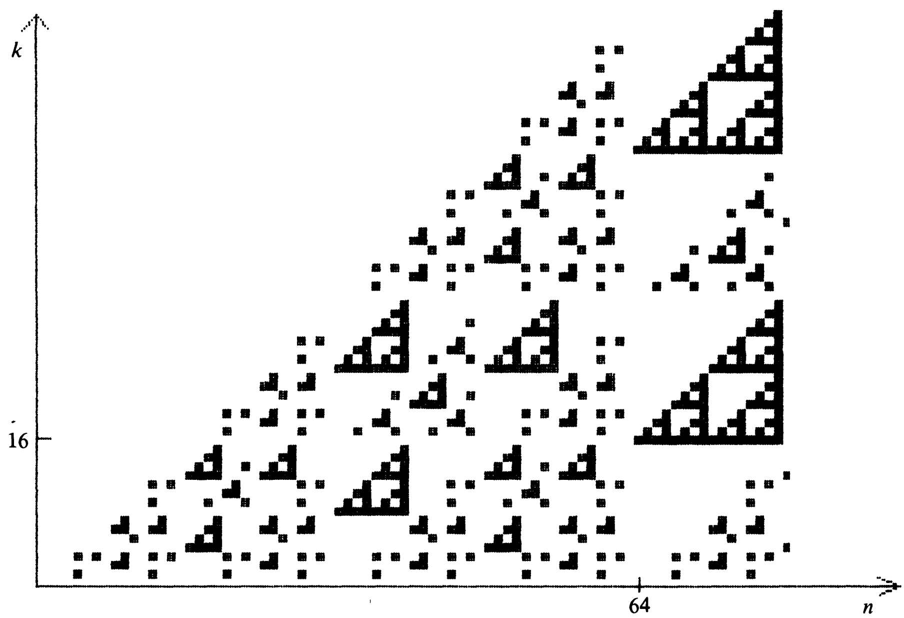

FIG. 3. Binomial coefficients divisible by 4 but not by 8.

(c) If $c_n < c_k$ and $b_n \neq b_k$, then $E(n, k) = E([n/p], [k/p]) + 1$.
(d) If $c_n < c_k$, $b_n = b_k$ and $b_n \neq 0$, then $E(n, k) = E([n/p] - 1, [k/p]) + 1$.
(e) If $c_n < c_k$ and $b_n = b_k = 0$, then $E(n, k) = E([n/p], [k/p] + 1) + 1$.

(Proofs of (b), (c), (d) and (e) are made using (1) and $[[n/p]/p^i] = [n/p^{i+1}]$).

(4) Denoting by $B(n, k)$ the number of borrow(s) in the subtraction $n - k$ in base p, we easily see that $B(n, k)$ satisfies the same induction relations as $E(n, k)$ and then is equal to $E(n, k)$.

3. Algorithm for computing the power of primes in the factorization. Let us note that formula (1) gives

$$E = 1, \quad \text{if } n - k < p \leqslant n,$$
$$E = 0, \quad \text{if } n/2 < p \leqslant n - k.$$

Furthermore if $p > n^{1/2}$, n and k have at most two digits in base p, and there is at most one borrow (on the least significant digits of n and k) in the subtraction $n - k$ in base p.

We denote by $\mathrm{INT}(x)$ the greatest integer lower than or equal to x, and by $\leftarrow$ the assignment operation.

The following algorithm gives the power E of the prime p in the factorization of $\binom{n}{k}$ into primes:

(1) Input n, k and p.
(2) $E \leftarrow 0, r \leftarrow 0$.
(3) If $p > n - k$ then $E \leftarrow 1$; end.
If $p > n/2$ then $E \leftarrow 0$; end.
If $p * p > n$ then if $n \bmod p < k \bmod p$ then $E \leftarrow 1$; end.
Repeat
(a) $a \leftarrow n \bmod p, n \leftarrow \mathrm{INT}(n/p)$,
$b \leftarrow (k \bmod p) + r, k \leftarrow \mathrm{INT}(k/p)$.
(b) If $a < b$ then $E \leftarrow E + 1, r \leftarrow 1$
else $r \leftarrow 0$.
until $n = 0$.
end.

4. Computation of $\binom{n}{k}$. From (R4), if a prime p satisfies $p > n$, n is not divisible by p. Then to factorize $\binom{n}{k}$ it is sufficient to compute the power of primes p such that $p \leqslant n$ which can be done quickly using the preceding algorithm. Knowing the factorization we can reconstruct the binomial coefficient by a classical multiprecision computation (large integers are stored as one-dimensional arrays of small integers, and the multiplication of two large integers is performed on the components of arrays taking care of the carries between successive components. For more details see [**6**, p. 332]) and we get easily $\binom{n}{k}$'s which are not available in tables for $n > 200$.

Example. The following computation, performed on a micro-computer with a 5 mhz 8086 CPU running CP/M, spends less than half a second to get the factorization and about 8 seconds for the exact value. (Program written in PASCAL)

Factorization of $\binom{1000}{353}$ into primes :

$2^3 \times 3^6 \times 5^3 \times 11 \times 19 \times 29 \times 31^2 \times 37 \times 41 \times 47 \times 59 \times 61 \times 71 \times$
$73 \times 83 \times 89 \times 97 \times 109 \times 131 \times 137 \times 139 \times 163 \times 179 \times 181 \times 191 \times 193 \times$
$197 \times 199 \times 223 \times 227 \times 229 \times 233 \times 239 \times 241 \times 331 \times 359 \times 367 \times 373 \times 379 \times$
$383 \times 389 \times 397 \times 401 \times 409 \times 419 \times 421 \times 431 \times 433 \times 439 \times 443 \times 449 \times 457 \times$
$461 \times 463 \times 467 \times 479 \times 487 \times 491 \times 499 \times 653 \times 659 \times 661 \times 673 \times 677 \times 683 \times$
$691 \times 701 \times 709 \times 719 \times 727 \times 733 \times 739 \times 743 \times 751 \times 757 \times 761 \times 769 \times 773 \times$
$787 \times 797 \times 809 \times 811 \times 821 \times 823 \times 827 \times 829 \times 839 \times 853 \times 857 \times 859 \times 863 \times$
$877 \times 881 \times 883 \times 887 \times 907 \times 911 \times 919 \times 929 \times 937 \times 941 \times 947 \times 953 \times 967 \times$
$971 \times 977 \times 983 \times 991 \times 997$.

Exact value of $\binom{1000}{353}$:

2 5229445633 0659742351 4408025205 5773735613 0435153119
5689363559 4388544559 6891848033 3018014952 8141512945 3596585561
6639939234 6118918439 7715091949 2045952055 6252295683 8053320988
8250237463 6769258037 6666922328 1259276867 8750591171 8832270161
1589146743 0491067982 6394724366 5313803538 2214107000 .

Truncated value : 2.52×10^{280} .

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THE TEACHING OF MATHEMATICS

EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

A Required Reading Program for Mathematics Majors

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A typical experience for a student in a college mathematics course is to buy a textbook at the beginning of the course, and then in a regular way read one section per lecture and work some problems at the end of the section. Rarely is another book consulted or is any research done in the library. Most teachers of college mathematics courses, especially the courses beyond calculus, would agree that the above scenario is not desirable, and wish that their students would read more widely in the mathematical literature.

Description of the program. At Wheaton College we have developed a program to ensure that our majors gain at least a minimal introduction to the literature. We choose a collection of 16 articles or chapters of books that sample such topics as history, biography, applications, surveys of broad areas, creativity, and the nature of mathematics. We would like our majors to be aware of the past contributions of people such as Cantor, Cauchy, Euler, Gauss and Hilbert as well as the present influence of people such as Halmos, Kline, Knuth and Polya. We would like them to become acquainted with the NCTM, MAA, ACM, Bell Labs and Göttingen.

We give the list of articles to our majors towards the end of their sophomore year while they are in the abstract algebra course and require that they finish the program before the end of their junior year. After reading each article, the student is to write a page of reactions to the material in the article. Unless there is some way to enforce this program, it will remain a good idea that died because students choose to concentrate their time on assignments that are required. If the 16 reports are not in by the end of the junior year, we assign a grade of incomplete in a major course that the student took during the spring term of the junior year. If the program is not completed within an additional 6 weeks, the incomplete is changed to a grade of F. While we have given a number of incompletes over the years, we have never had to resort to giving an F for this deficiency. Most students express appreciation for the value of the program when it is completed.

We place copies of these articles on reserve in the library and also make them available in the department area. If a department has a study area or seminar room, a better method is to place the books or journals on a table where they are readily visible. As the student reads one article in a book, the hope is that he or she will see other interesting articles and be encouraged to read beyond the requirement.

Content of the program. I will now list several of the books and articles that we have used in recent years. One has to fight the temptation to try to include too many articles, or selections that are too long. The program works best when the

faculty members are aware of the readings and try to include comments about them in their lectures.

1. *Historical Topics for the Mathematics Classroom*. This 31st yearbook of the National Council of Teachers of Mathematics was published in 1969 and contains several good survey articles. We have used "The History of Algebra" by John Baumgart, "The History of Geometry" by Howard Eves, "The History of the Calculus" by Carl Boyer, and "Development of Modern Mathematics" by Raymond Wilder. The entire book is a good one to put in the hands of prospective teachers of high school mathematics.
2. *Men of Mathematics* by Eric Temple Bell. Most students find this book to be very readable and a good way to become acquainted with the lives of many famous mathematicians of the past. It is difficult to limit oneself to only a few selections. We have used the brief biographies of George Boole for an introduction to logic, of Georg Cantor for a presentation of infinite set theory, of Karl Weierstrass for topics in analysis, and of Hamilton or Galois for topics in algebra.
3. *Mathematical People*: *Profiles and Interviews*, edited by Donald Albers and G. L. Alexanderson. This excellent book which came out in 1985 is a "20th Century Men (and Women) of Mathematics." The student can read about 25 modern mathematical scientists, most of whom are still alive, and the contributions they are presently making to mathematics. This book should certainly be available for students to browse through for information and inspiration.
4. "Professional Opportunities in the Mathematical Sciences." This 38-page publication of the Mathematical Association of America gives a brief survey of career opportunities for the mathematics major. It makes a good reference for students trying to decide on how they want to use their mathematical training.
5. *American Mathematical Monthly*. There have been many recent articles in this publication that are of general interest to the undergraduate student. It is a good idea to have several issues available for students to read in order to become familiar with the MAA. Some of the articles we have used are:
   (a) "Is Mathematical Truth Time-Dependent?" by Judith Grabiner in the April, 1974, issue. The setting for this discussion is in the historical development of calculus. The March, 1983, issue contains another article by this author on "Cauchy and the Origins of Rigorous Calculus."
   (b) "Computer Science and its Relation to Mathematics" by Donald Knuth, also in the April, 1974, issue.
   (c) "Are There Coincidences in Mathematics?" by Philip Davis in the May, 1981, issue.
   (d) "What is Mathematics?" by Ernst Snapper and "Mathematics as an Objective Science" by Nicolas Goodman, both in the August–September, 1979, issue.
6. *Mathematics in Western Culture* by Morris Kline. We have had students read the chapter "New Geometries, New Worlds" discussing the place of non-Euclidean

geometry, and the chapter "The Mathematical Theory of Ignorance: The Statistical Approach to the Study of Man." There are many other books by Kline which could be included in this list. The last chapter in his *Mathematical Thought from Ancient to Modern Times* is entitled "The Foundations of Mathematics" and provides a description of the schools of mathematical philosophy and the main results in logic and set theory that made it necessary to develop philosophies in the first place.

7. Some general readings on problem-solving approaches and creativity and discovery are welcomed by students. We have used successfully the brief article "Mathematical Creation" by Henri Poincaré, the opening chapter "The Creative Mind" in the book *Science and Human Values* by Jacob Bronowski and the opening chapter "Problem Solving" in the book *Patterns of Problem Solving* by Moshe Rubinstein. Part of one of George Polya's books on problem solving could also be used.
8. Source books of mathematics. Many individuals have urged modern students of mathematics to "read the masters." A good way to get started on this endeavor is to use one of the following source books. *A Source Book in Classical Analysis* by Garrett Birkhoff, *A Source Book in Mathematics* by David Eugene Smith or *A Source Book in Mathematics* 1200–1800 by Dirk Struik. We have used excerpts from the writings of Leibniz, Euler, Cauchy, Riemann and Weierstrass to give a glimpse of how these masters presented topics in the calculus. Pages 1–16 in Birkhoff and pages 270–291, 383–386 in Struik can serve as a brief introduction.
9. In addition to those mentioned above, there are several works which should be provided in a seminar room for students to browse through in leisure moments. They are more likely to do this if these books are not hidden on library shelves, but are displayed on a table.
   (a) *The World of Mathematics*, edited by James Newsman—4 volumes.
   (b) *Mathematics—People, Problems, Results*, edited by Douglas Campbell and John Higgins—3 volumes.
   (c) *The Mathematical Experience* by Philip Davis and Reuben Hersh.
   (d) *Great Moments in Mathematics* by Howard Eves—volumes 5 and 7 in the Dolciani Mathematical Exposition series.
   (e) *In Mathematical Circles* (2 volumes) and *Mathematical Circles Revisited* by Howard Eves.

Two things will probably occur to most people who have read this far. First, there is a fair amount of bookkeeping involved in setting up the readings and keeping track of the students' progress with the program. We have not found it to be excessive, but perhaps someone can think of a simpler way. Second, there are many readings which you like, but which I have omitted. If so, you can develop your own reading program and include those selections that you like. The specific content of what our students are reading is not nearly as important as the fact that they are reading.

Differentiation of Power Series

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A very simple, direct, and elementary proof on term-by-term differentiation of power series was given by Apostol [**1**]. The aim of this paper is to offer an at least equally simple proof which has the advantage that it is also applicable for complex differentiation. There is a certain similarity to Apostol's proof, particularly in that both proofs avoid uniform convergence and presume the knowledge of the theorem concerning the existence of the radius of convergence. We need the following lemma.

LEMMA. *For $x \in \mathbb{C}$, $h \in \mathbb{C}$, $0 < |h| \leqslant H$ and $n \in \mathbb{N}$*

$$|(x+h)^n - x^n - hnx^{n-1}| \leqslant \frac{|h|^2}{H^2}(|x| + H)^n \tag{1}$$

$$|nx^{n-1}| \leqslant \frac{1}{H}\{2(|x| + H)^n + |x|^n\}. \tag{2}$$

Proof. Inequality (2) follows from the inequality

$$H|nx^{n-1}| - (|x| + H)^n - |x|^n \leqslant |(x+H)^n - x^n - Hnx^{n-1}|$$

and from inequality (1) with $h = H$.

For the proof of (1) we use the binomial theorem

$$\begin{aligned} |(x+h)^n - x^n - nx^{n-1}h| &\leqslant |h|^2 \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k}|h|^{k-2} \\ &\leqslant \frac{|h|^2}{H^2} \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k} H^k \leqslant \frac{|h|^2}{H^2}(|x| + H)^n. \end{aligned}$$

THEOREM. *Let the power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ have a positive radius of convergence R (possibly ∞). Then $f_1(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ has the same radius of convergence and $f'(x) = f_1(x)$ for $0 \leqslant |x| < R$.*

Proof. Choose $H > 0$ such that $|x| + H < R$. It follows from (2) that

$$|nc_n x^{n-1}| \leqslant \frac{1}{H}\{2|c_n|(|x| + H)^n + |c_n||x|^n\},$$

and consequently $\sum_{n=1}^{\infty} nc_n x^{n-1}$ converges absolutely. Since x satisfied $0 \leqslant |x| \leqslant R$ but was otherwise arbitrary, it follows that the radius of convergence of f_1 is R.

It follows from (1) that

$$\left|\frac{f(x+h)-f(x)}{h}-f_1(x)\right| \leqslant \frac{|h|}{|H|^2}\sum_{n=1}^{\infty}|c_n|(|x|+H)^n.$$

By sending h to 0 we obtain $f'(x)=f_1(x)$.

On a less elementary level, we note that the preceding proof could, with appropriate modifications, be used in a normed ring.

REFERENCES

1. T. Apostol, Term-wise Differentiation of Power Series, Amer. Math. Monthly, 59 (1952) 323–326. Reprinted in Selected Papers on Calculus, Math. Assoc. of America, 1969.

PROBLEMS AND SOLUTIONS

For instructions about submitting ***proposed*** *problems for publication in this department see the inside front cover. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions *should be sent to the address given on the inside front cover.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed by August 31, 1987, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3201. *Proposed by Grahame Bennett, Indiana University, Bloomington.*

Let us say that a convergent series of positive terms, Σa_n, is rapidly convergent if there exists a sequence of positive numbers $\{b_n\}$ such that

$$\sum_{n=N}^{\infty} a_n \sum_{m=1}^{n} b_m = O(b_N).$$

Prove that the series Σa_n is rapidly convergent if and only if

$$\sum_{n=N}^{\infty} a_n = O\left(\frac{1}{N}\right).$$

(This shows that the termwise sum of two rapidly convergent series is rapidly convergent.)

E 3206. *Proposed by Chico Problem Group, California State University, Chico.*

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n nonzero vectors in R^d such that $x_i \cdot x_j \leqslant 0$ whenever $1 \leqslant i < j \leqslant n$.

(a) Prove that $n \leqslant 2d$ and characterize those S for which $n = 2d$.

(b) What if $x_i \cdot x_j \leqslant 0$ is replaced by $x_i \cdot x_j < 0$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Convex Brianchon Hexagons

E 3045 [1984, 310]. *Proposed by Calin P. Popescu, Bucharest, Romania.*

Let H be a hexagon inscribed in a circle. Show that H can be circumscribed about a conic if and only if the product of three alternate sides equals the product of the other three.

Solution by Aage Bondesen, Espergaerde, Denmark. A necessary and sufficient condition that H can be circumscribed about a conic is that H is a "Brianchon hexagon", i.e., that its main diagonals meet in a point, possibly at infinity. (See Graustein, *Introduction to Higher Geometry*, p. 261.) What we have to prove is therefore the following:

H is a Brianchon hexagon if and only if the product of three alternate sides equals the product of the other three.

Now, for this to be true, our concept of a hexagon must not be too wide. As the following example shows, self-intersecting hexagons cannot be tolerated.

Example. In Figure 1, $ABCDEF$ is a Brianchon hexagon inscribed in a circle c. AC is a diameter in c, and B' is the reflection of B in this diameter. $AB'CDEF$ is *not* a Brianchon hexagon (since EB' does not meet DA and FC in a point), yet the two hexagons have the same sequence of side-lengths.

We therefore assume H to be without self-intersections; since H is inscribed in a circle this means that H is a <u>convex hexagon</u>, say hexagon $ABCDEF$, with A, B, C, D, E, F on a circle c. Put $\overline{AB} = a_1$, $\overline{BC} = b_2$, $\overline{CD} = c_1$, $\overline{DE} = a_2$, $\overline{EF} = b_1$, and $\overline{FA} = c_2$.

(1) Let AD, BE, and CF meet in a point P, so that H is a Brianchon hexagon, and put

$$\overline{PF} = \alpha, \qquad \overline{PD} = \beta, \qquad \text{and} \qquad \overline{PB} = \gamma.$$

Then, by similar triangles, $a_2/a_1 = \beta/\gamma$, and similarly $b_2/b_1 = \gamma/\alpha$, and $c_2/c_1 = \alpha/\beta$. It follows that

$$\frac{a_2 b_2 c_2}{a_1 b_1 c_1} = 1,$$

E 3206. *Proposed by Chico Problem Group, California State University, Chico.*

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n nonzero vectors in R^d such that $x_i \cdot x_j \leqslant 0$ whenever $1 \leqslant i < j \leqslant n$.

(a) Prove that $n \leqslant 2d$ and characterize those S for which $n = 2d$.

(b) What if $x_i \cdot x_j \leqslant 0$ is replaced by $x_i \cdot x_j < 0$?

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Then, by similar triangles, $a_2/a_1 = \beta/\gamma$, and similarly $b_2/b_1 = \gamma/\alpha$, and $c_2/c_1 = \alpha/\beta$. It follows that

$$\frac{a_2 b_2 c_2}{a_1 b_1 c_1} = 1,$$

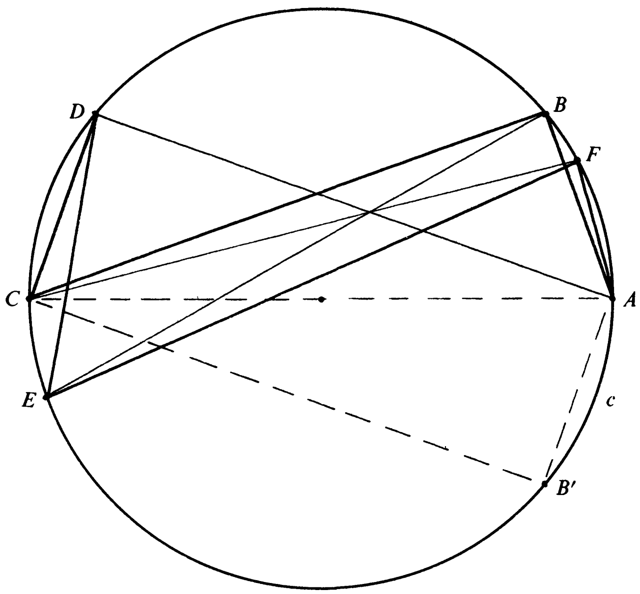

FIG. 1.

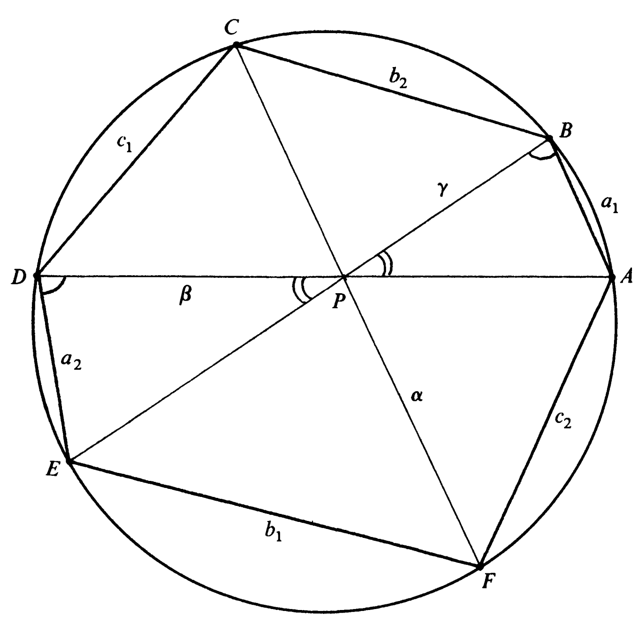

FIG. 2.

and we have proved the "only if" part of the theorem. (By the way, this part can be shown to be true also for self-intersecting hexagons. It is the "if" part that is restrictive.)

(2) Now assume that H is *not* a Brianchon hexagon, i.e., AD, BE, and CF do *not* meet in a point. Clearly, arcs ABC and DEF cannot both contain a half-circle, so, by symmetry, we may suppose arc ABC doesn't. Let AD and CF meet in P, and let $\overline{EP}$ meet the circle c again in B'. Then, clearly, $B' \in$ arc ABC. Put $\overline{AB'} = a_1'$ and $\overline{B'C} = b_2'$. Then, by (1),

$$a_2 b_2' c_2 = a_1' b_1 c_1.$$

Now there are two possibilities. Either, as in Figure 3, B' is between A and B, and then $b_2 < b_2'$, $a_1' < a_1$, and hence $a_2 b_2 c_2 < a_1 b_1 c_1$; or else, B' is between B and C, and we have the opposite inequalities. In either case

$$a_2 b_2 c_2 \neq a_1 b_1 c_1.$$

This proves the "if" part of the theorem.

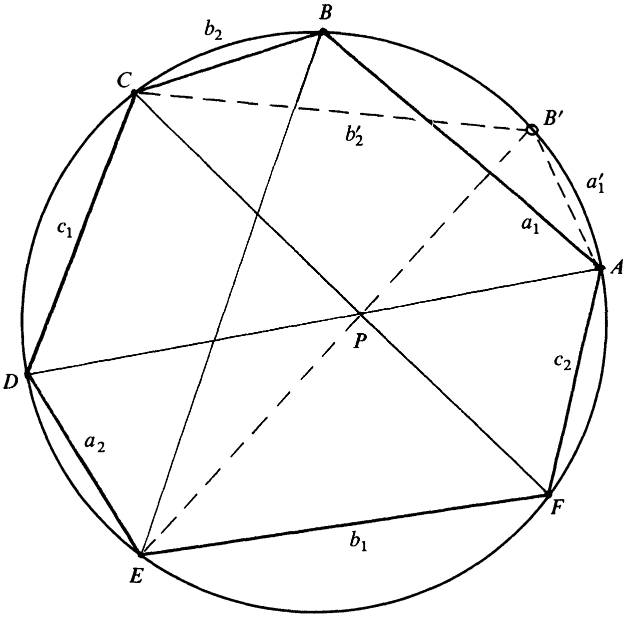

FIG. 3.

Also solved by H. Demir (Turkey), J. Dou (Spain), H. Eves, D. Jones, H. Kappus (Switzerland), L. Kuipers (Switzerland), O. P. Lossers (The Netherlands), E. Morgantini (Italy), L. Nair (Canada), W. A. Newcomb, I. J. Schoenberg, and the proposer.

A Versatile Identity

E 3062 [1984, 580]. *Proposed by Roger B. Nelsen, Lewis and Clark College.*

Show that for all integers $n \geqslant 0$ and all real x,

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^{n}=\frac{1}{2}\sum_{j=0}^{\infty}(x+j)^{n}2^{-j}.$$

Solution I by Donald E. Knuth, Stanford University.
If $0 \leqslant m \leqslant n$, we have

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{j}{m}=1=\frac{1}{2}\sum_{j=0}^{\infty}\binom{j}{m}2^{-j},$$

since $\Sigma_j(-1)^{k-j}\binom{k}{j}\binom{j}{m}=\delta_{km}$ and $\Sigma_{j\geqslant 0}\binom{j}{m}z^{j}=z^{m}(1-z)^{-1-m}$. Any polynomial $p(j)$ of degree $\leqslant n$ is a linear combination $\Sigma_{0\leqslant m\leqslant n}\binom{j}{m}a_{m}$, hence

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}p(j)=\frac{1}{2}\sum_{j=0}^{\infty}p(j)2^{-j}.$$

Solution II by J. C. Binz, University of Bern, Switzerland. We prove a more general equality: For all integers $n \geqslant 0$, for all reals x, and for all reals y with $|y|<1$

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\left(\frac{y}{1-y}\right)^{k}(-1)^{k-j}\binom{k}{j}(x+j)^{n}=(1-y)\sum_{j=0}^{\infty}(x+j)^{n}y^{j}.$$

Proof.

$$\sum_{j=0}^{\infty}(x+j)^{n}y^{j}=\sum_{s=0}^{n}\binom{n}{s}x^{n-s}\sum_{j=0}^{\infty}j^{s}y^{j}.$$

We have to compute

$$T_s=\sum_{j=0}^{\infty}j^{s}y^{j}.$$

First $T_0=1/(1-y)$; for $s\geqslant 1$

$$T_s=\sum_{j=0}^{\infty}\sum_{k=1}^{s}S(s,k)j(j-1)\cdots(j-k+1)y^{j},$$

where $S(s,k)$ are the Stirling numbers of the second kind. Hence,

$$T_s=\sum_{k=1}^{s}S(s,k)k!\sum_{j=0}^{\infty}\binom{j}{k}y^{j}=\sum_{k=1}^{s}S(s,k)k!y^{k}\sum_{i=0}^{\infty}\binom{k+i}{i}y^{i}$$

$$=\frac{1}{1-y}\sum_{k=1}^{s}S(s,k)k!\left(\frac{y}{1-y}\right)^{k}.$$

With $S(0,0) = 1$ it follows that

$$(1-y)\sum_{j=0}^{\infty}(x+j)^n y^j = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}\sum_{k=0}^{s}S(s,k)k!\left(\frac{y}{1-y}\right)^k$$
$$= \sum_{s=0}^{n}\binom{n}{s}x^{s}\sum_{k=0}^{n}S(n-s,k)k!\left(\frac{y}{1-y}\right)^k$$

(where $S(u,v) = 0$ for $u < v$). Replacing $S(n-s,k)k!$ by $\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{n-s}$, we obtain

$$(1-y)\sum_{j=0}^{\infty}(x+j)^n y^j = \sum_{k=0}^{n}\left(\frac{y}{1-y}\right)^k\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{s=0}^{n}\binom{n}{s}x^s j^{n-s}$$
$$= \sum_{k=0}^{n}\left(\frac{y}{1-y}\right)^k\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n.$$

Solution III by Gordon Williams, Virginia Military Institute. Using the difference operator E and several of its properties, the result is obtained easily. $Ef(x) = f(x+1)$. Other definitions and some properties are given below.

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}E^j x^n$$
$$= \sum_{k=0}^{n}(E-1)^k x^n$$
$$= \frac{1}{1-(E-1)}x^n$$
$$= \frac{1}{2}\frac{1}{1-\frac{E}{2}}x^n$$
$$= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{E}{2}\right)^j x^n$$
$$= \frac{1}{2}\sum_{j=0}^{\infty}(x+j)^n 2^{-j}.$$

Definitions and properties of operators: For operators A and B, $((A+B)f)(x) = (Af)(x) + (Bf)(x)$ and $((AB)f)(x) = (A(Bf))(x)$. Thus $A^m f(x) = A^{m-1}(Af(x))$ and

$$(A+B)^m f(x) = \sum_{i=0}^{m}\binom{m}{i}A^i B^{m-i}f(x), \quad \text{if } AB = BA.$$

Also solved by 23 other readers and the proposer. Several solvers used Stirling numbers, several used the difference operator, several used expansions involving e^x, and several used the operator $x(d/dx)$.

Another extension pertains to non-integer n. In this case the series is non-terminating

$$\sum_{j=0}^{\infty}(-1)^j\binom{k}{j}\binom{k'-1-j}{p-j}\bigg/(j+1)$$
$$=\left(\frac{1}{k+1}\right)\left[\binom{k'}{p+1}-\frac{\sin\pi(k'-p)}{\sin\pi k'}\binom{k-k'+p+1}{p+1}\right] \tag{4}$$

We can derive yet another formula that goes beyond even (3)

$$\sum_{j=0}^{\infty}(-1)^j\binom{k}{j-p-1}\binom{k'-1-j}{p-j}\bigg/(j+1)$$
$$=\binom{k}{-(p+2)}\binom{k'}{p+1}[\psi(k+p+2-k')-\psi(-k')], \tag{4$'$}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, i.e., $\Gamma'(z)/\Gamma(z)$. In particular, the choice of $k=-1/2$, $p=-(n+2)$ and $k'=-(n+1)=(p+1)$ gives

$$\sum_{j=0}^{\infty}(-1)^j\binom{-\frac{1}{2}}{n+j+1}\bigg/(j+1)=\binom{-\frac{1}{2}}{n}\left[\psi\left(\frac{1}{2}\right)-\psi(n+1)\right]. \tag{4$''$}$$

Also solved by 28 other readers and the proposer.

The first generalization noted by Rangarajan was also mentioned by S. Pedersen (Denmark).

Counting Partitions

E 3075 [1985, 147]. *Proposed by Duane M. Broline, University of Evansville.*

Let B_k be the kth Bell number, the number of partitions of k objects. For $k \leqslant n$, show that

$$\sum_{\substack{a_1,a_2,\dots,a_n\geqslant 0\\ a_1+2a_2+\cdots+na_n=n}}\frac{a_1^k}{1^{a_1}(a_1!)2^{a_2}(a_2!)\cdots n^{a_n}(a_n!)}$$
$$=\sum_{\substack{b_1,b_2,\dots,b_k\geqslant 0\\ b_1+2b_2+\cdots+kb_k=k}}\frac{k!}{(1!)^{b_1}(b_1!)(2!)^{b_2}(b_2!)\cdots(k!)^{b_k}(b_k!)}=B_k.$$

Editor's note. The last summation is a classical result. It is included to show the asymmetrical nature of the problem.

Solution I by Stephen M. Gagola, Jr., Kent State University. Consider the set F of functions from a k-element set S to the natural numbers up to n with $n \geqslant k$. Each such function f determines a partition of S, whose blocks are the nonempty inverse images of numbers. Two functions f and g determine the same partition if and only if $f = \sigma \circ g$ for some permutation σ of $(1, \ldots, n)$. Thus the number of partitions of S is the number of orbits in the natural action of S_n on F, by composition.

By a result often referred to as Burnside's Orbit Counting Formula (but which is not due to Burnside), this number is

$$|S_n|^{-1} \sum_{\sigma \in S_n} x(\sigma),$$

where $x(\sigma)$ is the number of functions in F fixed by the permutation σ. Clearly $\sigma \circ f = f$ if and only if f maps S into the fixed point set of σ, so that if σ has cycle structure described by the a_j then $x(\sigma) = a_1^k$.

There are

$$n!/(1^{a_1}(a_1!)2^{a_2}(a_2!) \cdots n^{a_n}(a_n!))$$

permutations in S_n which have this cycle type. When these terms are grouped together, and $n!$ is cancelled with $|S_n|^{-1}$, the first equality follows.

Solution II by Hang-Fai Yeung (student), University of New South Wales, Australia. Let $F(u, x)$ be defined by

$$F(u, x) = [\exp(u(e^x - 1))]/(1 - u).$$

Then we have, for $|u| < 1$,

$$\begin{aligned} F(u, x) &= \exp[ue^x - u - \log(1 - u)] \\ &= \exp[ue^x + (u^2/2) + (u^3/3) + \cdots] \\ &= \sum_{n=0}^{\infty} \left(\sum_{\substack{a_1, \ldots, a_n \geqslant 0 \\ a_1 + 2a_2 + \cdots + na_n = n}} \frac{n! e^{x a_1}}{a_1! \cdots a_n!} \frac{1}{1^{a_1} 2^{a_2} \cdots n^{a_n}} \right) \frac{u^n}{n!} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F(u, x) &= \left[\sum_{r=0}^{\infty} u^r (e^x - 1)^r \Big/ r! \right] (1 - u)^{-1} \\ &= \left[\sum_{r=0}^{\infty} u^r \sum_{m=0}^{\infty} S(m, r) x^m \Big/ m! \right] (1 - u)^{-1}, \end{aligned}$$

where $S(m, r)$ is a Stirling number of the second kind.

Differentiating these two expressions k times with respect to x, setting $x = 0$, and identifying coefficients of the nth power of u yields the desired result.

Also solved by S. Goodenough and the proposer.

Solutions to an Equation Involving Powers

E 3085 [1985, 287]. *Proposed by T. C. Lim, George Mason University, Fairfax, Virginia.*

Let $g(\mu)$ be the unique nonnegative solution of

$$\{\mu + g(\mu)\}^p + |\mu - g(\mu)|^p = 2\mu,$$

where $1 < p < 2$ and $0 \leqslant \mu \leqslant 1/2$. Prove that

$$\{1 - \mu + g(\mu)\}^p + |1 - \mu - g(\mu)|^p \leqslant 2(1 - \mu).$$

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. For $g \geqslant \mu$ the expression $(\mu + g)^p + (g - \mu)^p$ is an increasing function of g. Substituting μ for g we obtain $(2\mu)^p$, which is less than 2μ if $p > 1$ and $0 < \mu < 1/2$. For $g = 1 - \mu$ we obtain $1 + (1 - 2\mu)^p$ which is larger than 2μ if $\mu < 1/2$. Hence we have $\mu < g(\mu) < 1 - \mu$.

Now we introduce $x = \mu + g(\mu)$ and $y = g(\mu) - \mu$. We have to prove

$$(1 - y)^p + (1 - x)^p \leqslant 2 - 2\mu.$$

Using the definition of $g(\mu)$, this can be formulated as

$$x^p + (1 - x)^p + y^p + (1 - y)^p \leqslant 2,$$

which trivially follows from the convexity of the function $x^p + (1 - x)^p$.

Also solved by I. E. Leonard, S. Marivani, and the proposer.

A Counting Problem

E 3086 [1985, 287]. *Proposed by Dennis Spellman, Sacred Heart University, Bridgeport, CT.*

If c and m are positive integers each greater than 1, find the number $n(c, m)$ of ordered c-tuples $(n_1, n_2, \ldots, n_c)$ with entries from the initial segment $\{1, 2, \ldots, m\}$ of the positive integers such that $n_2 < n_1$ and $n_2 \leqslant n_3 \leqslant \cdots \leqslant n_c$.

Solution by the University of South Alabama Problem Group. If n_2 is fixed, there are $m - n_2$ choices for n_1 and the number of admissible $(c - 1)$-tuples $(n_2, n_3, \ldots, n_c)$ is the number of ways that $m - n_2$ indistinguishable balls can be distributed among urns $U_1, U_2, \ldots, U_{c-1}$. To see the latter, take

$$n_i = n_2 + N_1 + N_2 + \cdots + N_{i-2}, \qquad i = 3, \ldots, c,$$

where N_j is the number of balls in urn U_j (notice that if $c = 2$ there is only one way

to put $m - n_2$ balls in U_1). We have

$$n(c, m) = \sum_{i=1}^{m-1} (m - i)\binom{(m - i) + (c - 1) - 1}{(c - 1) - 1}$$
$$= (c - 1)\sum_{j=0}^{m-2}\binom{m - 2 - j + c - 1}{c - 1}$$
$$= (c - 1)\binom{m - 2 + c}{c},$$

because the last sum is just the number of ways to put $m - 2$ indistinguishable balls into $c + 1$ urns.

Also solved by J. C. Binz (Switzerland), R. B. Eggleton (Australia), U. Everling (West Germany), J. Fukuta (Japan), V. Hernandez (Spain), W. Janous (Austria), O. P. Lossers (Netherlands), H. M. Mahmoud, D. Neuenschwander (West Germany), V. Pambuccian (Romania), H. Prodinger (Austria), C. R. Rosentrater, A. J. Schwenk, J. S. Sumner, G. Sylvester, D. Tretter (student), J. T. Ward, W. P. Wardlaw, and the proposer.

Editor's note. The proposer also asked if the number $n(c, m)$, which is the number of simple (i.e., not compound) left normed basic commutators of weight c in a free group of rank m, has appeared previously. One place where this number is computed is in a paper by S. Bachmuth. *Trans. Amer. Math Soc.*, vol. 122 (1966), p. 5. There (with $c = n - 1$ and $m = q$) the number is computed in the course of determining the rank of a certain free abelian group of automorphisms.

Maximizing a Cyclic Sum of Powers of Differences

E 3087 [1985, 287]. *Proposed by Weixuan Li, University of Waterloo, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Let a_i, $i = 1, 2, \ldots, n$, be real numbers such that $0 \leqslant a_i \leqslant 1$, where $n \geqslant 2$. Find a best upper bound for

$$S_n = (a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_n - a_1)^2,$$

and determine all cases when this bound is attained.

Solution by M. S. Klamkin and A. Meir, University of Alberta, Edmonton, Alberta, Canada. More generally we find the maximum of the cyclic sum

$$S(n, p) = |a_1 - a_2|^p + |a_2 - a_3|^p + \cdots + |a_n - a_1|^p.$$

Clearly the case $n = 2m$ is trivial for which $\max S(2m, p) = 2m$ and achieved by either

$$a_1 = a_3 = \cdots = a_{2m-1} = 1, \quad a_2 = a_4 = \cdots = a_{2m} = 0,$$

or vice-versa. For the odd case, we show that

$$\max S(2m+1, p) = \begin{cases} 2m - 1 + 2^{1-p}, & \text{for } 0 < p \leqslant 1, \\ 2m, & \text{for } p > 1. \end{cases} \tag{1}$$

Proof. Suppose that the above maximum is achieved for the values $a_1, a_2, \ldots, a_{2m+1}$. Then either (i) all a_i are 0 or 1, or (ii) there exists an a_i with $0 < a_i < 1$. In case (i), $a_v = a_{v+1}$ for some v and thus $S(2m+1, p) \leqslant 2m$ for any $p > 0$. In case (ii), let $0 < a_v < 1$. Then $a_{v-1} < a_v$ and $a_{v+1} < a_v$ is impossible since $a_v = 1$ would yield a larger sum. Similarly $a_{v-1} > a_v$ and $a_{v+1} > a_v$ is impossible. Thus, $a_{v-1} > a_v > a_{v+1}$ or vice-versa. In both cases,

$$|a_{v-1} - a_v|^p + |a_v - a_{v+1}|^p \leqslant \begin{cases} |a_{v-1} - a_{v+1}| \leqslant 1, & \text{if } p > 1, \\ 2\left\{\dfrac{1}{2}|a_{v-1} - a_{v+1}|\right\}^p \leqslant 2^{1-p}, & \text{if } 0 < p \leqslant 1. \end{cases}$$

Since clearly $\sum_{i \neq v-1, v} |a_i - a_{i+1}|^p \leqslant 2m - 1$ for all $p > 0$, inequality (1) follows, the bound being attained in the case $0 < p \leqslant 1$ by taking, for example, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 1$, $a_{2k} = 0$, $a_{2k+1} = 1$, $2 \leqslant k \leqslant m$.

Also solved by 35 other readers and the proposers.

A Characterization of the Positive Integral Powers of Two

E 3089 [1985, 359]. *Proposed by Eliot T. Jacobson, Ohio University.*

Determine all positive even integers n having the following property: given any integer b with $1 < b < n$ and $\gcd(b, n) = 1$, there is a solution to the congruence $(b-1)x \equiv (n/2) (\text{mod } n)$.

Solution by Edwin M. Klein, University of Wisconsin-Whitewater. Let $n = 2m$. Then $(b-1)x \equiv m (\text{mod } n)$ is solvable if and only if $d \mid m$, where $d = \gcd(b-1, n)$.

If $n = 2^k$, then $d = 2^j$ for some $j, 1 \leqslant j < k$, and so $d \mid m$.

Conversely, suppose $n = 2^k c$, where $c > 1$ is odd. There exists t, $1 \leqslant t < c$, such that $2^k t \equiv 1 \pmod c$. Let $b = 2^k t + 1$. Then $1 < b < n$ and $\gcd(b, n) = 1$, but $d = 2^k \nmid m$.

Thus n has the specified property if and only if n is a power of 2.

Stephen M. Gagola, Jr., provided the following generalization. Determine all pairs of positive integers (n, d) with $d \mid n$, $d \neq n$ having the property: for any integer b satisfying $\gcd(b, n) = 1$ and $1 < b < n$, there is a solution to the congruence

$$(b-1)x \equiv d \pmod n. \tag{*}$$

The result is that (n, d) satisfies the stated property if and only if
(i) $n = p^a$ and $d = p^{a-1}$ where p is any prime and a is any positive integer, or
(ii) $n = 2p^a$ and $d = 2p^{a-1}$ where p is any odd prime and a is any positive integer.

R. B. McNeill provided the following generalization. Let $n = 2^\alpha q$, $\alpha \geqslant 1$, q odd, and $T(n) =$ the number of b's satisfying $1 < b < n$ and $\gcd(b, n) = 1$ for which the congruence $(b-1)x \equiv (n/2) \pmod{n}$ is solvable. Then $T(n) = (2^{\alpha-1} - 1)\phi(q)$, where ϕ is Euler's totient function.

Also solved by thirty-four other readers and the proposer.

Absolute Minima of Symmetric Functions

E 3093 [1985, 427]. *Proposed by William C. Waterhouse, Pennsylvania State University.*

Let $f(x_1, \ldots, x_n)$ be a real symmetric polynomial, and suppose that f has an absolute minimum. Bunyakovskii showed long ago that the minimum need not occur at a point where all x_i are equal (cf. this MONTHLY, 90 (1983) 378–387). Prove that $\nabla f = 0$ at some point where all x_i are equal. Show on the other hand that this statement need not be true for an arbitrary C^∞ symmetric function having an absolute minimum.

Solution by Daniel Neuenschwander (student), University of Bern, Switzerland.

(i) In the polynomial case if $f(x_1, \ldots, x_n)$ assumes an absolute minimum then so does $f(x, x, \ldots, x)$, in which case

$$\frac{d}{dx} f(x, x, \ldots, x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x, x, \ldots, x) = 0.$$

As f is symmetric all $\partial f / \partial x_i$ are equal, so they all vanish and $\nabla f(x, \ldots, x) = 0$.

(ii) In the C^∞ case $f(x_1, x_2) \equiv e^{x_1 + x_2}(\cos(x_1 - x_2) + 1)$ is a counterexample since f attains its absolute minimum 0 but $\nabla f(x, x) = 2(e^{2x}, e^{2x}) \neq 0$.

Also solved by E. Salamin, W. Taylor, and the proposer.

ADVANCED PROBLEMS

6542. *Proposed by Andrew M. Odlyzko, AT & T Bell Labs, Murray Hill, NJ.*

Suppose $x_1, x_2, x_3, \ldots$ is a sequence of numbers in $[0, 1)$ such that at least one of its sequential limit points is irrational. For given n, consider the 2^n numbers $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$, where each ε_i takes the two values ± 1. If $0 \leq a < b \leq 1$,

let $N_n(a,b)$ be the number of n-tuples $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ such that the fractional part of $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$ is in $[a,b)$. Prove that for every a,b we have

$$\lim_{n \to \infty} 2^{-n} N_n(a,b) = b - a.$$

6543. *Proposed by A. S. Cavaretta, Jr. and C. R. Selvaraj, Kent State University.*

Let $m_1, m_2, \dots, m_r$ be distinct odd integers. Show that the determinant

$$\begin{vmatrix} 1 & 2^{m_1} & 3^{m_1} & 4^{m_1} & 5^{m_1} & \cdots & (r+1)^{m_1} \\ & & \vdots & & & & \vdots \\ 1 & 2^{m_r} & 3^{m_r} & 4^{m_r} & 5^{m_r} & \cdots & (r+1)^{m_r} \\ 1 & 0 & -1 & 0 & 1 & \dots & d_{r+1} \end{vmatrix}$$

is nonzero, where the entries of the last row are

$$d_j = (-1)^{[j/2]} \left(\frac{1 - (-1)^j}{2} \right), \qquad j = 1, \dots, r+1.$$

6544. *Proposed by Daniel Goffinet, Lycée Claude Fauriel, St. Etienne, France.*

What is the partial fraction decomposition of $1/(x^n - 1)$ over the field Q of rational numbers?

SOLUTIONS OF ADVANCED PROBLEMS

Zero Distribution of Appell Sequence Polynomials

6498 [1985, 433]. *Proposed by I. J. Schoenberg, University of Wisconsin, Madison.*

Let $P_0(x) \equiv 1$, $P_1(x)$, $P_2(x), \dots$ be an Appell sequence of polynomials, i.e.,

$$P_n'(x) = P_{n-1}(x) \qquad (n = 1, 2, \dots).$$

If $x_{i,n}$ $(i = 1, 2, \dots, n)$ are the zeros of $P_n(x)$, show that

$$\lim_{n \to \infty} \max_i |x_{i,n}| = \infty,$$

unless

$$P_n(x) = (x - \alpha)^n / n! \qquad (n = 0, 1, \dots)$$

for some fixed α.

let $N_n(a,b)$ be the number of n-tuples $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$ such that the fractional part of $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$ is in $[a,b)$. Prove that for every a,b we have

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for some fixed α.

Solution by Eugene Salamin, Coherent, Inc., Palo Alto, California. We have

$$P_n(x) = \frac{x^n}{n!} + \sum_{k=1}^{n} c_k \frac{x^{n-k}}{(n-k)!}, \tag{1}$$

where the c_k are arbitrary constants of integration. With no loss of generality, we may assume $c_1 = 0$, since x can be replaced by a new variable $x' = x + c_1$. Thus the problem is to show that the above limit is true unless all $c_k = 0$.

Suppose some $c_k \neq 0$. Let c_r $(r \geq 2)$ be the first nonvanishing coefficient, i.e., $c_r \neq 0$, while $c_k = 0$ for $k < r$. For any $n \geq r$, let σ_m $(m = 1, 2, \ldots, n)$ denote the elementary symmetric function of degree m of the roots x_i $(i = 1, \ldots, n)$ of $P_n(x)$, i.e., σ_m is the sum of the $n!/m!(n-m)!$ distinct products $x_{i(1)}x_{i(2)} \cdots x_{i(m)}$. Then from (1) we have

$$\sigma_m = (-1)^m \frac{n!}{(n-m)!} c_m. \tag{2}$$

By the multinomial theorem and the theory of symmetric functions we see that

$$\sigma_1^r = \sum_{i=1}^{n} x_i^r + (-1)^r r \sigma_r + f(\sigma_1, \sigma_2, \ldots, \sigma_{r-1}), \tag{3}$$

where f is a polynomial with no constant term (if this step seems somewhat "sketchy", see the comments following the proof). By substituting (2) into (3) and using the hypothesis that $c_k = 0$ for $k < r$, we obtain

$$\sum_{i=1}^{n} x_i^r = -\frac{r\,n\,!}{(n-r)!} c_r$$

and, hence,

$$\sum_{i=1}^{n} |x_i|^r \geq \frac{r\,n\,!}{(n-r)!} c_r.$$

But this implies that

$$|x_i|^r \geq r(n-1)!|c_r|/(n-r)!,$$

for some i. Hence, for some constant $K > 0$ independent of n, there is a root $x_i = x_{i,n}$ of $P_n(x)$ such that

$$|x_i| > K[(n-1)!/(n-r)!]^{1/r},$$

and this approaches ∞ as $n \to \infty$ because $r \geq 2$.

The solutions presented by L. E. Mattics and Mauri Orjatsalo (Finland) were quite similar to Salamin's; they relied upon Newton's identities for power sums of roots at the step in the above solution that was referred to as seemingly "sketchy." The most crisp (though not the most elementary) treatment of this point was in the

solution of O. P. Lossers (The Netherlands). Lossers observed that if $r(z)$ is a polynomial of degree at most $n-k-1$, and

$$P(z) = z^n - az^{n-k} + r(z)$$

has roots $x_1, \dots, x_n$ in the interior of $|z| = R$, then

$$\sum_{i=1}^{n} x_i^k = \frac{1}{2\pi i}\int_{|z|=R} z^k \frac{P'(z)}{P(z)}\,dz = \frac{1}{2\pi i}\int_{|z|=R} \frac{z^k P'(z) - nz^{k-1}P(z)}{P(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{|z|=R} \frac{akz^{n-1} + z^k r'(z) - nz^{k-1}r(z)}{z^n - az^{n-k} + r(z)}\,dz = ak.$$

Only Chr. A. Meyer (Switzerland) found a really different solution. Meyer proves that if the limit is finite, then

$$|P_n(x)| \le \frac{(|x| + c)^n}{n!}$$

and

$$L(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$$

is an entire function of both x and t. Moreover,

$$\frac{\partial}{\partial x} L(x,t) = tL(x,t)$$

so

$$L(x,t) = f(t)e^{xt},$$

where $f(t)$ is an entire function of exponential type, and $f(0) = 1$. If $f(t_0) = 0$ then an adroit application of Grace's apolarity theorem to the sequence of auxiliary polynomials

$$Q_n(x) = \sum_{k=0}^{n} \frac{(-t_0)^k}{k!} x^k - \sum_{k=0}^{n} t_0^k P_k(0)$$

(together with a compactness argument) yields a contradiction. Hence, by the Hadamard factorization theorem,

$$f(t) = e^{-\alpha t}$$

for some fixed α, and the result follows from the addition formula for the exponential function!

Also solved by the proposer.

Positive Factorizations of e^z

6501 [1985, 595]. *Proposed by I. J. Schoenberg, Madison, Wisconsin.*

Suppose that we have an identity

$$e^z \equiv \left(\sum_{n=0}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right) \quad \text{for} \quad |z| < r, (r > 0), \tag{1}$$

where $a_n \geq 0$ and $b_n \geq 0$ for all n, and neither of the factors in (1) reduces to a constant. Show that the two factors in (1) are necessarily of the form

$$e^{az+c} \quad \text{and} \quad e^{bz-c}, \quad \text{with} \quad a > 0, \quad b > 0, \quad a + b = 1. \tag{2}$$

Solution by Peter Borwein, Dalhousie University, Halifax, Nova Scotia. Since $1 = e^0 = a_0 b_0$ both a_0 and b_0 are strictly positive and since

$$a_n b_0 + a_0 b_n \leq \sum_{k=0}^{n} a_{n-k} b_k = \frac{1}{n!},$$

we have

$$0 \leqslant a_n \leqslant \frac{b_0^{-1}}{n!} \quad \text{and} \quad 0 \leqslant b_n \leqslant \frac{a_0^{-1}}{n!}.$$

Thus,

$$f(z) := \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n$$

are both entire functions of order at most one and, since $f(z)g(z) = e^z$, neither have any zeros. It follows from the Hadamard Factorization Theorem that

$$f(z) = e^{p(z)} \quad \text{and} \quad g(z) = e^{q(z)},$$

where p and q are polynomials of degree at most one. The remaining conclusions are now straightforward.

Many solvers pointed out that this is essentially Raikov's theorem; the most commonly cited reference was E. Lukacs, *Characteristic Functions*, 2nd ed., Griffin, London (1970), pp. 243–245. In probabilistic terms, it says that the characteristic function of the Poisson distribution has only Poissonian factors. No "elementary" real variable solution was received. In fact, no solution was essentially different from Borwein's.

Also solved by L. E. Clarke (England), A. A. Jagers (The Netherlands), Gérard Letac (France), Michel Lobenberg, O. P. Lossers (The Netherlands), Howard C. Morris, Marcel F. Neuts, T. S. Norfolk, Adam Riese, Pei Yuan Wu (Taiwan), and the proposer.

F. In contrast to Borsuk's problem, in the covering problem, the disk does not require the greatest number of covering sets in the plane. It turns out that for a parallelogram P, four reduced copies of P are required to cover P. Surprisingly, no plane convex set requires more than four reduced copies to cover it, and in fact the number four is required only for parallelograms. This was proved by Levi in 1955 and by Gohberg and Markus in 1960. Furthermore, Hadwiger (1957), as well as Gohberg and Markus (1960), have conjectured that an analogue of the planar result holds in higher dimensions. The conjecture states that any n-dimensional convex body can be covered by 2^n or fewer reduced copies, and fully 2^n copies are required only for the n-dimensional parallelepipeds. The bounds obtained so far are much larger than those in the conjecture, although small bounds have been obtained for various special cases by Boltjansky (1960) and recently by Lassak (1984). The general case remains unsolved, even for $n = 3$.

One last problem which should be mentioned is the illumination problem for bounded convex sets. Let F be a plane bounded convex set, and let l be an arbitrary direction in the plane. Informally, we say that a boundary point A of F is a point of illumination with respect to l provided the parallel beam of rays having direction l 'illuminates' the point A and some neighborhood of A. The problem is that of finding the smallest number of directions which suffice to illuminate the whole boundary of F. For the disk, three directions are sufficient to illuminate its boundary. However, the parallelogram requires four directions to illuminate its boundary. Moreover, no plane convex set requires more than four directions, and the number four is required only for parallelograms. If this begins to sound familiar, it is for a good reason. As Boltjansky proved in 1960, the covering problem and the illumination problem are in fact equivalent, even in the n-dimensional case. That is, for F an n-dimensional bounded convex set, k directions are sufficient to illuminate the boundary of F if and only if k reduced copies of F are sufficient to cover F.

Deeper relationships between Borsuk's conjecture, the covering problem, and the illumination problem are investigated in the book. Throughout the book, the reader is impressed by the clarity of the presentation, as the text moves from basic definitions and examples to some elegant proofs. Historical information, as well as numerous diagrams and sketches, add to the enjoyment. The book is indeed for both layman and expert, and it invites the reader to try to solve some of these interesting problems himself.

Groups and Geometry. By Roger C. Lyndon. London Mathematical Society Lecture Note Series, No. 101. Cambridge University Press, Cambridge, 1985. 217 pp. (paperback).

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following definition of a group:

> Let $A, B, C, \ldots$ represent a set of operations, which can be performed on the same object or set of objects. Suppose this set of operations has the following characteristics.
>
> (α) The operations of the set are all distinct, so that no two of them produce the same change in every possible application.
>
> (β) The result of performing successively any number of operations of the set... is another definite operation of the set... .
>
> (γ) A being an operation in the set, there is always another operation A_{-1} belonging to the set, such that A followed by A_{-1} produces no change in any object.

I expected to see a binary operation, the identity element, and the associative law. What was all this about "operations" that can be "performed" on an object or objects so that no two "produce the same change"? A few pages later Burnside gives some examples of groups of order six (all isomorphic): inversions in three circles, rotations in 3-space, linear fractional transformations, linear transformations in two complex variables, affine transformations of the integers modulo 3, permutations on 3 symbols, and permutations on 6 symbols. Only the last two looked at all familiar. What was wrong? I had taken almost four semesters of algebra in college. I had read every page of Herstein, tried every exercise. Somehow, a message had been lost on me. Groups *act*. The elements of a group do not have to just sit there, abstract and implacable; they can *do* things, they can "produce changes." In particular, groups arise naturally as the symmetries of a set with structure. And if a group is given abstractly, such as the fundamental group of a simplical complex or a presentation in terms of generators and relators, then it might be a good idea to find something for the group to act on, such as the universal covering space or a graph.

This interplay between group and symmetry is well known to mathematicians, chemists, and physicists. It was certainly second nature to Burnside at the end of the 19th century. The trouble is that most undergraduate textbooks in algebra barely deign to mention it, other than a perfunctory chapter on permutation groups and Cayley's theorem. For example, in Saracino (*Abstract Algebra*: *A First Course*) one must wait until page 75 to see the symmetry group of the square, and that is the only example in the book that uses composition of functions as the underlying binary operation of a group, other than the full symmetric group or the automorphism group of a group. In fact, this single example on page 75 yields the only reference in the index for "symmetry." Other texts, such as Hillman and Alexanderson or Herstein, fare no better. Fraleigh, to his credit, does give symmetries of the triangle and square among his first examples of groups and even has a 9-page chapter on "Groups in Geometry and Analysis." Good old Mac Lane and Birkhoff, bless its soul, *begins* its chapter on groups with a section "Groups and Symmetry." It also mentions group actions and orbits and transitivity; Fraleigh, again, is the only other that does.

This brings me to Lyndon's *Groups and Geometry*. Now this is not an algebra textbook and it is not a geometry textbook. It is written, however, at an advanced undergraduate level and demands a minimum of background, none in geometry and only a little in groups. And it does show groups in action. I think it could make a

good course for mathematics majors to take after algebra, if only to show them where all those abstract groups come from. Normal subgroups, semidirect products, composition series, all become such natural ideas as the isometry group of the Euclidean plane is slowly assembled from translations, rotations, and orientation reversing isometries. Abstract group theory becomes concrete.

What does Lyndon cover? In a way, a lot of old-fashioned mathematics: group presentations, the isometry group of the Euclidean plane, the classification of the 17 Euclidean planar crystallographic groups, regular tesselations in two and higher dimensions, projective geometry and the projective group, inversive geometry and the group of Möbius transformations, the hyperbolic plane and its isometry group, and Fuchsian groups. Of course, I should be careful about the word "old-fashioned"; because of Thurston's reintroduction of hyperbolic geometry into low-dimensional topology, a lot of 19th century mathematics is coming back to haunt (or delight) 20th century mathematicians.

The treatment is also 19th, or even 18th, century. As the author says in his preface, "we have paid little attention to axioms. We feel that this is no real loss of rigor, since, where intuition is not found sufficient, the reader can always fall back on analytic geometry to verify elementary assertions." Well! I guess that takes care of the axiomatic approach. This does not mean, however, that Lyndon leaves out all the details. For example, when he introduces inversion in a circle, he describes it in terms of a metric, in terms of the complex plane, and in terms of a ruler and compass construction; he even includes a proof that an angle inscribed in a circle has measure one half of its subtended arc (again from the preface, "no technical knowledge of geometry is assumed"). Considering the present state of education in geometry, I suspect that Lyndon's cautiousness is necessary. In general, this is the gentlest introduction to this material that I have seen.

I do have a few complaints. It would have been nice to see a little about Kleinian groups and 3-dimensional hyperbolic geometry, especially in the light of Thurston's work. There is only the briefest glimpse of Riemann surfaces at the very end of the book. And the illustrations are meagre for a field so full of visual beauty. Magnus claimed that his lovely book, *Noneuclidean Tesselations and Their Groups*, was inspired just by the pictures in Klein and Fricke. At least, Lyndon could have included some interesting Arabic or Chinese patterns to illustrate the Euclidean crystallographic groups. Finally, references are few and typographical errors are many. But I really don't care! The author's heart was in the right place when he selected topics for this book. He has managed to give a self-contained presentation of material that I only picked up in bits and pieces from many sources over many years — a little Coxeter and Moser here, a little Magnus there, some Macbeath, some Ford, some Guggenheimer. Lyndon makes important but obscure mathematics accessible, and that's enough to ask of any book.

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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S:	Supplementary Reading	13:	Grade Level	??:	Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatabilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, P, L.** *The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics.* Ed: Richard E. Ewing, Kenneth I. Gross, Clyde F. Martin. Springer-Verlag, 1986, xvi + 214 pp, $24. [ISBN: 0-387-96414-2] Proceedings of the August 1985 symposium honoring Gail Young on the occasion of his 70th birthday. A spectacular collection of thoughtful papers on all aspects of mathematics, united by the example of Gail Young's life and by the unity of mathematics itself. Authors include Tom Banchoff, Harley Flanders, Peter Hilton, Daniel Kleitman, Anil Nerode, Henry Pollak, and Steve Smale. LAS

General, S(13), L. *Mathematical Formulae for Engineering and Science Students, Fourth Edition.* S. Barnett, T.M. Cronin. Longman (US Distr: Wiley), 1986, 77 pp, $9.95 (P). [ISBN: 0-582-44758-5] This edition includes new sections on Z-transforms, matrix algebra, orthogonal polynomials, and Walsh functions. Additional material has been included on maxima and minima of differential functions and Fourier transforms. Also a new list of symbols, revisions in the section on numerical solutions to differential equations, and some frequently used statistics tables have been added. (*Second Edition*, TR, November 1977; *Third Edition*, TR, June-July 1980.) CEC

General, P, L*. *M.C. Escher: Art and Science.* Ed: H.S.M. Coxeter, et al. Elsevier Science, 1986, xiii + 402 pp, $50. [ISBN: 0-444-70011-0] Proceedings of an international interdisciplinary congress held at the University of Rome in March 1985. 37 papers deal with links between Escher's work and symmetry, visual perception, geometry, computer graphics, the physical world, art, and humanities. An impressive illustration of the deep design implicit in Escher's work. LAS

Precalculus, T. *Fundamentals of Algebra and Trigonometry, Sixth Edition.* Earl W. Swokowski. Prindle, Weber & Schmidt, 1986, xi + 638 pp. [ISBN: 0-87150-877-X] Identical to *Algebra and Trigonometry with Analytic Geometry, Sixth Edition* except that this volume does not include the last chapter on analytic geometry, and, in this account, the unit circle definition of the trigonometric functions precedes the right triangle approach. (*Second Edition*, TR, February 1972; *Third Edition*, TR, August-September 1975; *Fourth Edition*, TR, June-July 1978.) LCL

Precalculus, T. *Plane Trigonometry, Fourth Edition.* Bernard J. Rice, Jerry D. Strange. Prindle, Weber & Schmidt, 1986, ix + 358 pp. [ISBN: 0-87150-913-X] Changes include delayed introduction to radian measures, expanded discussion of vectors, increased reliance on the calculator, new applications. (*First Edition*, TR, August-September 1975; *Second Edition*, TR, October 1978.) LCL

Education, P*, L*. *Second International Mathematics Study: Detailed Report for the United States.* Ed: Kenneth J. Travers. Stipes Pub (10-12 Chester St., Champaign, IL 61820), 1986, xix + 445 pp, $21 (P). Thorough analysis of the United States data from the Second International Mathematics Assessment, supplementing the 1985 *Preliminary Report* and the recent public summary *The Underachieving Curriculum* (TR, March 1987). Curriculum, teaching practice, student achievement, and attitudes for

eighth and twelfth grades. LAS

Education, P. *Mathematics Library: Elementary and Junior High School, Fifth Edition.* Margariete Montague Wheeler. NCTM, 1986, iv + 35 pp, $6.75 (P). [ISBN: 0-87353-228-7] An annotated bibliography of about 300 books suitable for school mathematics libraries to enrich mathematics from pre-school to grade 9. Books selected for children, not so much for their teachers. (*Fourth Edition*, TR, December 1978.) LAS

Education, P. *Ideas from the Arithmetic Teacher, Grades 4-6 Intermediate School, Second Edition.* Francis Fennell, David E. Williams. NCTM, 1986, iii + 140 pp, $6.50 (P). [ISBN: 0-87353-230-9] Selections from the "Ideas" department of *Arithmetic Teacher* from 1980 to 1985 in the areas of numeration, whole number computation, rational numbers, geometry, measurement and problem solving. JNC

History, L.** *International Mathematical Congresses: An Illustrated History, 1893-1986.* Donald J. Albers, G.L. Alexanderson, Constance Reid. Springer-Verlag, 1987, 63 pp, $29.95. [ISBN: 0-387-96479-7] A picture book of twentieth century mathematicians, from Poincaré, Klein, and Hilbert in the late nineteenth century to the Fields medalists from the 1986 International Congress in Berkeley. Brief descriptions of each ICM; photos of Fields Medalists; lists of plenary addresses. Revised from the edition sold at Berkeley to include the 1986 Congress itself. LAS

History, S*, P*, L.** *The Calculating Passion of Ada Byron.* Joan Baum. Archon Books, 1986, xix + 133 pp, $21.50. [ISBN: 0-208-02119-1] A warm, informative biography of "the world's first computer 'programmer'", Lord Byron's daughter Ada, Countess of Lovelace, and interpreter of Charles Babbage's "Analytical Engine." In 1843 Ada Byron, seeing in Babbage's plans the potential to "weave algebraic patterns just as the Jacquard loom weaves leaves and flowers," prepared notes on how to use the analytical engine (which of course had not been built) to compute Bernoulli numbers. In so doing, she outlined a century before Turing a program with internal control, loops, memory, and conditional choices. LAS

History, P, L.** *Norbert Wiener: Collected Works with Commentaries, V. IV.* Ed: P. Masani. Mathematicians of Our Time. MIT Pr, 1985, xx + 1083 pp, $70. [ISBN: 0-262-23123-9] Final volume of Wiener's *opus*, containing papers on cybernetics and an extraordinary variety of miscellany—science and society, aesthetics, literary criticism (e.g., of Rudyard Kipling), education, encyclopedia articles (e.g., on "alphabet" and "soul"), and book reviews. Contemporary interpretations by 25 commentators from a dozen different fields locate Wiener's incredible portfolio in the scheme of twentieth century scholarship. (*Volume I*, TR, May 1979; *Volume II*, TR, April 1980; *Volume III*, TR, December 1982.) LAS

Logic, T(18), S, P. *Introduction to Higher Order Categorical Logic.* J. Lambek, P.J. Scott. Stud. in Adv. Math., V. 7. Cambridge U Pr, 1986, ix + 293 pp, $49.50. [ISBN: 0-521-24665-2] "This book makes an effort to reconcile two different attempts to come to grips with the foundations of mathematics. One is mathematical logic, which traditionally consists of proof theory, model theory, and the theory of recursive functions; the other is category theory." The authors point to several close relationships and show that many ideas from these two approaches are essentially the same. LCL

Logic, P. *Modal Logic and Classical Logic.* Johan van Benthem. Indices: Mono. in Philo. Logic & Formal Ling., V. 3. Humanities Pr, 1985, 234 pp, $50. [ISBN: 88-7088-113-X] A survey of the model theory, algebra, and proof theory of modal logic, followed by a study of the interplay betwen the modal (intensional), and the classical (extensional) perspective. LCL

Logic, S(10-15). *Pensari.* Robert Katz. Kepler Pr (84 Main St., Rockport, MA 01966), 1986, v + 42 pp, $14.95 (P). [ISBN: 0-912938-10-2] A card game intended to teach scientific induction and logical deduction. Players (alone or in groups) make conjectures and test hypotheses in search of truth in one of fourteen "worlds" defined by various combinations of the game cards. A 42 page *Guidebook* explains rules of the game in the context of scientific exploration and logical verification. LAS

Foundations, P. *Foundations of Logic and Linguistics: Problems and Their Solutions.* Ed: Georg Dorn, P. Weingartner. Plenum Pr, 1985, xi + 715 pp, $95. [ISBN: 0-306-41916-5] Selected papers contributed to the International Congress of Logic, Methodology and Philosophy of Science, 1983, dealing with foundations of logic (e.g., completeness, decidability, reduction, probabilistic semantics), logic and language (e.g., categorial grammars and sense logic, quantification and relevant entailment, vagueness of concepts and information), and philosophy of language. LCL

Foundations, T(15-16: 1), S, L. *An Outline of Set Theory.* James M. Henle. Prob. Books in Math. Springer-Verlag, 1986, viii + 145 pp, $24 (P). [ISBN: 0-387-96368-5] Designed for use in a one-semester problem-oriented course in undergraduate set theory: Zermelo-Fraenkel axioms, axiom of choice, large cardinals, and nonstandard analysis. The necessary definitions and theorems are supplied, but the proofs are assigned as student projects. The second part of the book contains suggestions and comments regarding the projects, and complete solutions are provided

in the third part. Especially appropriate as an undergraduate seminar. LCL

Foundations, S(15-17), L. *The Philosophy of Mathematics: An Introductory Essay.* Stephan Körner. Dover, 1986, 198 pp, $5.95 (P). [ISBN: 0-486-25048-2] Unaltered paperback republication of the 1968 *Second Edition* of the original 1960 Hutchinson & Co. publication. A solid survey of classical approaches to the philosophy of mathematics leading to the author's own interpretations of a foundation for pure and applied mathematics. LAS

Foundations, P. *Lecture Notes in Mathematics-1182: Around Classification Theory of Models.* Saharon Shelah. Springer-Verlag, 1986, vii + 279 pp, $21.30 (P). [ISBN: 0-38-16448-0] A sequence of papers generally concerned with extending classification theory in various directions. LCL

Graph Theory, S(15-17), P, L. *The Four-Color Problem: Assaults and Conquest.* Thomas L. Saaty, Paul C. Kainen. Dover, 1986, vi + 217 pp, $6 (P). [ISBN: 0-486-65092-8] A corrected paperback republication of the 1977 original McGraw-Hill edition (TR, May 1980). Includes a new preface and a few additions to the bibliography. LAS

Combinatorics, T(16-18: 1, 2), S, P*, L. *Combinatorics: Theory and Applications.* V. Krishnamurthy. Math. & Its Applic. Halsted Pr, 1986, xxxv + 483 pp, $95. [ISBN: 0-470-20345-5] A valuable contribution to the combinatorics literature. The first book, at this level, to emphasize the various ramifications of Polyá's enumeration theory and to give a self-contained exposition of Schur functions and allied topics (e.g., applications to the character theory of the symmetric group). Pedagogically sound, with motivation and forerunner problems at the beginning of the section, and application and drill problems at the end. Note price! LCL

Combinatorics, P, L*. *Percy Alexander MacMahon: Collected Papers, Volume II: Number Theory, Invariants, and Applications.* Ed: George E. Andrews. Mathematicians of Our Time. MIT Pr, 1986, xxv + 952 pp, $95. [ISBN: 0-262-13214-1] Concluding eight chapters of the work of Major Percy MacMahon on determinants, number theory, and invariant theory, "unquestionably the greatest single untapped inheritance from our great-grandfathers" (G.-C. Rota). Richly supported by introductions, commentary, supporting papers, references and summaries. (TR, *Volume I*, April 1979.) LAS

Combinatorics, T*(15-17: 1), S*, P, L*. *Constructive Combinatorics.* Dennis Stanton, Dennis White. Undergrad. Texts in Math. Springer-Verlag, 1986, x + 183 pp, $19.80. [ISBN: 0-387-96347-2] A unique and innovative textbook featuring algorithmic explanations for combinatorial phenomena (e.g., bijective proofs, involutions). The text builds on a familiarity with basic counting principles (e.g., generating functions, inclusion/exclusion) to give an introduction to an increasingly important area of mathematics. The exercises and projects, many involving the computer, encourage active investigation and mathematical discovery. Perfect for an advanced seminar! Also, can't beat the price. LCL

Discrete Mathematics, T(13-15: 1, 2), S*, L. *Introduction to Difference Equations.* Samuel Goldberg. Dover, 1986, xii + 260 pp, $6.95 (P). [ISBN: 0-486-65084-7] Corrected republication of the 1958 John Wiley edition, itself an enlarged version of material prepared in 1954 for the Social Science Research Council. A classic treatment of mathematical ideas whose importance has steadily increased since the original publication. LAS

Number Theory, T(18: 1, 2), P. *Local Fields.* J.W.S. Cassels. Math. Soc. Stud. Texts, V. 3. Cambridge U Pr, 1986, xiv + 360 pp, $49.50; $16.95 (P). [ISBN: 0-521-30484-9; 0-521-31525-5] A readable, self-contained introduction to p-adic methods, with many applications and exercises. (The p-adics are entirely natural counterparts to the reals and complex numbers.) Cassels' incisive preface and chapter-ending notes are particularly worth reading. BC

Number Theory, T(18), P. *Local Class Field Theory.* Kenkichi Iwasawa. Math. Mono. Oxford U Pr, 1986, viii + 155 pp, $39.95. [ISBN: 0-19-504030-9] The author develops local class field theory (l.c.f.t.) via the Lubin-Tate theory of formal groups. Assumes some knowledge of algebra and topological groups, but gives a thorough presentation to local fields and formal groups. The main results of l.c.f.t. are then proved and applied to prove the explicit reciprocity laws of Wiles. SG

Number Theory, P. *Lecture Notes in Mathematics-1205: Analytic Arithmetic in Algebraic Number Fields.* B.Z. Moroz. Springer-Verlag, 1986, vii + 177 pp, $15.70 (P). [ISBN: 0-387-16784-6] Detailed treatment of analytic methods, largely classical, in algebraic number theory, focused on analytic continuation of Euler products and their natural boundaries. BC

Number Theory, P. *Intégration et théorie des nombres.* Jean-Loup Mauclaire. Hermann, 1986, 152 pp, 160F (P). [ISBN: 2-7056-6035-6] A research treatise in which results in harmonic analysis and probability theory are derived and applied to number theory, especially to the theory of additive and multiplicative functions. SG

Number Theory, P. *Über die Klassenzahl abelscher Zahlkörper.* Helmut Hasse. Springer-Verlag, 1985, xii + 190 pp, DM 64. [ISBN: 0-387-13628-2] A reprinting (with minor corrections) of Hasse's important 1952 monograph. Includes a brief introduction

by Martinet in which some recent developments are discussed. SG

Group Theory, T(17-18: 1), S, P. *Local Representation Theory.* J.L. Alperin. Stud. in Adv. Math. II. Cambridge U Pr, 1986, x + 178 pp, $29.95. [ISBN: 0-521-30660-4] Nice quick introduction to the modular representation theory of finite groups G over algebraically closed fields k of characteristic $p > 0$, using minimum of machinery to gain depth quickly. Treats Green correspondence between indecomposable kG modules and modules over p-local subgroups (normalizers of p-subgroups), Brauer correspondence, defects, blocks, trees. RM

Algebra, T*(17: 1), S*, P*, L*. *Finite Fields.* Rudolf Lidl, Harald Niederreiter. Ency. of Math. & Its Appl., V. 20. Addison-Wesley, 1983, xx + 755 pp, $69.50. [ISBN: 0-521-30240-4] This book, the first in this series devoted to finite fields, presents both the classical and the applications-oriented aspect of the subject. A comprehensive and encyclopaedic look at finite fields which does not consider algebraic geometry or the theory of algebraic function fields. Requires linear algebra and a bit of mathematical maturity. Outstanding lists of exercises and references. CEC

Algebra, P. *Thirteen Papers in Algebra.* I.G. Dmitriev, et al. Transl: Ben Silver. AMS Transl. Ser. 2, V. 132. AMS, 1986, vii + 104 pp, $40. [ISBN: 0-8218-3107-0] Thirteen papers which have appeared in Russian journals since 1980 are presented in translation. Actually, four of the papers are in graph theory, two are in algebraic geometry, one is in the metric theory of continued fractions, and the remaining papers are algebraic. CEC

Algebra, P. *Lecture Notes in Mathematics-1197: Ring Theory.* Ed: F.M.J. van Oystaeyen. Springer-Verlag, 1986, 231 pp, $19.10 (P). [ISBN: 0-387-16496-0] Proceedings of an international conference held in Antwerp, 1985. Papers from France and Belgium algebraists on a wide variety of topics. MR

Algebra, P. *Lecture Notes in Mathematics-1187: Categories of Boolean Sheaves of Simple Algebras.* Yves Diers. Springer-Verlag, 1986, vi + 168 pp, $14.30 (P). [ISBN: 0-387-16459-6] This book studies and classifies categories of commutative regular algebra equipped with structure (such as order, lattice order, differential structure, continuous group representation, with integral, algebraic, or separable elements, etc.). It uses the axiomatic method and this leads to a unified treatment which highlights their specific features and permits a classification based on universal properties. LCL

Algebra, P. *Clones in Universal Algebra.* Ágnes Szendrei. Pr U Montreal, 1986, 166 pp, $19 (P). [ISBN: 2-7606-0770-4] A self-contained set of lectures (modulo the rudiments of universal algebra and lattice theory) whose aim is to present several techniques in clone theory and to introduce some results showing how clones contribute to the understanding of the structure of algebras. LCL

Algebra, T(17-18: 1), S, P. *Lectures on Rings and Modules, Third Edition.* Joachim Lambek. Chelsea, 1986, viii + 183 pp, $14.95. [ISBN: 0-8284-2283-4] Apart from one rewritten proof and correction of misprints, the book is unchanged from the *Second Edition* (*First Edition*, TR, April 1969; Extended Review, November 1969; *Second Edition*, TR, June-July 1977). JS

Algebra, P. *Mathematical Studies of Lie-Admissible Algebras.* Reprint Ser. in Math., V. 10-13. Ed: Hyo Chul Myung. Hadronic Pr, $60 each (P). *Volume I,* 1985, viii + 248 pp [ISBN: 0-911767-13-4]; *Volume II,* 1985, vii + 321 pp [ISBN: 0-911767-14-2]; *Volume III,* 1986, vii + 328 pp [ISBN: 0-911767-14-2]; *Volume IV,* 1986, viii + 350 pp. [ISBN: 0-911767-16-9] Reprints, arranged chronologically, of all mathematical papers (physics papers are excluded) in Lie-admissible algebras published, sometimes in rather inaccessible journals, from 1948 to 1983. LCL

Calculus, T(15-17), S, P*, L.** *Aspects of Calculus.* Gabriel Klambauer. Undergrad. Texts in Math. Springer-Verlag, 1986, x + 515 pp, $38. [ISBN: 0-387-96274-3] A valuable resource of stimulating examples and exercises from calculus (limits, continuity, differentiation, integration, series), complete with detailed discussion of theoretical underpinnings. An instructive introduction to elegant tricks of the craft. Recommended to all teachers of undergraduate mathematics and to students in transition to rigorous courses in analysis. LCL

Calculus, C*(12-14). *MicroCalc (Version 2.0).* IBM PC. Harley Flanders. MathCalcEduc, 1986 (1449 Covington Dr., Ann Arbor, MI 48103; (313)761-4666), $250.00. Although written to accompany the author's calculus text, this software is entirely self-contained and easy to use. It consists of 26 interactive programs for the IBM-PC family and their clones. The programs do symbolic differentiation and approximate integration and graphing in two- and three-dimensions along with much more. This version includes routines which do implicit differentiation, space curves, and gradient fields. Improved menu makes this package even more friendly (*Version 1.0,* TR, April 1986); upgrade price: $50. The programs handle any function from calculus with ease. The graphics are impressive visually but are extremely slow in some instances. Output may be sent to a printer. Disk includes a *Manual* which may be printed or viewed on the screen. Version using 8087 coprocessor also available. Institutional discounts for class use. CEC

Real Analysis, T(13-14: 1). *An Introduction to Mathematical Analysis.* John B. Reade. Clarendon Pr, 1986, vii + 169 pp, $35; $18.95 (P). [ISBN: 0-19-853258-X; 0-19-853257-1] Elementary introduction to the theory of calculus, covering sequences and series, continuity, differentiation and integration of functions of a single variable at advanced (honors) freshman level. LC

Real Analysis, S(18), P. *Measures on Infinite Dimensional Spaces.* Y. Yamasaki. Ser. in Pure Math., V. 5. World Science, 1985, viii + 256 pp, $28. [ISBN: 9971-978-52-0] Lecture notes covering two basic topics: construction of an infinite dimensional measure as the limit of finite dimensional ones on direct product of σ-compact metric spaces, and invariance of measures on infinite dimensional spaces. Almost no references. PH

Complex Analysis, T*(15: 1). *Complex Variables: Harmonic and Analytic Functions.* Francis J. Flanigan. Dover, 1983, x + 353 pp, $7.50 (P). [ISBN: 0-486-61388-7] Corrected republication of the 1972 Allyn and Bacon edition (TR, October 1972). BH

Complex Analysis, S(18), P. *Entire Functions of Several Complex Variables.* Pierre Lelong, Lawrence Gruman. Grund. der math. Wissenschaften, B. 282. Springer-Verlag, 1986, xi + 270 pp, $64.50. [ISBN: 0-387-15296-2] Entire holomorphic functions of several variables with prescribed growth arise (e.g., as Fourier transforms) in many other areas of mathematics. This monograph treats basic theory (growth scales, currents, plurisubharmonic functions, etc.) and some applications, including analytic number theory. Historical notes, extensive bibliography, and three short appendices are included. PZ

Complex Analysis, T(17: 2). *Complex Analysis in One Variable.* Raghavan Narasimhan. Birkhauser Boston, 1985, xvi + 266 pp, $29.95. [ISBN: 0-8176-3237-9] An introductory graduate level text which also includes chapters on several complex variables, compact Riemann surfaces, the Corona theorem, and subharmonic functions. Presented in definition-theorem-proof format with extensive references, and informative notes at the end of each chapter. No problems. BH

Differential Equations, P. *Oscillation Theory, Computation, and Methods of Compensated Compactness.* Ed: Constantine Dafermos, et al. Inst. for Math. & Its Applic., V. 2. Springer-Verlag, 1986, ix + 395 pp, $35. [ISBN: 0-387-96401-0] Proceedings of a workshop held as part of the 1984-85 IMA program on continuum physics and partial differential equations at the University of Minnesota. The papers survey recent work in both the numerical and analytical aspects of nonlinear systems of conservation laws. AO

Differential Equations, T*(15: 1), L. *Differential Equations: An Introduction With Applications.* Lothar Collatz. Transl: E.R. Dawson. Wiley, 1986, xv + 372 pp, $41.95. [ISBN: 0-471-90955-6] Mainly a translation of the 1980 German edition. The original German edition was based on lecture notes from 1945-46. Additional material on the Laplace and Fourier transforms by the translator and a brief section on non-linear partial differential equations by Collatz. The presentation is refreshingly different from most, if not all, currently available textbooks in this country. There are fewer worked-out examples and fewer exercises and problems. Discussions are brief but clear. Applications are not the usual superficial variety but are at a higher level of sophistication. Only 372 pages, including the index, but no important topics are omitted and more than lip service is paid to theory. For example, singular points and solutions are covered and appropriate elements of functional analysis are included. A fine book. Unfortunately, most of us will deem it not suitable as a textbook for our courses. JK

Differential Equations, S(14-17), C. *Differential and Difference Equations through Computer Experiments; PHASER: An Animator/Simulator for Dynamical Systems.* IBM PC. Hüseyin Koçak. Springer-Verlag, 1986, xv + 224 pp, $44 (P). [ISBN: 0-387-96239-5] A handbook for the program PHASER, preceded by a brief introduction to differential and difference equations (with references to major texts for a more complete treatment). The final half of the book consists of a reference guide to the equations and parameters stored on the program disk. Includes thorough discussion of graphical experimentation with differential and difference equations. Equations and parameters for them are stored in text files which may be added to, changed, or deleted by the user. These equations may be explored graphically by simultaneously displaying direction fields and solution curves. The user has a wide selection of options including the choice of numerical methods (Euler or Runge-Kutta). This otherwise sound package is seriously weakened by design decisions: low resolution color graphics; screen displays that are cutsey in color and hard to read in monochrome; large letter text occupying valuable screen space. It also seems too easy to choose a *very* slow option (say a fine grid direction field) and to get stuck with no non-violent way out except to wait and wait... . Results are presented without discussion of the potential foibles of machine implementation of algorithms. It would be nice to have at least an appendix on the implementations used so as to introduce the interested reader to a non-trivial area of new mathematics. JAS

Partial Differential Equations, T(16-17). *Ele-*

mentary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Second Edition. Richard Haberman. Prentice-Hall, 1987, xii + 547 pp. [ISBN: 0-13-252875-4] Excellent introduction to partial differential equations and boundary value problems. Would be an appropriate text to follow an undergraduate course in ordinary differential equations taught at the level of Boyce and DiPrima. Discusses the heat equation, wave equation, Laplace's equation, Sturm-Liouville theory, Green's functions, and the method of characteristics. Also contains detailed worked examples and exercises with solutions to selected problems. (*First Edition*, TR, August-September 1983; Extended Review, June-July 1985.) AM

Partial Differential Equations, P. *Free Boundary Problems: Applications and Theory.* A. Bossavit, A. Damlamian, M. Fremond. Pitman, 1985. *V. III,* Res. Notes in Math., V. 120, 303 pp, $44.95 (P) [ISBN: 0-273-08677-4]; *V. IV,* Res. Notes in Math., V. 121, 613 pp, $44.95 (P). [ISBN: 0-273-08678-2] Free boundaries refer to moving frontiers between spatial zones with different physical characteristics. The books contain the proceedings of the International Colloquium: "Problèmes à Frontières Libres: Applications et Théorie," held in Maubuisson, France from June 7-16, 1984. They contain over 60 invited papers concerning free boundary concepts and methods. Covers solid mechanics, fluid mechanics, reaction diffusion, control theory, and math-numerical methods. AM

Numerical Analysis, T*(14-15: 1-2), L. *Numerical Methods for Computer Science, Engineering, and Mathematics.* John H. Mathews. Prentice-Hall, 1987, xiii + 507 pp. [ISBN: 0-13-626656-8] Standard topics: iteration, root finding, linear systems of equations, interpolation and approximation, least-squares curve fitting, differentiation, minimization of a function, integration, nonlinear systems of equations, solution of ordinary differential equations. Very readable. Assumes computer programming ability; uses pseudo-code for the algorithms. Nice mix of computation and theory. Many exercises, most of a computational nature. DFA

Functional Analysis, S(18), P. *Fredholm Theory in Banach Spaces.* Anthony F. Ruston. Tracts in Math., V. 86. Cambridge U Pr, 1986, x + 293 pp, $59.50. [ISBN: 0-521-24846-9] Using the theme of Fredholm theory adapted to Banach spaces and making use of results of Riesz, the author develops a theory of operators of finite rank, then studies special classes of operators. Treatment is aimed toward graduate students. Includes notes and comments together with a 34-page bibliography and an index. JS

Functional Analysis, P. *Differential Analysis in Infinite Dimensional Spaces.* Ed: Kondagunta Sundaresan, Srinivasa Swaminathan. Contemp. Math., V. 54. AMS, 1986, ix + 122 pp, $18 (P). [ISBN: 0-8218-5059-8] A collection of papers submitted by the participants in the special session on differential analysis on infinite dimensional spaces held at the summer meeting of the American Mathematical Society at SUNY, Albany, New York, August 8-11, 1983. AM

Functional Analysis, T(18: 1), S, P. *Completely Bounded Maps and Dilations.* Vern I. Paulsen. Res. Notes in Math. Ser., V. 146. Longman Scientific & Technical (US Distr: Wiley), 1986, 187 pp, $38.95 (P). [ISBN: 0-470-20369-2] Assumes a familiarity with operator theory in a Banach space and basic properties of C^* algebras, the intent is to present a unified approach to completely positive and completely bounded operators with applications. Some classical theory along with more recent results of Stinespring, Arveson, and others. Exercises, bibliography, index. JS

Analysis, P. *Lectures on Topics in One-Parameter Bifurcation Problems.* P. Rabier. Springer-Verlag, 1985, vi + 286 pp, $9.20 (P). [ISBN: 0-387-13907-9] Let X be a real vector space, $L : X \to R$ linear. A simple example of a bifurcation problem is to determine the nature of the zero set of $H(\lambda, x) = x - \lambda Lx$ near $(\lambda_o, 0)$, where λ_o is a characteristic value of L. Text examines various methods, including the Lyapunov-Schmidt method and other "newer" approaches, for attacking more general versions of this problem. BH

Analysis, S(18), P. *Nonstandard Methods in Stochastic Analysis and Mathematical Physics.* Sergio Albeverio, et al. Pure & Appl. Math., V. 122. Academic Pr, 1986, xi + 514 pp, $34.50 (P); $59.50. [ISBN: 0-12-048861-2; 0-12-048860-4] A book on applied nonstandard analysis. The first part gives a complete and self-contained introduction to nonstandard methods in calculus, topology, probability, with some examples from differential equations, linear space theory, and Brownian motion. The second part presents selected applications to stochastic analysis and mathematical physics. LCL

Analysis, P. *Potential Theory: An Analytic and Probabilistic Approach to Balayage.* J. Bliedtner, W. Hansen. Universitext. Springer-Verlag, 1986, xiii + 435 pp, $40 (P). [ISBN: 0-387-16396-4] Describes the connection between analytic and probabilistic potential theory using the concept of a balayage space as a link and then uses this concept to develop balayage theory which is then applied to the Dirichlet problem. BH

Analysis, P. *Beijing Lectures in Harmonic Analysis.* Ed: E.M. Stein. Annals of Math. Stud., No. 112. Princeton U Pr, 1986, vii + 424 pp, $22.50 (P); $65. [ISBN: 0-691-08419-X; 0-691-08418-

1] Seven long expository papers from the September 1984 summer symposium on analysis in China, treating current topics in Fourier analysis, pseudo-differential and singular integral operators, partial differential equations, real analysis, and several complex variables. Basic material as well as recent results are covered. PZ

Analysis, P. *Lecture Notes in Mathematics-1199: Analytic Theory of Continued Fractions II.* Ed: W.J. Thron. Springer-Verlag, 1986, 299 pp, $26.40 (P). [ISBN: 0-387-16768-4] Proceedings of a seminar-workshop held in Pitlochry and Aviemore, Scotland, June 13-29, 1985. Primarily concerned with convergence theory of continued fractions and continued fraction methods in the solution of strong moment problems. BH

Analysis, S(18), P. *Real and Stochastic Analysis.* Ed: M.M. Rao. Wiley, 1986, xi + 347 pp, $39.95. [ISBN: 0-471-82969-2] Five papers on the connections between stochastic processes and functional analysis: bimeasures and nonstationary processes, Besicovitch-Orlicz spaces of almost periodic functions, diffusion processes in Hilbert space and likelihood ratios, chains in Banach spaces, two-parameter stochastic differential equations. Assumes basic knowledge of real analysis and a graduate course in probability. BH

Algebraic Geometry, P. *Compactification of Siegel Moduli Schemes.* Ching-Li Chai. London Math. Soc. Lect. Note Ser., V. 107. Cambridge U Pr, 1985, xvi + 326 pp, $29.95 (P). [ISBN: 0-521-31253-1] The main body of the author's thesis (David Mumford, Harvard, advisor). The main result is a proof of the existence of the toroidal compactification over $Z[1/2]$. LCL

Algebraic Geometry, P. *Le problème des modules pour les branches planes.* Oscar Zariski. Hermann, 1986, 212 pp, 180F (P). [ISBN: 2-7056-6036-4] Notes of lectures delivered by Zariski in 1973 in France, presenting the known results on the problem of determining the moduli space of branches of a given equisingularity class for an algebraic curve. SG

Differential Geometry, P. *Lecture Notes in Mathematics-1209: Differential Geometry, Peñíscola 1985.* Ed: A.M. Naveira, A. Ferrández, F. Mascaró. Springer-Verlag, 1986, viii + 306 pp, $27.50 (P). [ISBN: 0-387-16801-X] Proceedings of the second international symposium on differential geometry held in June 1985 at Peñíscola, Spain. A sequel to *LNM-1045,* the proceedings of the first such symposium from October 1982. LAS

Differential Geometry, T*(16-17: 1, 2). *An Introduction to Differential Manifolds and Riemannian Geometry, Second Edition.* William M. Boothby. Pure & Appl. Math., V. 120. Academic Pr, 1986, xvi + 430 pp, $32.95 (P); $59. [ISBN: 0-12-116053-X; 0-12-116052-1] *Second Edition* incorporates new exercises, enlarged historical notes, updated references, and improvements to proofs from the *First Edition* (TR, January 1976). AM

Differential Topology, P. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Paul H. Rabinowitz. CBMS Reg. Conf. Ser. in Math., No. 65. AMS, 1986, vii + 100 pp, $16 (P). [ISBN: 0-8218-0715-3] Minimax methods are methods that characterize a critical value of a functional as a minimax over a suitable class of sets. The book applies minimax methods to prove the existence of critical points of real valued functionals; in differential equations, these critical points correspond to weak solutions of the equation. AM

Differential Topology, T(16-18: 1). *An Introduction to Global Analysis.* Victor P. Snaith. Papers in Pure & Appl. Math., No. 72. Queen's U, 1986, 129 pp, (P). The index problem explores the relationship betweeen the topology of a differentiable manifold and the index of an elliptic differential equation posed on that manifold. The book presents an introduction to the Index Theorem and several of its corollaries. It includes background material in Hilbert space, elliptic differential operators, vector bundles, and Riemannian manifolds. AM

Differential Topology, S*(16-17).** *Catastrophe Theory, Second, Revised and Expanded Edition.* V.I. Arnold. Transl: G.S. Waserman, R.K. Thomas. Springer-Verlag, 1986, xiii + 108 pp, $13 (P). [ISBN: 0-387-16199-6] Revised and expanded edition (*First Edition*, TR, August-September 1984). New edition includes a chapter on Riemann surfaces, vanishing cycles, and monodromy as well as a description of several recent results (results on normal forms for implicit differential equations and the theory of relaxation oscillations, results on boundary singularities, and new results on caustics). AM

Topology, T(17: 3), S. *Complements d'Analyse, Volume 1: Topologie, Premiere Partie.* F. Guenard, G. Lelievre. l'ENS (31, Avenue Lombart, 92260 Fontenay Aux Roses, France), 1985, 264 pp, 75,00 F (P). An attractive introduction to topology that includes basic properties of topological and metric spaces which are frequently used in analysis. Numerous examples and exercises. An appendix including brief exposition of axiomatic set theories and models. PH

Topology, S(17), P. *Topological Methods in Bifurcation Theory.* Kazimierz Gęba, Paul H. Rabinowitz. Pr U Montreal, 1985, 112 pp, $16 (P). [ISBN: 2-7606-0695-3] Notes from a series of lectures by the authors at the Mathematical Seminar at the University of Montreal, June 27-July 15, 1983. Concerns problems in bifurcation in which the cohomotopy groups provide most natural tools (by the first author), and

the global structure of the zero-set of a continuous map from one real Banach space into another (by the second author). PH

Topology, T(17: 2), S, P*, L. *Modern General Topology, Second Revised Edition.* Jun-iti Nagata. Math. Lib., V. 33. Elsevier Science, 1985, x + 522 pp, $92.50. [ISBN: 0-444-87655-3] The classical editions of 1968 and 1974 have been fundamentally revised to include recent developments in general topology. The introductory part, Chapters I-III comes from the old edition. There are many new topics included in Chapters IV-VII, among them: Wallman-Shanin compactification, real compact spaces, generalized metric spaces, Dugundji type extension theory. New Chapter VIII was added to cover linearly ordered spaces, cardinal functions, and dyadic spaces. (*First Edition*, TR, April 1971; Extended Review, May 1971.) PH

Optimization, P. *Lecture Notes in Mathematics-1190: Optimization and Related Fields.* Ed: R. Conti, E. De Giorgi, F. Giannessi. Springer-Verlag, 1986, viii + 419 pp, $32.50 (P). [ISBN: 0-387-16476-6] Seventeen contributions to a conference in Sicily, 1984. Topics include optimal control, calculus of variations, and convex analysis. Some papers in rather awkward English. BC

Optimization, P. *Convexity and Optimization in Banach Spaces, Second Revised and Extended Edition.* V. Barbu, Th. Precupanu. Math. & Its Applic. D Reidel, 1986, xvii + 397 pp, $64. [ISBN: 90-277-1761-3] First published in Romania (1975) and translated into English in 1976, this revised and expanded edition further develops the idea of extending the methods of convex analysis into infinite dimensional optimization, especially as they may be applied to optimal control problems that start in finite dimensional space. (*First Edition,* TR, August-September 1979.) AWR

Optimization, P. *Lecture Notes in Control and Information Sciences-76: Stochastic Programming.* Ed: F. Archetti, G. Di Pillo, M. Lucertini. Springer-Verlag, 1986, v + 285 pp, $22 (P). [ISBN: 0-387-16044-2] Contributions to a 1983 conference in Gargano, Italy on stochastic modelling, simulation, and stochastic optimization. Papers on queueing networks, random differential equations, stochastic linear and integer programming, and methodology. RM

Optimization, T(15-17: 1), L. *Mathematical Programming: An Introduction to Optimization.* Melvyn W. Jeter. Pure & Appl. Math., V. 102. Dekker, 1986, ix + 342 pp, $34.50. [ISBN: 0-8247-7478-7] An introductory textbook for students with a background that includes courses in linear algebra and multivariable calculus. Covers both linear and nonlinear optimization techniques. Interesting because level is between that of most introductory texts (with too little mathematics) and that of a typical graduate-level text. AO

Dynamical Systems, P. *Lecture Notes in Control and Information Sciences-78: Stochastic Differential Systems.* Ed: N. Christopeit, K. Helmes, M. Kohlmann. Springer-Verlag, 1986, v + 372 pp, $28.20 (P). [ISBN: 0-387-16228-3] Proceedings of the third Bad Honnef conference held June 3-7, 1985 on optimal control, filtering and stochastic analysis. Special emphases of the conference were applications and connections between probability theory and quantum physics. AO

Dynamical Systems, P. *Systèmes Dynamiques non Linéaires: Intégrabilité et Comportement Qualitatif.* Pavel Winternitz. Pr U Montreal, 1986, 339 pp, $30 (P). [ISBN: 2-7606-0777-1] Papers given at a seminar held at the University of Montreal, June 29-August 16, 1985. JD-B

Dynamical Systems, T(17-18: 1), S, P. *The Stability and Control of Discrete Processes.* J.P. LaSalle. Appl. Math. Sci., V. 62. Springer-Verlag, 1986, 150 pp, $22 (P). [ISBN: 0-387-96411-8] Published posthumously with assistance of Kenneth Meyer, the subject is stability and control for a discrete dynamical system. Approach is by way of Liapunov's direct method, and development is accessible to advanced undergraduates who are proficient in analysis and linear algebra. Exercises, references, index. JS

Probability, T(15-16: 1), S, L. *Introduction to Probability.* John B. Thomas. Springer-Verlag, 1986, x + 247 pp, $24 (P). [ISBN: 0-387-96319-7] An introduction to the theory of probability including a chapter on sequences of random variables and the various sorts of convergence. FLW

Probability, S(17-18), P. *Random Measures.* Olav Kallenberg. Academic Pr, 1983, 187 pp, $22.50; $15 (P). [ISBN: 0-12-394960-2] A systematic account of those parts of modern random measure (point processes) theory which do not require any particular order or metric structure of the state space. Little overlap with other books in the field. Numerous exercises, historical remarks and an appendix containing basic facts from topology, measure theory, and probability. (1976 Akademie-Verlag edition, TR, March 1978.) PH

Probability, P. *Parameter Estimation for Stochastic Processes.* Yu. A. Kutoyants. Res. & Expos. in Math., V. 6. Transl: B.L.S. Prakasa Rao. Heldermann Verlag, 1984, viii + 206 pp, $28 (P). [ISBN: 3-88538-206-7] A translation of the 1980 work in Russian. Maximum likelihood and Bayes estimates are developed for parameters associated with Gaussian, diffusion and nonhomogeneous Poisson processes. Excellent bibliography. TAV

Probability, P. *Lecture Notes in Mathematics-1204: Séminaire de Probabilités XX 1984/85.* Ed: J. Azéma, M. Yor. Springer-Verlag, 1986, v + 639 pp, $51.60 (P). [ISBN: 0-387-16779-X] Twentieth in the series of Strasbourg seminars on probability theory. Includes complete tables of contents of all previous seminars. LAS

Probability, T(15-16: 1), S, L. *Probability and Random Variables.* G.P. Beaumont. Math. & Its Applic. Halsted Pr, 1986, 345 pp, $69.95. [ISBN: 0-470-20307-2] An introduction to probability up to the central limit theorem plus chapters on point estimation, sampling from finite populations, and machine generation of random variables. FLW

Probability, P. *Random Matrices and Their Applications.* Ed: Joel E. Cohen, et al. Contemp. Math., V. 50. AMS, 1986, xiv + 358 pp, $33 (P). [ISBN: 0-8218-5044-X] The proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held at Bowdoin College, June 17-23, 1984. The papers are equally divided between basic theory and application; the emphasis is on exposition and example rather than abstraction and generalization. Many open problems. LCL

Statistics, S(15-18), C*. *Statistical Analysis Package (Version 6.0).* IBM PC. David S. Walonick. Walonick Assoc., 1986 (6500 Nicollet Ave. S., Minneapolis, MN 55423; (800)328-4907), $495.00. A professional-level statistical package containing three disks: system, execution, and utility. Disk swapping needed unless a hard disk drive is present. Minimum 198K of memory and two disk drives required. Package is for IBM PC, XT, AT, compatibles and certain other MS DOS computers. Programs menu driven. User creates codework containing labels and other information about data. Up to 254 variables may be used. Codebook may be added to, edited, or printed out. Data may be entered into data file in fixed or free format. Data files may be edited and printed. When codebook has been created and data entered, various statistical analyses may be performed and printed (with graphs) including frequency distributions, descriptive statistics, cross-tabulation, multiple regression, analysis of variance, probit regression, factor analysis, and principle components analysis. Supported by a thorough 300 page manual. Intended as a simple alternative to SPSS. KK

Statistics, T(17-18: 1), S, P. *Experimental Design: Sample Size Determination and Block Designs.* Dieter Rasch, Günter Herrendörfer. D Reidel, 1985, vii + 200 pp, $39. [ISBN: 90-277-1684-6] Presents a model independent definition of experimental designs and details of block designs. Results used from the theories of Galois fields and finite geometries are presented in appendices. FLW

Statistics, T(16-18: 1), S, P. *Series of Irregular Observations: Forecasting and Model Building.* Robert Azencott, Didier Dacunha-Castelle. Appl. Prob. Springer-Verlag, 1986, vii + 236 pp, $32. [ISBN: 0-387-96263-8] Spectral analysis and ARMA (Autoregressive-Moving Average process) models for one-dimensional processes. FLW

Statistics, C*(14-17). *Statpal: A Statistical Package for Microcomputers.* IBM PC. Bruce J. Chalmer, David G. Whitmore. Dekker, 1985 (Statpal Associates, POB 991, Burlington, VT 05402). A package of programs to do standard statistical analysis of data stored in disk files. Provides utilities for creating and manipulating such files and for transformations on the data; provides enough information so that data files could be created easily from other databases. Standard statistical tools include analysis of variance, nonparametric tests, regression, graphical routines (histograms and scatterplots). Appears to be relatively generic MS-DOS, using a standard ANSI terminal driver. Although there is no discussion of the robustness of the algorithms, there is some discussion of accuracy and error trapping. Intended to be used in conjunction with Chalmer's *Understanding Statistics*, but can easily be used to supplement any elementary text. JAS

Computer Literacy, T(13: 1). *Computers & Information Processing.* Gerald A. Silver, Myrna L. Silver. Harper & Row, 1986, xxiv + 600 pp. [ISBN: 0-06-046159-4] A computer literacy textbook covering the standard topics. Easy to read with many photographs and drawings. Covers microcomputers extensively. AO

Computer Programming, T?(14: 1, 2). *Ada Programming with Applications.* Eugen N. Vasilescu. Allyn & Bacon, 1986, x + 539 pp, (P). [ISBN: 0-205-08744-2] Introductory Ada text (requiring programming experience) oriented towards computer information systems, business and commercial applications. Early and heavy emphasis on types and specifications, light treatment of generics, implementation-dependent features. Annoying simulated dot-matrix type for program examples; too few exercises. RM

Software Systems, S(15-17), P, L. *UNIX for Super-Users.* Eric Foxley. Intern. Comput. Sci. Ser. Addison-Wesley, 1985, xiv + 213 pp, $22.95 (P). [ISBN: 0-201-14228-7] A description of those aspects of UNIX (shell, devices, booting, security, logging-in) of special interest to system managers. The author is in the mathematics department of Nottingham University. LAS

Computer Science, T(18: 1), P. *Fairness.* Nissim Francez. Springer-Verlag, 1986, xiii + 295 pp, $45. [ISBN: 0-387-96235-2] Concurrency and nondeterminism have implications for the semantics of

programming languages. One problem is fairness (e.g., freedom from starvation, reachability, finite delay, eventual success on repeated retrial). Study of the effect of adding fairness assumptions to a language (a blend of Dijkstra's guarded commands and Hoare's language CSP for communicating sequential processes, using fairness of termination as a prototype problem. RM

Applications (Biology), P, L. *Some Mathematical Questions in Biology: DNA Sequence Analysis.* Ed: Robert M. Miura. Lect. on Math. in the Life Sci., V. 17. AMS, 1986, x + 124 pp, $28 (P). [ISBN: 0-8218-1167-3] Five papers from the 1984 AAAS symposium on mathematical biology dealing with probability distributions and sequence analysis in DNA, and the three-dimensional folding of RNA. LAS

Applications (Biology), S(16), P, L. *Mathematical Ecology, An Introduction.* Ed: Thomas G. Hallam, Simon A. Levin. Biomath., V. 17. Springer-Verlag, 1986, xii + 457 pp, $80. [ISBN: 0-387-13631-2] Sixteen papers, populated with an abundance of difference and differential equations, stemming from a course on ecology held in Trieste, Italy, 1982. Four basic areas are considered: physiological, population, community, and ecosystem ecology. A readable "idiosyncratic overview" introduces them and the role of mathematics. BC

Applications (Biology), P. *Some Mathematical Questions in Biology: Plant Biology.* Ed: Louis J. Gross, Robert M. Miura. Lect. on Math. in the Life Sci., V. 18. AMS, 1986, xi + 267 pp, $36 (P). [ISBN: 0-8218-1168-1] Six papers from the traditional AMS-sponsored symposium on mathematical biology at the AAAS annual meeting in May 1985: topics range from cell biology to plant ecology. LAS

Applications (Chemistry), S*(16-17). *Mathematical Concepts in Organic Chemistry.* Ivan Gutman, Oskar E. Polansky. Springer-Verlag, 1986, x + 212 pp, $71. [ISBN: 0-387-16235-6] Applications of topology, graph theory, and group representations to organic chemistry. Specifically, the book uses graph theory and group theory to provide a means for analyzing the π-electron systems of organic molecules (familiarity with HMO theory is assumed). Also discusses recent work aimed at correlating physical quantities (e.g., heat of vaporization or molar volumes) with topological indices associated to molecular structure. The book is written at an introductory level and is quite readable. AM

Applications (Economics), P. *Lecture Notes in Economics and Mathematical Systems-265: Dynamic Games and Applications in Economics.* Ed: T. Başar. Springer-Verlag, 1986, ix + 288 pp, $24.60 (P). [ISBN: 0-387-16435-9] Eleven articles (in-depth surveys, theoretical discussions, modeling applications) on different aspects of dynamic and differential game theory, presented at the conference on economic dynamics and control in London, June 26-28, 1985. LCL

Applications (Economics), S*, P*, L.** *Studies in Mathematical Economics.* Ed: Stanley Reiter. Stud. in Math., V. 25. MAA, 1986, xiii + 441 pp, $42. [ISBN: 0-88385-1278-x] A sampler of mathematical methods in modern economic theory by eight well-known mathematical economists, covering game theory, general equilibrium, welfare economics, decentralization and stochastic alternatives to optimal models. Mathematical methods range (in different papers) from sophomore mathematics to differential topology, combinatorial topology, and stochastic processes. An excellent primer for mathematics and economics graduate students. LAS

Applications (Engineering), T(16-17: 1). *Finite Element Methods in Mechanics.* Noboru Kikuchi. Cambridge U Pr, 1986, xiii + 418 pp, $39.50. [ISBN: 0-521-30262-5] A textbook for a first course in finite element methods for mechanical engineering students. Focuses on the solution of engineering problems rather than on mathematical theory. AO

Applications (Engineering), P. *Shell Theory: General Methods of Construction.* I.N. Vekua. Transl: Ts. Gabeskiria. Mono. Adv. Texts & Surveys in Pure & Appl. Math., V. 25. Pitman, 1986, xv + 287 pp, $120. [ISBN: 0-273-08692-8] Monograph representing the Tbilisi University in USSR, proponents of use of generalized functions to express stress field. Chapters 1 and 2 constitute introduction to subject and mathematical prerequisites. Chapters 3-5 treat special problems in shell equilibrium theory and new methods yielding elliptic systems. MR

Applications (Engineering), P. *Theoretical Approaches to Turbulence.* Ed: D.L. Dwoyer, M.Y. Hussaini, R.G. Voigt. Appl. Math. Sci., V. 58. Springer-Verlag, 1985, xii + 373 pp, $36 (P). [ISBN: 0-387-96191-7] The proceedings of a workshop sponsored by the Institute for Computer Applications in Science and Engineering and NASA Langley Research Center, Hampton, Virginia, October 10-12, 1984. Includes two expository articles, twelve papers dealing with specific aspects of theoretical approaches to fluid turbulence, and five position papers by panel members. PH

Applications (Engineering), T?(16-17), P. *System Reliability: Evaluation and Prediction in Engineering.* Alain Pagès, Michel Gondran. Transl: E. Griffin. Springer-Verlag, 1986, xvii + 351 pp, $66. [ISBN: 0-387-91276-2] Mathematical treatment of evaluation and prediction of system reliability, maintainability, etc. New critical operating states method (analysis of states with transitions to failures) gives methods for calculating reliability proba-

bilities for large repairable systems and aids in optimization of safety. Motivated by work in mechanical systems (e.g., travelling-bridge crane), large electrical networks, nuclear power plants. RM

Applications (Information Theory), P. *Analysis and Design of Stream Ciphers.* Rainer A. Rueppel. Commun. & Control Eng. Ser. Springer-Verlag, 1986, xii + 244 pp, \$47. [ISBN: 0-387-16870-2] Stream ciphers are secret-key ciphers with memory in the enciphering transformation. This book investigates several problems in the design of stream ciphers. AO

Applications (Information Theory), P. *Lecture Notes in Computer Science-218: Advances in Cryptology—CRYPTO '85.* Ed: Hugh C. Williams. Springer-Verlag, 1986, x + 548 pp, \$32.50 (P). [ISBN: 0-387-16463-4] Proceedings of the fifth annual conference devoted to cryptologic research. Six to ten papers in each of the following sections: signatures and authentication, protocols, copy protection, single-key cryptology, two-key cryptology, randomness and other problems, and final section on impromptu talks. LCL

Applications (Physics), T(1), S, P, L. *Initial Boundary Value Problems in Mathematical Physics.* Rolf Leis. Wiley, 1986, viii + 266 pp, \$44.95. [ISBN: 0-471-90863-0] Classical scattering theory and time-dependent theory of linear equations (wave, linear acoustics and electromagnetism, Schrödinger with a Coulomb potential, linear elasticity, plate, thermoelasticity). Time-asymptotic behavior of solutions via Hilbert space methods. Bounded domains and domains with bounded complement only. From graduate mathematics/physics lectures at Bonn and Strathclyde, Glasgow. Provides required functional analysis background. DFA

Applications (Physics), T(17-18: 1-3), S, P. *The Mathematical Theory of Combustion and Explosions.* Ya. B. Zeldovich, et al. Transl: Donald H. McNeill. Consultants Bureau, 1985, xxi + 597 pp, \$95. [ISBN: 0-306-10974-3] Presuming a strong background in applied mathematics, this book surveys the current level of understanding, organizing the many individual contributions (primarily Soviet and Western) into a comprehensive theory. In addition, it includes a history of the development of each topic, some fundamentals of thermochemistry and kinetics, and a very extensive bibliography. MU

Applications (Physics), P. *Lecture Notes in Physics-257: Statistical Mechanics and Field Theory: Mathematical Aspects.* Ed: T.C. Dorlas, N.M. Hugenholtz, M. Winnink. Springer-Verlag, 1986, vii + 328 pp, \$32.50. [ISBN: 0-387-16777-3] Papers from the international conference on mathematical aspects of statistical mechanics and field theory, held in Groningen, The Netherlands, August 1985. Topics include first order phase transitions and perturbations, renormalization group methods, and quantum field theory. MR

Applications (Physics), T(15-16: 1, 2), S, P, L. *Ideal and Incompressible Fluid Dynamics.* M.E. O'Neill, F. Chorlton. Math. & Its Applic. Halsted Pr, 1986, 412 pp, \$89.95. [ISBN: 0-470-20273-4] A relatively self-contained text dealing primarily with the theory of incompressible, frictionless fluids. The first chapter provides a review of some mathematics, including such things as Stokes' Theorem. Later, conformal transformations are discussed and used. The text is quite readable and is laced with mathematics which, for the most part, should delight even the purest mathematician. MU

Applications (Physics), S(18), P. *Scattering by Obstacles.* A.G. Ramm. Math. & Its Applic. D Reidel, 1986, xiv + 423 pp, \$89. [ISBN: 90-277-2103-3] This book develops the theory of scattering by small bodies as well as by obstacles with infinite boundaries and inverse scattering. The contents are taken from research journals and presented for the first time in a larger coherent framework. The book contains a wealth of high-level mathematics concentrating on such topics as existence and uniqueness of solutions to the scattering problem, eigenfunction expansions, and analytic properties of the S-matrix. MU

Applications (Physics), T(15-16: 1, 2), P. *New Foundations for Classical Mechanics.* David Hestenes. Fund. Theories of Physics. D Reidel, 1986, xi + 644 pp, \$89. [ISBN: 90-277-2090-8] An introduction to geometric algebra—a mathematical language for physics that integrates vectors, complex numbers, and matrices into a single framework. Covers standard intermediate mechanics course material (and more) from this new point of view. AO

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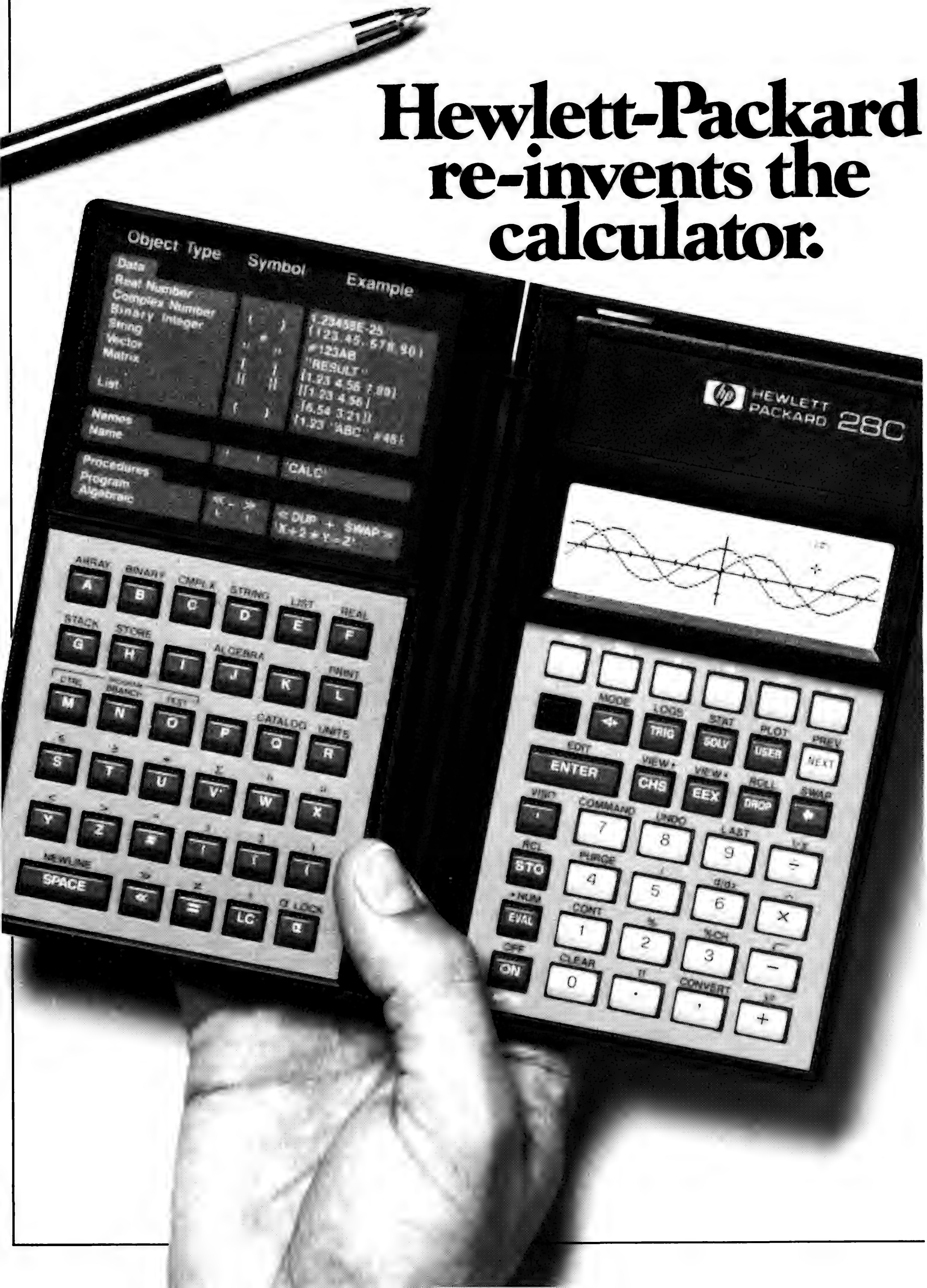

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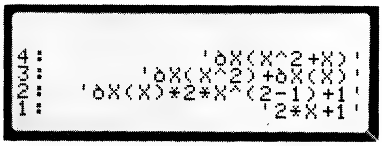

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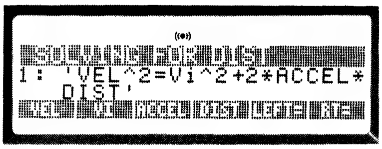

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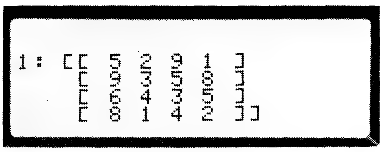

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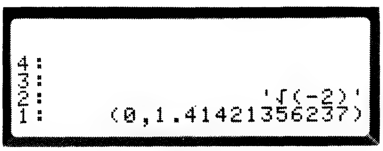

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THE AMERICAN
MATHEMATICAL MONTHLY

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Subscription correspondence, changes of address, and inquiries about nondelivered or defective issues: Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

Extreme Points of Convex Sets in Infinite Dimensional Spaces

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A 1953 graduate of Mount Holyoke College, Nina M. Roy received an M.A. in 1955 from the University of California at Berkeley and a Ph.D. in 1972 from Bryn Mawr College, where her thesis director was Frederic Cunningham, Jr. Since then she has authored or coauthored seven papers in Banach space theory and has risen through the ranks to Professor at Rosemont College, where she was the 1985 recipient of the Lindback Foundation Award for Distinguished Teaching.

1. Introduction. If C is a subset of a linear space, a point x in C is said to be an *extreme point* of C if whenever $x = \lambda y + (1 - \lambda)z$ with $0 < \lambda < 1$ and y, z in C, then $x = y = z$. Thus if C is convex, then x is an extreme point of C provided x is not an inner point of any line segment contained in C. As simple examples, the extreme points of a closed triangular region are its vertices, while those of a closed solid ball are its surface points.

The notion of extreme point goes back to H. Minkowski [**41**, Vol. II, pp. 157–161], who proved that if C is a compact convex set in $\mathbb{R}^3$, then each point of C can be expressed as a convex combination of extreme points of C, that is, as a sum $\sum_{i=1}^{m}\lambda_i x_i$, where $\lambda_i > 0$, $\sum_{i=1}^{m}\lambda_i = 1$, and each x_i is an extreme point of C. This result, known (in $\mathbb{R}^n$) as Minkowski's theorem, was sharpened by Carathéodory, who showed that if C is a compact convex subset of $\mathbb{R}^n$, then each point of C can be expressed as a convex combination of at most $n + 1$ extreme points of C. (A proof and an application to doubly stochastic matrices may be found in [**47**].)

In 1940, M. Krein and D. Milman extended Minkowski's theorem to infinite dimensional spaces by proving that if C is a compact convex subset of a locally convex Hausdorff linear space, then C is the closure of the set of all convex combinations of extreme points of C. The Krein-Milman theorem served as the starting point for virtually all modern research into the external structure of convex sets in infinite dimensional spaces. The purpose of this article is to describe some of the notions, examples, theorems, and applications which this research has produced.

Following the preliminaries in Section 2, we begin in Section 3 with some applications of the Krein-Milman theorem and Klee's generalization. Section 4 contains examples of extremal structure (varying from quite rich to very poor) of closed unit balls in Banach spaces. The Arens-Kelley theorem on the extreme points of the dual ball of $C(K)$ is presented in Section 5 and then used to prove the Banach-Stone theorem. Section 6 contains an extreme point characterization of the class of L_1-preduals and of one of its subclasses ($C(K)$-spaces). Extreme points in spaces of operators are considered in Section 7; included here are Milman's proof that a surjective isometry is an extreme operator (which is essentially a proof that every nice operator is extreme) and some results on the old question: When is an extreme operator nice? In Section 8, we give references for certain topics, dealing with extreme points, which are not included in this paper.

It is hoped that the reader is familiar with the elements of point set topology and real analysis. Useful references for background material are [**55**] and [**60**].

2. Preliminaries. A *normed linear space* is a linear space X, with either real or complex scalars, for which each x in X has a nonnegative norm $\|x\|$ satisfying $\|x\| = 0 \Leftrightarrow x = 0$, $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|ax\| = |a|\|x\|$ for all scalars a and all x, y in X. A *Banach space* is a normed linear space which is complete with respect to the metric $d(x, y) = \|x - y\|$. The *unit ball* of a normed linear space, denoted here by $B(X)$, is the set of those x in X such that $\|x\| \leqslant 1$.

Let X and Y be normed linear spaces with the same scalars. A linear map T of X into Y is continuous if and only if it is bounded—that is, there is a positive real number M such that $\|Tx\| \leqslant M$ for all x in $B(X)$; in this case we define $\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$. With this norm, the space $\mathscr{L}(X, Y)$ of all bounded linear maps of X into Y is a normed linear space; if Y is a Banach space, then $\mathscr{L}(X, Y)$ is also a Banach space. A map T in $\mathscr{L}(X, Y)$ is called a *linear isometry* if $\|Tx\| = \|x\|$ for all x in X. If Y is $\mathbb{R}$ or $\mathbb{C}$ (the field of scalars for X) with the absolute value norm, then $\mathscr{L}(X, Y)$ is denoted by X^* and is called the *dual space* (or simply, *dual*) of X. The natural embedding τ of X into its second dual X^{**} is defined by $\tau x(f) = f(x)$ for x in X and f in X^*, and is a linear isometry. If τ maps X onto X^{**}, then X is said to be *reflexive*.

There are several useful topologies on the dual of a normed linear space X. The *weak* topology* on X^* is the weakest (i.e., smallest) topology in which all the functionals in X (or more precisely, $\tau(X)$) are continuous. A net (f_α) converges weak* to f if and only if $\lim_\alpha f_\alpha(x) = f(x)$ for every x in X. The *strong topology* on X is the metric topology induced by the norm. In this paper, when no topology is specified for a normed linear space, it is understood to be the strong topology.

In the following theorem, the scalars may be real or complex. A proof for real scalars is in [**55**, p. 202].

BANACH-ALAOGLU THEOREM. *If X is a normed linear space, then $B(X^*)$ is weak* compact.*

If S is a subset of a linear topological space, the *convex hull* of S, which we denote by $\operatorname{co}(S)$, is the smallest convex set containing S; it consists of all convex combinations $\sum_{i=1}^n \lambda_i x_i$, where $x_i \in S$, $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. The *closed convex hull* of S is the closure of its convex hull and is equal to the intersection of all closed convex sets containing S.

By a *locally convex space* we mean a locally convex Hausdorff linear topological space. In what follows, when a statement is made about a locally convex space (in particular, a Banach space) and there is no mention of the field of scalars, it is understood that the statement is true in both the real and complex cases.

Our notation for the set of extreme points of a set C is $\operatorname{ext} C$.

It will be convenient at times to use the fact that if C is convex, then a point x of C is in $\operatorname{ext} C$ if and only if, whenever $x = \frac{1}{2}(y + z)$ with y, z in C, then $x = y = z$.

The following lemma contains another simple characterization of extreme points, to be used in the sequel.

LEMMA. *Let C be a convex subset of a linear space, and let $x \in C$. Then $x \in \operatorname{ext} C$ if and only if the conditions $x + y \in C$ and $x - y \in C$ imply $y = 0$.*

Proof. Observe that $x = \frac{1}{2}[(x+y) + (x-y)]$. Thus if $x \in \operatorname{ext} C$, $x + y \in C$, and $x - y \in C$, then $x + y = x - y = x$, which implies $y = 0$. Conversely, if $x + y \in C$ and $x - y \in C$ with $y \neq 0$, then $x + y \neq x - y$ and so $x \notin \operatorname{ext} C$.

3. Applications and a generalization of the Krein-Milman theorem. We begin with a formal statement of the theorem.

KREIN-MILMAN THEOREM [**30**]. *Let C be a nonempty compact subset of a locally convex space. Then* ext C *is not empty. If C is also convex, then C is the closed convex hull of* ext C.

Krein and Milman proved their theorem for the case of a weak* compact convex subset C of the dual space of a normed linear space, and they used transfinite induction to establish the existence of extreme points of C. Simpler proofs using Zorn's Lemma were given independently in 1951 by J.L. Kelley [**24**] and J. Hotta [**19**]. Kelley's proof is in [**55**, p. 207].

An intermediate consequence of the Krein-Milman and Banach-Alaoglu theorems is that for a normed linear space X, $B(X^*)$ is the weak* closed convex hull of its extreme points. Hence a normed linear space whose unit ball has no extreme points cannot be linearly isometric to a dual space. Examples of such spaces will be given in the next section.

To paraphrase Victor Klee [**28**], the fundamental reasons for the importance of extreme points are Bauer's Minimum Principle and the theorems of Krein and Milman. We will show how the latter may be used to prove the former, following some preliminary definitions.

Let f be a real valued function defined on a convex subset C of a real linear space. We say that f is *concave* provided

$$f(\lambda x + (1-\lambda)y) \geqslant \lambda f(x) + (1-\lambda) f(y)$$

whenever $0 \leqslant \lambda \leqslant 1$ and $x, y \in C$. The function f is called *convex* if its negative is concave. A subset K of C is said to be *extremal* if whenever $x, y \in C$ and $\lambda x + (1-\lambda)y \in K$ for some λ, $0 < \lambda < 1$, then $x, y \in K$. Clearly an extreme point of K is necessarily an extreme point of C.

THEOREM (Bauer's Minimum Principle [**4**]). *If C is a nonempty compact convex subset of a real locally convex space and f is a concave lower semicontinuous function defined on C, then f attains its minimum at an extreme point of C.*

Proof. The fact that f attains its minimum can be proved in the classical way. (See, for example, p. 161 of [**55**]). Let $\alpha = \inf\{f(x) : x \in C\}$ and let $K = \{x \in C : f(x) = \alpha\}$. We will show that K is compact and extremal. Observe that

$K = \{x \in C : f(x) \leqslant \alpha\}$; therefore by lower semicontinuity, K is closed, hence compact. To see that K is extremal, let $x, y \in C$ with $\lambda x + (1 - \lambda)y \in K$ for some λ, $0 < \lambda < 1$. Then

$$\begin{aligned} \alpha &= f(\lambda x + (1-\lambda)y) \\ &\geqslant \lambda f(x) + (1-\lambda)f(y) \quad \text{(because } f \text{ is concave)} \\ &\geqslant \lambda\alpha + (1-\lambda)\alpha \\ &= \alpha. \end{aligned}$$

Thus, $\alpha = \lambda f(x) + (1 - \lambda)f(y)$. Since $f(x) \geqslant \alpha$ and $f(y) \geqslant \alpha$, we conclude that $\alpha = f(x) = f(y)$. Hence $x, y \in K$ and therefore K is extremal. Since K is compact, it follows from the Krein-Milman theorem that K has an extreme point; because K is extremal, this point must be an extreme point of C.

Bauer's Minimum Principle implies, of course, that a convex upper semicontinuous function defined on a compact convex set C attains its maximum at an extreme point of C. As a simple illustration, let C be a closed rectangular region in $\mathbb{R}^2$, let $p \in \mathbb{R}^2$, and define f on C by $f(x) = d(x, p)$ for all x in C, where d is Euclidean distance. Then f attains its maximum at one of the four vertices of C.

Other prominent theorems which can be proved using the Krein-Milman theorem are the Stone-Weierstrass theorem [**33**, §11.4], Bernstein's theorem [**46**, §2], and the "Bang-Bang Principle" of time optimal control theory, equivalent to Liapunov's theorem on the range of a vector measure [**17**, §8].

The Krein-Milman theorem has also found application in the theory of infinite doubly stochastic matrices ([**26**], [**49**]). Finite doubly stochastic matrices and their extreme points, as well as an application to the optimal assignment problem, can be found in [**23**, §5.8]. Karlin's book also contains (in §2.4) an interesting characterization of the extreme-point optimal strategies in the theory of two-person matrix games.

We close this section with Klee's generalization of the Krein-Milman theorem and an application of it to linear programming.

THEOREM (V. Klee [**27**]). *If C is a closed, convex and locally compact subset of a locally convex space and if C contains no straight line, then C is the closed convex hull of the set of its extreme points and extreme rays.*

(An *extreme ray* of C is a closed half-line R contained in C such that every open interval in C which meets R is entirely contained in R.)

Klee's theorem is proved in [**29**, §25].

In linear programming, the objective is to find the value of $x = (x_1, x_2, \ldots, x_n)$ which maximizes (or minimizes) the function $f(x) = \sum_{j=1}^n c_j x_j$ subject to the constraints $\sum_{j=1}^n a_{ij}x_j \leqslant b_i$ $(i = 1, 2, \ldots, m)$ and $x_j \geqslant 0$ $(j = 1, 2, \ldots, n)$. The set C defined by these constraints is a polyhedral convex subset of $\mathbb{R}^n$; hence the extreme points of C are its vertices. The function f is linear and so the maximum, if it exists, must be attained at an extreme point of C. (This can be proved using Klee's

theorem. See [**51**, Cor. 32.3.1].) The simplex method consists of systematically determining and testing the extreme points until the vector $x = (x_1, x_2, \dots, x_n)$ is found or it is clear that there is no solution.

4. Examples. In the following examples, we shall need the fact that if S is a finite subset of a linear topological space X, then $\mathrm{co}(S)$ is compact. To see this, let $S = \{x_1, x_2, \dots, x_n\}$, let I be the interval $[0,1]$, and let $f\colon \mathbb{R}^n \to X$ be defined by $f(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i x_i$. The set K of those $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in I^n such that $\sum_{i=1}^n \lambda_i = 1$ is a closed subset of the compact space I^n, hence is compact. The function f is continuous (because f is linear and $\mathbb{R}^n$ is finite dimensional [**9**, Cor. I.4.4]), and $f(K) = \mathrm{co}(S)$. Therefore $\mathrm{co}(S)$ is compact.

For our first example, let l_1 denote the Banach space of all absolutely summable real sequences $x = (x_n)$ with the norm $\|x\| = \sum |x_n|$. Then $B(l_1)$ is the closed convex hull of its extreme points. Note that this does not follow from the Krein-Milman theorem because $B(l_1)$ is not compact; only finite dimensional spaces have compact unit balls [**10**, Thm. IV.3.5]. Let $E = \mathrm{ext}\, B(l_1)$. To prove that $B(l_1)$ is the closure of $\mathrm{co}(E)$, we show first that E consists of the sequences e_n, $n = 1, 2, \dots$ and their negatives, where e_n is the sequence with 1 in the nth position and zeros elsewhere. Let us first verify that each e_n is in E. Suppose that $e_n = \frac{1}{2}(x + y)$, where $x, y \in B(l_1)$. Say $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Then $1 = \frac{1}{2}(x_n + y_n)$ and $|x_n| \leqslant 1$, $|y_n| \leqslant 1$. This implies that $x_n = y_n = 1$. Since x and y have norm at most 1, it follows that $x_j = y_j = 0$ for $j \neq n$. Thus $x = y = e_n$. Hence $e_n \in E$. For the reverse inclusion, let $x = (x_n) \in B(l_1)$ and suppose that $x \neq \pm e_n$ for all n. We may assume that $\|x\| = 1$ since otherwise x would clearly not be an extreme point. Then we can write $x = y + z$, where $y = (x_1, x_2, \dots, x_k, 0, 0, \dots)$, $z = (0, 0, \dots, 0, x_{k+1}, x_{k+2}, \dots)$, and neither y nor z is the zero sequence. Let $u = (1/\|y\|)y$ and $v(1/\|z\|)z$. Then, since $\|y\| + \|z\| = \|x\| = 1$, it follows that $x = \|y\|u + (1 - \|y\|)v$. Thus $x \notin E$. We now show that an arbitrary sequence $x = (x_n)$ in $B(l_1)$ is in the closure of $\mathrm{co}(E)$. For each n, let $y_n = \sum_{k=1}^n x_k e_k$. Then $y_n \in \mathrm{co}(E)$. To see this, consider the point $(x_1, x_2, \dots, x_n)$ in the space l_1^n (which is $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n |x_i|$). Let $E_n = \mathrm{ext}\, B(l_1^n)$. Then the members of E_n are the vectors $\pm e_k$ of length n. Hence by the Krein-Milman theorem and the remarks made before this example, $B(l_1^n) = \mathrm{co}(E_n)$. Thus $(x_1, x_2, \dots, x_n) \in \mathrm{co}(E_n)$ and consequently $y_n \in \mathrm{co}(E)$. Since $x = \lim_{n \to \infty} y_n$, we have the desired result.

More generally, if Γ is any nonempty set, the space $l_1(\Gamma)$ of all real absolutely summable functions on Γ has the property that $B(l_1(\Gamma))$ is the closed convex hull of its extreme points, and this property characterizes $l_1(\Gamma)$-spaces among real Banach spaces of a certain type ([**54**]). For definition and properties of these spaces, the reader may consult [**9**, §II.2] and [**25**, Problem 2G].)

Let us now examine the extremal structure of $C(K)$-spaces. Let K be a compact Hausdorff space and let $C(K, \mathbb{R})$ (respectively, $C(K, \mathbb{C})$) denote the Banach space of all continuous real (resp., complex) valued functions f on K, with the uniform

norm $\|f\| = \sup\{|f(k)| : k \in K\}$. The following lemma is due to Krein and Milman [**30**].

LEMMA. *Let C denote either $C(K, \mathbb{R})$ or $C(K, \mathbb{C})$. Then* $\operatorname{ext} B(C) = \{f \in C : |f(k)| = 1$ *for all k in K*$\}$.

Proof. Let $f \in C$ with $|f(k)| = 1$ for all k in K, and suppose that $f = \frac{1}{2}(g + h)$, where $g, h \in B(C)$. Then, since for each k, $f(k)$ is an extreme point of the unit ball in the field of scalars, it follows that $g(k) = h(k) = f(k)$. Thus $g = h = f$, and so $f \in \operatorname{ext} B(C)$. Conversely, assume that $f \in B(C)$ with $|f(k_0)| < 1$ for some k_0 in K. Let $g = 1 - |f|$. Then $f + g \in B(C)$ and $f - g \in B(C)$, but $g(k_0) \neq 0$. Hence by the lemma at the end of Section 2, $f \notin \operatorname{ext} B(C)$.

We can now see that K is connected if and only if $B(C(K, \mathbb{R}))$ has only two extreme points, namely, the constant functions 1 and -1. (Hence, for example, $C([0,1], \mathbb{R})$ is not a dual space.) An abundance of extreme points in $B(C(K, \mathbb{R}))$ will force K to be very disconnected. More precisely, we have the following theorem.

THEOREM (W. Bade [**2**]). *Let K be a compact Hausdorff space. Then $B(C(K, \mathbb{R}))$ is the closed convex hull of its extreme points if and only if K is totally disconnected.*

("Totally disconnected" means that the only connected subsets of K are the singletons.)

In the case of complex scalars, the situation is quite different.

THEOREM (R. R. Phelps [**45**]). *If K is a compact Hausdorff space, then $B(C(K, \mathbb{C}))$ is the closed convex hull of its extreme points.*

A "neat measure-theoretic proof" (to quote the MR reviewer) of this theorem has been given by L. G. Brown [**6**] in this MONTHLY.

There are Banach spaces in which every point of norm 1 is an extreme point of the unit ball. Such spaces are called *strictly convex* or *rotund*. For example, $\mathbb{R}^n$ with the Euclidean norm is strictly convex; in fact, every Hilbert space is strictly convex [**16**, Problem 3]. A good source of information and references for strict convexity is the recent survey by V.I. Istrăţescu [**20**].

At the other extreme (!) there are Banach spaces whose unit balls have no extreme points. Such is the case, for example, of the space c_0 of all real sequences which converge to zero. To prove this, let $x = (x_n) \in c_0$ with $\|x\| = \sup|x_n| = 1$ and let k be an integer such that $|x_k| < 1/2$. Define the sequences (y_n) and (z_n) by $y_n = z_n = x_n$ if $n \neq k$, and $y_k = x_k - 1/2$, $z_k = x_k + 1/2$. Setting $y = (y_n)$, $z = (z_n)$, we have y, z in $B(c_0)$, $y \neq z$, and $x = \frac{1}{2}(y + z)$. Thus $x \notin \operatorname{ext} B(c_0)$. Therefore $\operatorname{ext} B(c_0) = \varnothing$.

The following problem was open for many years: If C is a nonempty compact convex subset of a Hausdorff linear topological space, must C have an extreme point? In 1977, J. W. Roberts [**50**] constructed an example showing that the answer is negative. The interested reader may consult [**22**, Ch. 9] for details and related open problems.

5. The Arens-Kelley theorem. Let K be a compact Hausdorff space and let $C(K)$ denote the Banach space of all continuous real (or complex) valued functions on K, with the uniform norm. As Elton Lacey has remarked [**31**, p. 53], the extreme points of the unit ball of $C(K)^*$ play an important role in the analysis of K, $C(K)$, and $C(K)^*$. For each k in K, let ε_k be the evaluation functional in $C(K)^*$ defined by $\varepsilon_k(f) = f(k)$ for all f in $C(K)$. R. F. Arens and J. L. Kelley [**1**] proved that in the real case, the evaluation functionals and their negatives are precisely the extreme points of the unit ball of $C(K)^*$. In the statement of their theorem below, the scalars may be real or complex. A proof is in [**10**, p. 441].

ARENS-KELLEY THEOREM. Ext $B(C(K)^*) = \{a\varepsilon_k : |a| = 1,\ k \in K\}$.

Generalizations of the Arens-Kelley theorem, in various directions, have been given by I. Singer [**61**, Lemma 1.7, p. 197], F. Cunningham, Jr., and N. M. Roy [**8**], A. and C. Ionescu Tulcea [**20**, Thm. 2.13.8], and R. R. Phelps [**44**].

An interesting application of the Arens-Kelley theorem, and one which first appeared in [**1**], is its use in proving the following theorem due to S. Banach [**3**, p. 170] in the separable case and M. H. Stone [**62**, p. 469] in the general case.

BANACH-STONE THEOREM. *Let K and H be compact Hausdorff spaces. Then $C(K)$ and $C(H)$ are linearly isometric if and only if K and H are homeomorphic.*

The following lemma, to be used also in the next section, will facilitate the proof of the Banach-Stone theorem.

LEMMA. *Let $\hat{K} = \{\varepsilon_k : k \in K\}$. Then $\hat{K}$ is weak* compact and the map $k \mapsto \varepsilon_k$ is a homeomorphism of K onto $\hat{K}$.*

Proof. The map $k \mapsto \varepsilon_k$ is clearly surjective, and Urysohn's Lemma may be applied to show that it is injective. To see that it is continuous, let $\{k_\alpha\}$ be a net in K which converges to k in K. Then $\{f(k_\alpha)\}$ converges to $f(k)$ for every f in $C(K)$. Consequently $\{\varepsilon_{k_\alpha}\}$ converges weak* to ε_k. Thus the map is continuous. We now have that $\hat{K}$ is a weak* compact and the map is a homeomorphism of K onto $\hat{K}$ [**25**, Thm. 5.8].

Proof of Banach-Stone Theorem. Let $X = C(K)$ and $Y = C(H)$. Suppose first that there is a homeomorphism τ of K onto H. Define the map T on Y by $Tf(k) = f(\tau(k))$ for f in Y and k in K. One can routinely check that $Tf \in X$ and $T\colon Y \to X$ is a linear isometry. To see that T is surjective, let $g \in X$ define f by setting $f(h) = g(\tau^{-1}(h))$, $h \in H$. Then $f \in Y$ and $Tf = g$.

For the converse, assume that T is a linear isometry of Y onto X. The conjugate map $T^*\colon X^* \to Y^*$ is given by $T^*\Phi(f) = \Phi(Tf)$ for Φ in X^* and f in Y. It is not hard to verify that T^* is weak* continuous and is a linear isometry of X^* onto Y^*. (To see that T^* is surjective, let $\Psi \in Y^*$ and define $\Phi = \Psi \circ T^{-1}$. Then $\Phi \in X^*$ and $T^*\Phi = \Psi$.) Thus T^* is a bijection of ext $B(X^*)$ onto ext $B(Y^*)$. Hence, by the Arens-Kelley theorem, for each k in K, there are unique h in H and

scalar $a(k)$ with $|a(k)| = 1$ such that $T^*\varepsilon_k = a(k)\varepsilon_h$. Let τ: $K \to H$ be defined by $\tau(k) = h$ if and only if $T^*\varepsilon_k = a(k)\varepsilon_h$. Then τ is the desired homeomorphism of K onto H. To verify this, let us first check that τ is continuous. Let $\{k_\alpha\}$ be a net in K which converges to k in K. Then by the lemma preceding this proof, the net $\{\varepsilon_{k_\alpha}\}$ converges weak* to ε_k. Let $h_\alpha = \tau(k_\alpha)$ for each α, and let $h = \tau(k)$. Because T^* is weak* continuous, the net $\{a(k_\alpha)\varepsilon_{h_\alpha}\}$ converges weak* to $a(k)\varepsilon_h$. Hence $\{a(k_\alpha)\}$ converges to $a(k)$ (take $f = 1$ to see this), and consequently $\{\varepsilon_{h_\alpha}\}$ converges weak* to ε_h. Then by the above lemma, $\{h_\alpha\}$ converges to h. It is easy to see that τ is bijective, and so the proof is complete.

6. Extreme point characterizations of L_1-preduals. If (S, φ, μ) is a measure space, we denote by $L_1(\mu)$ the Banach space of all real or complex valued measurable functions f on S for which $|f|$ is integrable, with the norm given by $\|f\| = \int |f|\, d\mu$, and with two functions regarded as equivalent if they are equal almost everywhere. An L_1*-space* is a Banach space which is linearly isometric to $L_1(\mu)$ for some measure space (S, φ, μ). An L_1*-predual* is a Banach space whose dual is an L_1-space. An L_1-predual is also called a *Lindenstrauss space* because these spaces were first studied extensively by Lindenstrauss in [**36**]. Convexity and extreme point structure play important roles in the isometric theory of L_1-preduals.

An example of an L_1-predual is a $C(K)$-space, which is a Banach space linearly isometric to the space $C(K)$ of all continuous real- or complex-valued functions on some compact Hausdorff space K. To see that $C(K)^*$ is an L_1-space, recall that by the Riesz Representation Theorem, $C(K)^*$ may be viewed as the space of all regular Borel signed or complex measures on K, hence is an abstract L-space (in the sense of Kakutani). Then by Kakutani's representation theorem, $C(K)^*$ is linearly isometric to $L_1(\mu)$ for some measure μ. (For details, the reader may consult [**31**, §8 and §15].)

The space $C(K)$ has the following two properties:

(i) Ext $B(C(K)) \neq \varnothing$.

(ii) Ext $B(C(K)^*)$ is weak* closed.

Property (i) holds because, as we recall from the lemma in Section 4, the constant function 1 is an extreme point. Property (ii) follows easily from the Arens-Kelley theorem and the lemma to the proof of the Banach-Stone theorem in the preceding section.

Properties (i) and (ii) actually characterize $C(K)$-spaces among L_1-preduals. This is due to J. Lindenstrauss [**36**, p. 76] in the real case and to B. Hirsberg and A. J. Lazar [**18**] when the scalars are complex.

Space does not permit the description here of the other classes of L_1-preduals and their extreme point characterizations. References, besides those given above in this section, are [**9**], [**11**], [**12**], [**31**], [**52**], [**63**], [**34**], [**38**], and [**53**]. The last three contain diagrams indicating how the classes are related.

We close this section with an interesting characterization of L_1-preduals given by Å. Lima. Some preliminaries are needed first. A *projection* in a linear space X is a linear map P of X into itself such that $P \circ P = P$. An *L-projection* in a Banach

space X is a projection P of X such that $\|x\| = \|Px\| + \|x - Px\|$ for all x in X. An L-projection is bounded, in fact, has norm 1 (if not 0). An *L-summand* is the range of an L-projection. As an example, let (X, φ, μ) be a measure space, let χ be the characteristic function of a measurable subset of S, and define P on $L_1(\mu)$ by $P(f) = f\chi$ for all f in $L_1(\mu)$. Then P is an L-projection in $L_1(\mu)$.

L-projections were first studied by F. Cunningham, Jr. [7], who showed that a real L_1-space has the property that its nonzero L-projections form a maximal abelian family of projections of norm 1, and this property characterizes L_1-spaces among real Banach spaces. (Cunningham's theorem is also true in the complex case.)

Let X be a Banach space and let $E = \operatorname{ext} B(X^*)$. For each L-projection P in X^*, define

$$N_P = \{f \in B(X^*) : P(f) = f \quad \text{or} \quad P(f) = 0\},$$

and let $N = \cap \{N_P\colon P \text{ is an } L\text{-projection in } X^*\}$. We will show that $[0,1]E \subset N$ (the easy part of Lima's theorem below). For this, it suffices to show that $E \subset N$. Let $f \in E$ and let P be an L-projection in X^*. Since $\|f\| = 1$, we have $1 = \|Pf\| + \|f - Pf\|$. Suppose that $Pf \neq 0$ and $Pf \neq f$. Then $\|Pf\| > 0$ and $\|f - Pf\| > 0$, hence $0 < \|Pf\| < 1$. Let $\lambda = \|Pf\|$. We then have the convex combination

$$f = \lambda(1/\|Pf\|)Pf + (1 - \lambda)(1/\|f - Pf\|)(f - Pf).$$

Since $f \in E$, it follows that $f = (1/\|Pf\|)Pf$. Applying P to both sides gives $Pf = (1/\|Pf\|)Pf$, which implies that $\|Pf\| = 1$, a contradiction. Thus $[0,1]E \subset N$. The inclusion also goes the other way if X is an L_1-predual and, as the following theorem shows, this characterizes L_1-preduals among those Banach spaces for which $\operatorname{span}(f)$ is an L-summand for every f in E. (Span(f) denotes the linear span of f.)

THEOREM (Å. Lima [**35**, Thm. 5.8, p. 35]). *A Banach space X is an L_1-predual if and only if $[0,1]\ E = N$ and $\operatorname{span}(f)$ is an L-summand for every f in E.*

7. Extreme points in spaces of operators. Let X and Y be Banach spaces over the same scalar field, and let $\mathscr{B}(X, Y)$ denote the unit ball in the space $\mathscr{L}(X, Y)$ of all bounded operators of X into Y. An extreme point of $\mathscr{B}(X, Y)$ is called an *extreme operator*. The following theorem shows, in particular, that $\mathscr{B}(X, Y)$ has at least one extreme point.

THEOREM (D. Milman [**40**]). *If T is a linear isometry of X onto Y, then T is an extreme operator.*

Proof (Milman). The conjugate map T^*, defined by $T^*f = f \circ T$ for all f in Y^*, is a linear isometry of Y^* onto X^*, hence it maps $\operatorname{ext} B(Y^*)$ onto $\operatorname{ext} B(X^*)$. To see that T is an extreme operator, suppose $T = \frac{1}{2}(U + V)$, where $U, V \in \mathscr{B}(X, Y)$. Let $f \in \operatorname{ext} B(Y^*)$. Then $T^*f \in \operatorname{ext} B(X^*)$, and $T^*f = \frac{1}{2}(U^*f + V^*f)$. Consequently $T^*f = U^*f = V^*f$. Hence $f(Tx) = f(Ux) = f(Vx)$ for all x in X and every f in $\operatorname{ext} B(Y^*)$. This implies that for each x in X, $f(Ux - Vx) = 0$ for all f in $\operatorname{ext} B(Y^*)$; hence by the Krein-Milman theorem, $f(Ux - Vx) = 0$ for all f in

$B(Y^*)$. Then $Ux - Vx = 0$ because Y^* is total over Y [**10**, p. 418]. Thus $U = V$ and the proof is complete. We note for future reference that what makes T an extreme operator, in Milman's proof, is the fact that T maps $\operatorname{ext} B(Y^*)$ into $\operatorname{ext} B(X^*)$.

As a partial converse, we have the following theorem, which seems to have first appeared in [**37**].

THEOREM. *Let X denote n-dimensional Euclidean space. If $T \in \operatorname{ext} \mathscr{B}(X, X)$, then T is an isometry.*

Proof (adapted from [**14**] and [**15**]). Let $T \in \operatorname{ext} \mathscr{B}(X, X)$. Then $\|T\| = 1$. We recall that the adjoint T^* is the operator on X defined by $\langle T^*x, y\rangle = \langle x, Ty\rangle$ for all x, y in X. Let $Y = \{x \in X : \|Tx\| = \|x\|\}$ and let $Z = \{x \in X : \|T^*x\| = \|x\|\}$. Our goal is to show that $Y = X$. Let us first see that Y is a linear subspace of X. If $x \in Y$, then

$$\begin{aligned}\|T^*Tx - x\|^2 &= \langle T^*Tx - x, T^*Tx - x\rangle \\ &= \|T^*Tx\|^2 - 2\langle T^*Tx, x\rangle + \|x\|^2 \\ &= \|T^*Tx\|^2 - 2\langle Tx, Tx\rangle + \|x\|^2 \\ &= \|T^*Tx\|^2 - 2\|x\|^2 + \|x\|^2 \\ &\leqslant \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0.\end{aligned}$$

Hence $T^*Tx = x$ for all x in Y. Thus if $x, y \in Y$, then $\|T(x + y)\|^2 = \langle T^*Tx + T^*Ty, x + y\rangle = \|x + y\|^2$. Since we also have that $\|Tax\| = |a|\|Tx\| = |a|\|x\| = \|ax\|$ for $a \in \mathbb{R}$, $x \in Y$, it follows that Y is a linear subspace of X. The same argument as above shows that $TT^*x = x$ for all x in Z; it follows readily that $T(Y) = Z$. It is not hard to verify that $T(Y^\perp) \subset Z^\perp$, where $Y^\perp$ and $Z^\perp$ denote the orthogonal complements of Y and Z, respectively. Suppose that T is not an isometry, i.e., $Y \neq X$. Then $Y^\perp \neq \{0\}$ because $X = Y \oplus Y^\perp$; and $Z^\perp \neq \{0\}$ because $X = Z \oplus Z^\perp$ and $T(Y) = Z$. Consider the two cases: $T(Y^\perp) = \{0\}$ and $T(Y^\perp) \neq \{0\}$. If $T(Y^\perp) = \{0\}$, choose u in $Y^\perp$ and v in $Z^\perp$ with $\|u\| = \|v\| = 1$. Let S in $\mathscr{L}(X, X)$ be defined by $S(x) = \langle x, u\rangle v$ for all x in X. Then $S \neq 0$ because $S(u) = v$. Further, $\|T \pm S\| \leqslant 1$. To see this, let $x \in X$ with $\|x\| \leqslant 1$. Then $x = y + z$, where $y \in Y$ and $z \in Y^\perp$. We have

$$\begin{aligned}\|Tx \pm Sx\|^2 &= \|Ty + Tz \pm (\langle y, u\rangle v + \langle z, u\rangle v)\|^2 \\ &= \|Ty \pm \langle z, u\rangle v\|^2 \\ &= \|Ty\|^2 + \|\langle z, u\rangle v\|^2 \quad (\text{because } Ty \in Z \text{ and } \langle z, u\rangle v \in Z^\perp) \\ &\leqslant \|y\|^2 + \|z\|^2\|u\|^2\|v\|^2 \\ &= \|y\|^2 + \|z\|^2 = \|x\|^2 \leqslant 1.\end{aligned}$$

Thus by the lemma at the end of Section 2, $T \notin \operatorname{ext} \mathscr{B}(X, X)$, a contradiction. Now let us consider the case $T(Y^\perp) \neq \{0\}$. We shall need the fact, not difficult to verify,

that $\|T|Y^{\perp}\| < 1$ because $Y^{\perp}$ is finite dimensional. Then $\|T|Y^{\perp}\| \leqslant 1/(1+\varepsilon)$, where $0 < \varepsilon \leqslant 1$. Let $R = \varepsilon TP$, where P is the orthogonal projection onto $Y^{\perp}$. Then $R \neq 0$ because if $x \in Y^{\perp}$ with $Tx \neq 0$, then $Rx = \varepsilon TPx = \varepsilon Tx \neq 0$. Further, $\|T \pm R\| \leqslant 1$. (We omit the proof of this because it is very similar to the proof above that $\|T \pm S\| \leqslant 1$.) Hence $T \notin \operatorname{ext} \mathscr{B}(X, X)$, again a contradiction. The proof of the theorem is now complete.

In general, an extreme operator need not be an isometry, even when it maps a finite dimensional space into itself. For example, let X be $\mathbb{R}^2$ with the norm $\|(x, y)\| = \max\{|x|, |y|\}$. Then $B(X)$ is the square with vertices $(\pm 1, \pm 1)$. Define T: $X \to X$ by $T(x, y) = (x, x)$. Then $\|T\| = 1$, hence $T \in \mathscr{B}(X, X)$. Further, T is an extreme operator. For suppose $T = \frac{1}{2}(U + V)$, where U, $V \in \mathscr{B}(X, X)$. Then $(1,1) = T(1,0) = \frac{1}{2}(U(1,0) + V(1,0))$. But $(1,1) \in \operatorname{ext} B(X)$, hence $U(1,0) = V(1,0) = (1,1)$. Similarly $U(1,-1) + V(1,-1) = (1,1)$. Since U and V agree at elements of a basis for $\mathbb{R}^2$, it follows that $U = V$. Thus $T \in \operatorname{ext} \mathscr{B}(X, X)$. And T is clearly not an isometry.

Following P. D. Morris and R. R. Phelps [**43**], if X and Y are Banach spaces, an operator T in $\mathscr{L}(X, Y)$ is called *nice* if T^* maps $\operatorname{ext} B(Y^*)$ into $\operatorname{ext} B(X^*)$. The proof of Milman's theorem at the beginning of this section shows that every nice operator is extreme. The converse is true for certain types of spaces. For example, R. M. Blumenthal, J. Lindenstrauss, and R. R. Phelps [**5**] proved that if K and H are compact Hausdorff spaces with K metrizable and if the scalars are real, then every extreme operator in $\mathscr{L}(C(K), C(H))$ is nice. The hypothesis that K is metrizable cannot be removed, as was shown by M. Sharir [**59**]. Sharir has also proved that in the complex case, for each non-dispersed compact Hausdorff space K, there exists a compact Hausdorff space H such that $\mathscr{L}(C(K), C(H))$ contains an extreme non-nice operator [**58**]. Another result of Sharir is that for real or complex scalars, if X and Y are both L_1-spaces, then every extreme operator in $\mathscr{L}(X, Y)$ is nice [**57**]. More results of this kind and many references can be found in [**20**, §2.13 and §2.14].

8. Other topics. Certain topics which have been omitted from this paper are discussed elsewhere in very readable expositions. We recommend [**47**] and its references for the classic integral representation theorems and the theory of simplexes. Also by R. R. Phelps, [**48**] is an excellent survey of results related to the Radon-Nikodym property (RNP) and includes a discussion of an important unsolved problem concerning the relationship between the RNP and the Krein-Milman Property. Finally, T. H. MacGregor's article [39] in this MONTHLY is recommended for extreme points of families of analytic functions.

The author thanks the referees for their helpful comments and suggestions.

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A Matrix With Applications to Random Walk, Brownian Motion, and Ring Theory

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1. Introduction. We shall be studying the powers of the infinite matrix $A = A(a;\, b, c;\, p, q)$ defined by

$$A = A(a;\, b, c;\, p, q) = \begin{pmatrix} 2a & b & \cdot & \cdot & \cdot & \cdot & \cdots \\ c & \cdot & p & \cdot & \cdot & \cdot & \cdots \\ \cdot & q & \cdot & p & \cdot & \cdot & \cdots \\ \cdot & \cdot & q & \cdot & p & \cdot & \cdots \\ \cdot & \cdot & \cdot & q & \cdot & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

so that the matrix $A = [A_{m,n}]\,(m, n = 0, 1, 2, \dots)$, where $A_{0,0} = 2a$, $A_{0,1} = b$, $A_{1,0} = c$, $A_{m+1,\,m+2} = p$, $A_{m+2,\,m+1} = q$, and otherwise $A_{m,n} = 0$, indicated by $\cdot$ in (1.1).

We denote by $A^{(s)}_{m,n}(a;\, b, c;\, p, q)$ the (m, n)th element of the sth power of the matrix $A = A(a;\, b, c;\, p, q)$. Our primary goal is to find a number of closed expressions for $A^{(s)}_{m,n}(a;\, b, c;\, p, q)$ and to apply the results in widely diverse contexts. Examples of such expressions are to be found in Sections 7, 10 and 11 below.

One application of the present study is to the problem in abstract ring theory proposed in Section 2 below and solved in Section 13. Another application of a totally different nature, developed in Section 12, is to the theory of random walk with a single barrier including, as a particular case, part of the theory of Brownian motion as presented by Mark Kac in his masterly exposition [**1**]. Section 12 includes corrections for a number of errors in [**1**] which are repeated in the authoritative texts [**2**] and [**3**]. Other surprises, such as the remark after (7.2), are also revealed in the course of the present investigation.

2. A problem in ring theory. The problem propounded in [**4**] can be stated more generally as follows, where we change the notation in order to conform with [**1**]:

Determine the constant term $w_d(s)$ in the expansion of

$$\left(\sum_{i=1}^{d} \left(x_i + x_i^{-1}\right)\right)^{s} \qquad (d \geq 2) \tag{2.1}$$

where the x_i are noncommuting variables.

Any word in the set of $2d$ symbols $\{x_i, x_i^{-1}\}$ is equivalent by cancellation to some unique irreducible word and, for fixed d and for $k = 0, 1, 2, \ldots$, we denote by $v_k(s)$ the number of terms in the expansion of (2.1) which are equivalent to an irreducible word of length k. Then $v_k(0) = \delta_{k,0}$ (Kronecker delta), $v_0(s+1) = v_1(s)$, $v_1(s+1) = 2dv_0(s) + v_2(s)$ and, for $k \geq 2$, $v_k(s+1) = (2d-1)v_{k-1}(s) + v_{k+1}(s)$.

Hence the vector $\langle v_0(s), v_1(s), v_2(s), \ldots \rangle$ is the first row of the matrix A^s, where $A = A(a; b, c; p, q)$ of (1.1) with

$$a = 0, \qquad b = p + 1 = 2d \quad \text{and} \quad c = q = 1, \tag{2.2}$$

it follows that the solution to our problem is

$$w_d(s) = v_0(s) = A_{0,0}^{(s)}(0; 2d, 1; 2d-1, 1) \tag{2.3}$$

and it may be remarked that $(2d)^{-1}A$ is a stochastic matrix.

In Section 13 we provide four different, not obviously equivalent, expressions for $w_d(s)$, together with a table of values of this function.

3. A transformation and a preliminary reduction. In the first instance (cf. Section 7 *et seq.*) we confine attention to the case

$$pq > 0, \qquad bc \neq 0, \tag{3.1}$$

we write

$$\begin{gathered} a = u\sqrt{pq}\,, \qquad b = px, \qquad c = qy, \qquad t = xy = bc/pq \neq 0, \\ Z(u; x, y; p, q) = A(a; b, c; p, q), \\ Z_{m,n}^{(s)}(u; x, y; p, q) = A_{m,n}^{(s)}(a; b, c; p, q), \end{gathered} \tag{3.2}$$

and we further define

$$\begin{gathered} B = B(u; t) = Z(u; t, 1; 1, 1) = A(u; t, 1; 1, 1), \\ B_{m,n}^{(s)}(u; t) = \text{the } (m, n)\text{th element of the matrix } B^s, \\ C(m, n; s) = C_{m,n}^{(s)}(u; t) = t^{\delta_{n,0}} B_{m,n}^{(s)}(u; t) \text{ (Kronecker delta)}. \end{gathered} \tag{3.3}$$

Then, for a suitable (unique) diagonal matrix D, we have $Z(u; x, y; p, q) = (pq)^{1/2} D^{-1} B D$ and consequently

$$Z_{m,n}^{(s)}(u; x, q; p, q) = y^{\delta_{n,0} - \delta_{m,0}} \left(\frac{p}{q}\right)^{(n-m)/2} (pq)^{s/2} B_{m,n}^{(s)}(u; t), \tag{3.4}$$

so that

$$Z_{m,n}^{(s)}(u; x, y; p, q) = x^{-\delta_{n,0}} y^{-\delta_{m,0}} \left(\frac{p}{q}\right)^{(n-m)/2} (pq)^{s/2} C(m, n; s). \tag{3.5}$$

By (3.2) and (3.5), our goal of evaluating $A_{m,n}^{(s)}(a; b, c; p, q)$ will be achieved by finding an expression for $C(m, n; s) = C_{m,n}^{(s)}(u; t) = C_{m,n}^{(s)}(u; xy)$, and we now take up this task.

4. An alternative characterization of $C(m,n;s) = C^{(s)}_{m,n}(u;xy) = C^{(s)}_{m,n}(u;t)$. From the relations $B^0 = I$ and $B^{s+1} = B \cdot B^s$, together with (3.3), we deduce that, for given values of (u, t),

$$\begin{aligned} C(m,n;s+1) &= C(m-1,n;s) + C(m+1,n;s) \qquad (m \geq 1), \\ C(0,n;s+1) &= 2uC(0,n;s) + tC(1,n;s), \\ C(m,n;0) &= t^{\delta_{n,0}}\delta_{m,n} \qquad \text{(Kronecker delta)}, \end{aligned} \tag{4.1}$$

and the value of $C(m, n; s)$ is uniquely determined by (4.1).

For fixed u, t we define two functions g, h of the complex variable z by

$$g(m,n;s)(z) = z^{n-1}(z+z^{-1})^s(z^{-m} + z^m h(z)),$$

where

$$h(z) = h(u;t)(z) = \frac{(t-1) + 2uz - z^2}{1 - 2uz - (t-1)z^2}, \tag{4.2}$$

and we expand $h(u; t)(z)$ as a Maclaurin series by writing

$$h(u;t)(z) = t - 1 + \sum_{i=1}^{\infty} a_i z^i = t - 1 + \sum_{i=1}^{\infty} b_i z^{2i-1} + \sum_{i=1}^{\infty} c_i z^{2i} \tag{4.3}$$

observing that, in the particular case $u = 0$, we have

$$b_i = 0, \qquad c_i = t(t-2)(t-1)^{i-1} \qquad (i = 1,2,3,\ldots). \tag{4.4}$$

Taking $s = 0$ we find that the residue of the function $g(m, n; 0)$ at its (potential) pole $z = 0$ is equal to $C(m, n; 0)$ (cf. (4.1)), and furthermore we have

$$\begin{aligned} g(m,n;s+1) &= g(m-1,n;s) + g(m+1,n;s) \qquad (m \geq 1), \\ g(0,n;s+1) &= 2ug(0,n;s) + tg(1,n;s). \end{aligned} \tag{4.5}$$

Comparing (4.5) with (4.1), we deduce that $C(m, n; s)$ is equal to the residue at $z = 0$ of the function $g(m, n; s)$, and we further deduce that if

$$r = \left\lfloor \tfrac{1}{2}(s-m-n) \right\rfloor \qquad \text{(integer part or floor function)}, \tag{4.6a}$$

if b_i and c_i are as defined in (4.3), and if also $s - m - n = 2r + 1$ is odd then, in terms of binomial coefficients,

$$C(m,n;s) = C(n,m;s) = \sum_{i=0}^{r} \binom{s}{r-i} b_{i+1} \qquad (s-m-n = 2r+1), \tag{4.6b}$$

while if $s - m - n = 2r$ is even, then $C(n, m; s) = C(m, n; s)$, where

$$C(m,n;s) = \binom{s}{r+m} + \binom{s}{r}(t-1) + \sum_{i=1}^{r} \binom{s}{r-i} c_i \qquad (s-m-n = 2r). \tag{4.6c}$$

The sums in (4.6) are empty if $m + n \geq s$.

5. The cases $u = 0$ and $xy = 1$. If $u = 0$ we have (4.4) and from (3.5) and (4.6) we obtain, in terms of binomial coefficients, the polynomial expression (5.1) below. In fact

$$A^{(s)}_{m,n}(0; b, c; p, q) = Z^{(s)}_{m,n}(0; x, y; p, q) = 0 \quad \text{if } s - m - n \text{ is odd} \tag{5.1a}$$

while, if $s - m - n = 2r$ is even, then

$$\begin{aligned} x^{\delta_{n,0}} y^{\delta_{m,0}} \left(\frac{p}{q}\right)^{(m-n)/2} & (pq)^{-s/2} Z^{(s)}_{m,n}(0; x, y; p, q) \\ = C^{(s)}_{m,n}(0; xy) &= \binom{s}{r+m} + \binom{s}{r}(xy - 1) \\ &\quad + xy(xy - 2) \sum_{i=1}^{r} \binom{s}{r-i}(xy - 1)^{i-1}. \end{aligned} \tag{5.1b}$$

The last member of (5.1b) can alternatively be expressed as

$$\binom{s}{r+m} + \binom{s}{r}(xy - 1) + xy(xy - 2) \sum_{i=0}^{r-1} \binom{s}{i}(xy - 1)^{r-i-1}. \tag{5.1c}$$

If $xy = 1$ then $C^{(s)}_{m,n}(u; 1)$ is a polynomial in u:

$$(2u)^{-1} C^{(s)}_{m,n}(u; 1) = \binom{s}{r} + (4u^2 - 1) \sum_{i=1}^{r} \binom{s}{r-i}(2u)^{2i-2}$$

$$(s - m - n = 2r + 1),$$

$$C^{(s)}_{m,n}(u; 1) = \binom{s}{r+m} + (4u^2 - 1) \sum_{i=1}^{r} \binom{s}{r-i}(2u)^{2i-2}$$

$$(s - m - n = 2r). \tag{5.2}$$

6. The case $u^2 \neq 1 - t$. Trigonometric and polynomial expressions, etc. For $t < 1$ define

$$\begin{aligned} d_j &= (u \sec\theta)^{j-1} \frac{\sin j\theta}{\sin\theta} \quad \text{where} \quad \cos^2\theta = u^2/(1 - t) < 1, \\ d_j &= (u\, \mathrm{sech}\,\phi)^{j-1} \frac{\sinh j\phi}{\sinh\phi} \quad \text{where} \quad \cosh^2\phi = u^2/(1 - t) > 1, \end{aligned} \tag{6.1}$$

observing ((6.4), (6.5)) that these definitions are equivalent. From (6.1) and (4.3) we see that $d_0 = 0$, $d_1 = 1$, $a_1 = 2ut$, $a_2 = 4u^2t + t^2 - 2t$, and the sequences $\langle d_j \rangle$ and $\langle a_j : j \geq 1 \rangle$ satisfy the linear recurrence relation:

$$x_{j+2} = 2ux_{j+1} + (t - 1)x_j, \tag{6.2}$$

also satisfied by the sequence $\langle x_j \rangle = \langle a_j - t(d_{j+1} - d_{j-1}) : j \geq 1 \rangle$.

Since $x_1 = x_2 = 0$ it follows from (6.2) that, for all $j \geq 1$, $a_j = t(d_{j+1} - d_{j-1})$ and consequently, from (4.3),

$$b_j = t(d_{2j} - d_{2j-2}) \quad \text{and} \quad c_j = t(d_{2j+1} - d_{2j-1}) \qquad (j \geq 1). \tag{6.3}$$

An algebraic expression for d_j is provided by

$$d_j = \frac{\left(u + \sqrt{u^2 + t - 1}\right)^j - \left(u - \sqrt{u^2 + t - 1}\right)^j}{2\sqrt{u^2 + t - 1}} \tag{6.4}$$

and an explicit polynomial expression by

$$d_j = \sum_{k=0}^{\lfloor \frac{1}{2}(j-1) \rfloor} \binom{j}{2k+1} u^{j-2k-1}(u^2 + t - 1)^k \qquad (j = 0, 1, 2, \dots). \tag{6.5}$$

It may be observed that (6.5), (6.3) agrees with (4.4) if $u = 0$.

Equations (6.3), (4.6) and (3.5), with (6.1), (6.4) or (6.5), provide several alternative expressions for $Z_{m,n}^{(s)}(u; x, y; p, q)$ and hence (by (3.2)) for $A_{m,n}^{(s)}(a; b, c; p, q)$.

7. Explicit polynomials in the general case. We define

$$r = \left\lfloor \tfrac{1}{2}(s - m - n) \right\rfloor \text{ (cf. (4.6a)) and } k = k(i, j) = r - i - j \qquad (i, j \geq 0). \tag{7.1a}$$

Collecting like powers of u and $u^2 + t - 1$ in (4.6), using (6.3) and (6.5), we find that, if $s - m - n = 2r + 1$ is odd, then

$$C_{m,n}^{(s)}(u; t) = \sum_{\substack{i+j+k=r \\ i,j,k \geq 0}} \binom{s}{k} \binom{2i + 2j + 2}{2j + 1} \frac{s + 1 - 2k}{s + 1 - k} t u^{2i+1} (u^2 + t - 1)^j, \tag{7.1b}$$

while if $s - m - n = 2r$ is even,

$$C_{m,n}^{(s)}(u; t) = \binom{s}{r + m} - \binom{s}{r} + \sum_{\substack{i+j+k=r \\ i,j,k \geq 0}} \binom{s}{k} \binom{2i + 2j + 1}{2j + 1} \frac{s + 1 - 2k}{s + 1 - k} t u^{2i} (u^2 + t - 1)^j. \tag{7.1c}$$

A detailed derivation of (7.1) is given in Section 14.

The value of $Z_{m,n}^{(s)}(u; x, y; p, q)$ follows immediately from (3.5), and the value of $A_{m,n}^{(s)}(a; b, c; p, q)$ is given in equation (7.2) below.

In terms of the Kronecker δ-function $\delta_{m,0}$ we define

$$m^* = 1 - \delta_{m,0} = (m > 0) = \min\{1, m\}, \qquad n^* = \min\{1, n\}. \tag{7.2a}$$

With $r = \lfloor \frac{1}{2}(s - m - n) \rfloor$, we deduce from (3.2), (3.5) and (7.1) that, if $s - m - n$

$= 2r + 1$ is odd, then

$$A^{(s)}_{m,n}(a;\, b, c;\, p, q) = p^{n-n^*}q^{m-m^*}b^{n^*}c^{m^*}$$
$$\times \sum_{\substack{i+j+k=r\\ i,j,k\geq 0}} \binom{s}{k}\binom{2i+2j+2}{2j+1}\frac{s+1-2k}{s+1-k} \tag{7.2b}$$
$$\times a^{2i+1}(a^2+bc-pq)^j(pq)^k,$$

while if $s - m - n = 2r$ is even

$$A^{(s)}_{m,n}(a;\, b, c;\, p, q) = p^{r+n}q^{r+m}\left(\binom{s}{r+m} - \binom{s}{r}\right) + p^{n-n^*}q^{m-m^*}b^{n^*}c^{m^*}$$
$$\times \sum_{\substack{i+j+k=r\\ i,j,k\geq 0}} \binom{s}{k}\binom{2i+2j+1}{2j+1}\frac{s+1-2k}{s+1-k} \tag{7.2c}$$
$$\times a^{2i}(a^2+bc-pq)^j(pq)^k.$$

We observe that, for given m, n, s, p, q with $m + n + s$ odd and $mn \neq 0$, the value of the expression $(abc)^{-1}A^{(s)}_{m,n}(a;\, b, c;\, p, q)$ is expressible as a polynomial in the two variables a^2 and $(a^2 + bc - pq)$ which is symmetric in its two arguments. It would be interesting to find an explanation for this surprising phenomenon.

The sums in (7.1) and (7.2) are empty if $m + n > s$ and otherwise contain $\frac{1}{2}(r + 1)(r + 2)$ terms, which is less than $\frac{1}{8}(s + 3 - m - n)^2$ terms. In contrast with Section 3, the polynomial expressions (7.1), (7.2) and (7.3) below are valid, by analytic continuation, for all values of their arguments.

A particularly valuable case of (7.2c) (see Section 13 below) is

$$A^{(2r)}_{0,0}(0;\, b, c;\, p, q) = \sum_{k=0}^{r}\binom{2r}{k}\frac{2r+1-2k}{2r+1-k}(bc-pq)^{r-k}(pq)^k. \tag{7.3}$$

Figures 1, 2, and 3 list APL functions for evaluating the equations (7.1), (3.5), (7.2) and (7.3), with examples of their use. These functions illustrate the elegance of APL, and its power in handling intricate expressions without the necessity for looping or branching. The sequence of statements in the function *CMNS* corresponds in order exactly to the processes expressed in the formula (7.1), and a similar remark applies to the functions *ZPOWER*, *APOWER* and *SPECIALCASE*. The three Figures, as well as the Table in Section 13, were themselves automatically typeset, using APL programming to produce a TEX source file.

8. The special case $u = 0, m = n = 0$. An alternative approach. For $|\lambda| < 1/2$, $|t| < 1/2$ we define the matrix $K = [K_{m,n}]$ by

$$K = K(\lambda) = (I - \lambda B)^{-1}, \tag{8.1}$$

where B is defined in (3.3), observing that this inverse matrix is well-defined (e.g., as

```
     ∇C ← MNS  CMNS  UT ; S ; U ; T ; W ; R ; E ; J ; I ; L ; B ; K
[1]    ⍝                   THIS FUNCTION COMPUTES (7.1)
[2]    S ← 2 ↓ MNS                ⍝         SETTING UP
[3]    U ← 1 ↑ UT                 ⍝                THE
[4]    T ← 1 ↓ UT                 ⍝                 MAIN
[5]    W ← ¯1+T+U*2               ⍝                  VARIABLES
[6]    E ← 0=2| +/MNS             ⍝  E ← 1  IF  M+N+S  IS EVEN.
[7]    R ← ⌊ S-( +/MNS ) ÷2       ⍝  (7.1A)
[8]    J ← ( ⍳ 0 ⌈ 1+R ) -⎕IO     ⍝         ⎕IO  SHOULD BE  0.
[9]    I ← 1+2×J                  ⍝      J  GETS  0 , 1 , 2 , ..., R.
[10]   L ← ( I-E ) ∘.+ I          ⍝             THREE
[11]   B ← ( I×0 ) ∘.+ I          ⍝                 SQUARE
[12]   K ←  R-( J ∘.+ J )         ⍝  (7.1A)               ARRAYS.
[13]   ⍝
[14]   ⍝  WE NOW MOVE STEADILY THROUGH
[15]   ⍝                    EQUATIONS  (7.1B)  AND  (7.1C):
[16]   ⍝
[17]   C ← T×( K ! S )×( B ! L )            ⍝  BINOMIAL COEFFS
[18]   C ← C×1-K÷( S+1 -K )                 ⍝  RATIONAL FACTOR
[19]   C ← C×( U* I-E ) ∘.× W* J            ⍝   POWERS OF  U,W
[20]   C ← +/ +/ C                              ⍝  RESULT  (7.1B)
[21]   C ← C+E× -/ ( R+2 ↑ 1 ↑ MNS ) ! S        ⍝  RESULT  (7.1C)
     ∇
       4  6  21  CMNS  2  7
826955136
       3  5  20  CMNS  ¯2  ¯7
¯2963232
```

Fig. 1

```
     ∇Z  ← MNS  ZPOWER  UXYPQ ; XY ; PQ ; NM ; UT
[1]    ⍝                   THIS FUNCTION COMPUTES  (3.5)
[2]    XY ← 2 ↑ 1 ↓ UXYPQ                 ⍝  ALL
[3]    PQ ← 3 ↓ UXYPQ                     ⍝      THIS
[4]    NM ← ⌽2 ↑ MNS                      ⍝          EXPLAINS
[5]     → ( 0∊PQ , XY ) / FAIL            ⍝              ITSELF
[6]    Z  ← XY ×.* -(NM=0 )
[7]    Z  ← Z×( ÷/ PQ ) * ( -/NM) ÷2
[8]    Z  ← Z×( ×/ PQ ) * ( 2 ↓ MNS) ÷2
[9]    UT ← ( 1 ↑ UXYPQ ) , ( ×/ XY )     ⍝          EQUN  (3.2)
[10]   Z  ← Z×( MNS  CMNS  UT )           ⍝ PARENS UNNECRY
[11]    → 0
[12]  FAIL:    Z ← ' FAILS BECAUSE   0  =  P×Q×X×Y '
     ∇
       2  4  12  ZPOWER  3  4  5  1  2
191080064
       3  5  11  ZPOWER  3  4  ¯5  1  2
¯10861.16016
```

Fig. 2

```
∇Y ← MNS  APOWER  ABCPQ ; S ;NM; A ; BC ; PQ ; W; R ; E ; J ; I ; L ; B ; K
    ⍝                                    THIS FUNCTION COMPUTES (7.2)
    S ← 2↓MNS      ◇ NM← ⌽2↑MNS
    A ← 1↑ABCPQ ◇ BC ← 2↑1↓ABCPQ ◇ PQ ← 3↓ABCPQ
    W← (A*2)+(×/BC)-(×/PQ)          ◇  E ← 0=2|R ← S-(+/NM)
    R ← ⌊R÷2       ◇   I ← 1+2×J ← (⍳0⌈1+R)-⎕IO    ⍝  J ← 0 , 1 ,..., R
    L ← I ∘.+ I-E ◇   B ← I ∘.+ I×0      ◇  K ← R-( J ∘.+ J )     ⍝ (7.1A)
    Y ← (K!S)×(B!L)×1-K÷(S+1-K) ◇   Y ← Y×(W*J) ∘.× (A*I-E)
    Y ← +/ , Y×(×/PQ)*0⌈K ◇ Y ← Y×(PQ ×.* 0⌈NM-1)×(BC ×.* 1⌊NM)
    Y ← Y +   ( E × (PQ ×.* R+NM) × (-/(R+2↑1↑NM)!S) )
∇
    2 3 12 APOWER 1 2 3 4 5
761072640
    0 3 9 APOWER 4 3 2 1 0
1197192


∇A ← R   SPECIALCASE   BCPQ ; K ; Y ; W
    ⍝                      STARTS WITH  K ← 0 , 1 , ... , R  AND COMPUTES  (7.3)
    K ← (⍳1+R)-⎕IO    ◇   Y ← ×/2↓BCPQ    ◇   W← -/×/(2 2⍴BCPQ)
    A ←  +/  (K!2×R)  ×  (1+K÷K-1+2×R)  ×  (W*R-K)  ×  (Y*K)
∇
    7  SPECIALCASE  8 1 7 1
489517344
```

Fig. 3

an infinite series) and that

$$[K_{0,0}] = \sum_{s=0}^{\infty} \lambda^s B_{0,0}^{(s)}. \tag{8.2}$$

The first column $\langle k_m \rangle = [K_{m,0}]$ of the matrix K, in the present context $u = 0$, is bounded for small $|\lambda|$ and satisfies the relations

$$k_0 - \lambda t k_1 = 1, \qquad k_m = \lambda(k_{m-1} + k_{m+1}) \qquad (m \geq 1). \tag{8.3}$$

The unique bounded solution of (8.3) is given by

$$k_m = \frac{\alpha^m}{1 - \lambda t \alpha} \quad \text{where} \quad \alpha = \frac{1 - \sqrt{1 - 4\lambda^2}}{2\lambda} \tag{8.4}$$

so that $\alpha = 0$ if $\lambda = 0$. In particular, by rationalizing the denominator, we obtain

$$[K_{0,0}] = k_0 = \left(1 - \frac{t^2\lambda^2}{t-1}\right)^{-1}\left(1 - \frac{t}{2(t-1)}\left(1 - \sqrt{1 - 4\lambda^2}\right)\right). \tag{8.5}$$

Expanding (8.5) as a product of two power series in λ and referring to (8.2), (3.4) we deduce that $Z_{0,0}^{(s)}(0; x, y; p, q) = 0$ if s is odd, and otherwise $Z_{0,0}^{(2r)}(0; x, y; p, q) = (pq)^r B_{0,0}^{(2r)}(0; xy)$ where

$$(t-1)^r B_{0,0}^{(2r)}(0; t) = t^{2r} - \sum_{k=1}^{r} \binom{2k}{k} \frac{t^{2r+1-2k}}{4k-2} (t-1)^{k-1}. \tag{8.6}$$

It is difficult, in view of (8.6), to believe that $B_{0,0}^{(2r)}(0;\, t)$ is in fact a polynomial in t of degree r with nonnegative coefficients. This observation may be compared with the result of [**5**].

9. Definition of the functions F and G, and an identity relating them. The arguments of this section, which lead eventually to a conclusion conflicting with statements in the standard literature, may well be omitted on a first reading. The results (9.3) and (9.8) will be used in the next Section. Motivation for (9.1), (9.2) is provided immediately after (10.5).

Supposing the values of u, t to be given, we define two functions $F_m(z)$, $G_{m,n}(z)$ of the complex variable z, namely

$$F_m(z) = (z^{m+1} - z^{-m-1}) - 2u(z^m - z^{-m}) + (1-t)(z^{m-1} - z^{1-m}), \tag{9.1}$$

$$G_{m,n}(z) = \frac{zg(m,n;s)(z)}{(z+z^{-1})^s} = z^{n-m} + z^{n+m}h(z), \tag{9.2}$$

where the functions $h(z)$ and $g(z) = g(m, n; s)(z)$ are defined in (4.2). We note that

$$g(z) + z^{-2}g(z^{-1}) = z^{-1}(z+z^{-1})^s\big(G_{m,n}(z) + G_{m,n}(z^{-1})\big). \tag{9.3}$$

There is a remarkable identity (9.4) relating (9.1), (9.2), whose genesis is to be found, perhaps, in (4.5). It is readily verified, in fact, that we have

$$\begin{aligned}(1 - 2uz^{-1} + (1-t)z^{-2})(1 - 2uz + (1-t)z^2)\big(G_{m,n}(z) + G_{m,n}(z^{-1})\big)&\\ = -F_m(z)F_n(z).&\end{aligned} \tag{9.4}$$

We now define two even functions of θ, namely

$$k = k(u,t;\theta) = \frac{t-2}{t} + \frac{2u}{t}\sec\theta \tag{9.5}$$

and

$$f_m(\theta) = f_m(u,t;\theta) = \cos m\theta - k\frac{\sin m\theta}{\tan\theta} \tag{9.6}$$

observing that, in the special case $z = e^{i\theta}$, we have

$$\begin{aligned}F_m(e^{i\theta}) &= 2i(\sin(m+1)\theta - 2u\sin m\theta + (1-t)\sin(m-1)\theta)\\ &= 2it\sin\theta f_m(\theta)\end{aligned} \tag{9.7}$$

and, again in the case $z = e^{i\theta}$, the first factor of (9.4) is equal to $te^{-i\theta}(i\sin\theta - k\cos\theta)$, so that the product of the first two factors is equal to $t^2(\sin^2\theta + k^2\cos^2\theta)$. Hence, if $z = e^{i\theta}$, (9.4) can be written

$$(1 + k^2\cot^2\theta)\big(G_{m,n}(e^{i\theta}) + G_{m,n}(e^{-i\theta})\big) = 4f_m(\theta)f_n(\theta). \tag{9.8}$$

10. The representation of $C_{m,n}^{(s)}(u;t)$ as an integral. Although for computational purposes a finite sum such as (7.1) or (7.2) is clearly preferable to an integral, yet the

latter form of expression has considerable theoretical interest. We therefore define the integral

$$I_{m,n}^{(s)}(u;t) = \frac{2}{\pi}\int_0^{\pi} \frac{(2\cos\theta)^s}{1 + k^2\cot^2\theta} f_m(\theta) f_n(\theta)\, d\theta, \tag{10.1a}$$

where, as in (9.5) and (9.6), the even functions $k(u, t; \theta)$ and $f_m(\theta)$ are defined by

$$k = k(u,t;\theta) = \frac{t-2}{t} + \frac{2u}{t}\sec\theta \tag{10.1b}$$

and

$$f_m(\theta) = f_m(u,t;\theta) = \cos m\theta - k\frac{\sin m\theta}{\tan\theta} = \cos m\theta - \frac{t-2}{t}\,\frac{\sin m\theta}{\tan\theta} - \frac{2u}{t}\,\frac{\sin m\theta}{\sin\theta}. \tag{10.1c}$$

In this section we confine attention to the case where the point (t, u) lies inside the isosceles triangle defined by the inequalities

$$|u| < 1 - \tfrac{1}{2}t < 1. \tag{10.2}$$

Then the function $g(z) = g(m, n; s)(z)$ has no singularity inside or on the unit circle except at $z = 0$, so that the sum (4.6) for $C(m, n; s)$, which is equal to the residue at this pole, can be expressed as a contour integral round the unit circle. In fact we have, using (9.3),

$$\begin{aligned} C_{m,n}^{(s)}(u;t) &= \frac{1}{2\pi i}\oint g(z)\,dz = \frac{1}{2\pi i}\oint z^{-2}g(z^{-1})\,dz \\ &= \frac{1}{2\pi i}\oint (z+z^{-1})^s\left(G_{m,n}(z) + G_{m,n}(z^{-1})\right)\frac{dz}{2z}, \end{aligned} \tag{10.3}$$

provided that (10.2) holds.

Writing $z = e^{i\theta}$ and using (9.8) and (10.1) we obtain, if $|u| < 1 - \frac{1}{2}t < 1$,

$$C(m,n;s) = C_{m,n}^{(s)}(u;t) = I_{m,n}^{(s)}(u;t) = \frac{2}{\pi}\int_0^{\pi}\frac{(2\cos\theta)^s}{1+k^2\cot^2\theta} f_m(\theta) f_n(\theta)\,d\theta. \tag{10.4}$$

An immediate corollary of (10.4) is

$$C_{m,n}^{(s)}(u;t) = \frac{2}{\pi}\int_0^{\pi}\cos(m\theta + \eta(\theta))\cos(n\theta+\eta(\theta))(2\cos\theta)^s\,d\theta, \tag{10.5a}$$

where

$$\eta(\theta) = \tan^{-1}(k\cot\theta) \quad \text{and} \quad k = k(u,t;\theta) = \frac{t-2}{t} + \frac{2u}{t}\sec\theta. \tag{10.5b}$$

My first approach was based on the observation that $|h(e^{i\theta})| = 1$, so that we may write $h(e^{i\theta}) = e^{2i\eta}$. This approach yields (10.5) prior to (10.4), and the identity (9.4) is seen to be an elaborate version of the identity $\cos(x + y) + \cos(x - y) = 2\cos x \cos y$.

The formula (10.4), or (10.5), isolates in an unexpected way the separate contributions of the variables m, n, t, u, s. Similarly, the variables p, q, a, b, c are introduced in Section 3.

Since the integrand commonly takes values far in excess of the integral, the use of (10.4) or (10.5) can involve significant computational error.

11. The cases $t = 1 - u^2$, $u = 0$ and $t = 1$. The results of the preceding Section rely on the fact that the function $g(z)$ has no nonzero singularity inside the unit circle, and this condition obtains (in particular) if $1 - u^2 = t > 0$. If however $1 - u^2 = t < 0$ the function $g(z)$ has a (double) pole inside the unit circle at $z = u^{-1}$, and the value of the integral (10.1) includes, in addition to $C_{m,n}^{(s)}(u; 1 - u^2)$, the residue at this pole.

Since $1 + h(z) = t(1 - z^2)/(1 - 2uz + (1 - t)z^2)$, this residue is equal to that of the function

$$\frac{tz^{m+n-s-1}(1 + z^2)^s(1 - z^2)}{(1 - uz)^2} \tag{11.1}$$

and it is readily computed by writing $z = u^{-1} + h$ and expanding in powers of h up to the first power giving, in place of (10.4), if $t = 1 - u^2$,

$$C_{m,n}^{(s)}(u; 1 - u^2) = I_{m,n}^{(s)}(u; 1 - u^2) + \left(\max\{0, u^2 - 1\}\right)(u^2 + 1)^{s-1} \times \frac{(m + n)(u^4 - 1) - s(u^2 - 1)^2 - (u^2 + 1)^2}{u^{m+n+s+2}}. \tag{11.2}$$

If $u = 0$ and $m + n + s$ is odd, then clearly $C_{m,n}^{(s)}(0; t) = 0$. If $u = 0$ and $t^2 < 2t$ we have (10.4) and (10.5). If however $u = 0$ and $t^2 > 2t$, then the function g has two simple poles, with equal or opposite residues, inside the unit circle, so the appropriate modification of (10.4) is

$$C_{m,n}^{(s)}(0; t) = I_{m,n}^{(s)}(0; t) + \left(\max\{0, t^2 - 2t\}\right)\frac{\frac{1}{2}\left(1 + (-1)^{m+n+s}\right)t^s}{(t - 1)^{(m+n+s+2)/2}}. \tag{11.3}$$

Finally, if $t = 1$ and $4u^2 > 1$, we have to account for a single simple pole, yielding the result

$$C_{m,n}^{(s)}(u; 1) = I_{m,n}^{(s)}(u; 1) + \left(\max\{0, 4u^2 - 1\}\right)\frac{(4u^2 + 1)^s}{(2u)^{m+n+s+2}}. \tag{11.4}$$

12. Random walk and Brownian motion. A comparison with earlier results. Throughout this section we confine attention to a case of (1.1) which is

appropriate to the study of one-dimensional random walk with a single barrier, namely the case

$$p > 0, \qquad q > 0, \qquad p + q = 1 \tag{12.1a}$$

and for convenience we shall also assume that

$$b = p \quad \text{and} \quad c = 1. \tag{12.1b}$$

[This choice instead of $b = 1$, $c = q$ facilitates comparison with equation (16) of [**1**].]

Then (10.4) shows that, in the case

$$q - p > 2|a|\sqrt{q/p} \geq 0, \tag{12.2a}$$

we have

$$A_{m,n}^{(s)}(a; p, 1; p, q)$$

$$= \frac{2}{\pi} q^{\delta_{m,0}} \left(\frac{q}{p}\right)^{(m-n)/2} (4pq)^{s/2} \int_0^\pi \frac{\cos^s\theta \tan^2\theta}{k^2 + \tan^2\theta} f_m(\theta) f_n(\theta)\, d\theta, \tag{12.2b}$$

where

$$k = p - q + 2a \sec\theta$$

and

$$f_m(\theta) = \cos m\theta + (q - p)\frac{\sin m\theta}{\tan\theta} - 2a\frac{\sin m\theta}{\sin\theta}\sqrt{\frac{q}{p}}. \tag{12.2c}$$

This result is applicable only if $q > p$, but if $a = 0$ it can be extended to general p, q subject to (12.1). In fact, (11.3) yields

$$A_{m,n}^{(s)}(0; p, 1; p, q) = \tfrac{1}{2}(1 + (-1)^{m+n+s})(\max\{0, p - q\})\frac{q^{\delta_{m,0}}}{pq}\left(\frac{q}{p}\right)^m$$

$$+ \frac{2}{\pi} q^{\delta_{m,0}} \left(\frac{q}{p}\right)^{(m-n)/2} (4pq)^{s/2} \tag{12.3a}$$

$$\times \int_0^\pi \frac{\cos^s\theta \tan^2\theta}{(p - q)^2 + \tan^2\theta} f_m(\theta) f_n(\theta)\, d\theta,$$

where

$$f_m(\theta) = \cos m\theta + (q - p)\frac{\sin m\theta}{\tan\theta}. \tag{12.3b}$$

It is remarkable (cf. Section 11) that (for given parity of $m + n + s$) the first term on the right-hand side of (12.3a) is independent both of n and of s.

Comparing (16) of [**1**] with (1.1) above, we see that the case $p \geq q$ of (12.3) is addressed in [**1**], which deals with the question of Brownian motion. Equation (41) of [**1**], quoted also in [**2**] and [**3**], is the exact analogue of (12.3a) above. Unfor-

tunately, however, the expression in [**1**] corresponding to (12.3b) is erroneous in that the denominator of the last fraction is given as $\sin\theta$ instead of $\tan\theta$. In fact, in eight unnumbered equations on pages 377–8 of [**1**], affecting also (37), (38), (40) and (41), the functions $\sin\theta$, $\sinh\theta$ should be replaced, respectively, by $\tan\theta$, $\tanh\theta$. Referring to (12.2c), we observe that the function $\sin m\theta/\sin\theta$ does indeed play a part in the theory, but here we need $\sin m\theta/\tan\theta$.

13. Solutions to the problem in Section 2. Substituting (2.3) in (10.4), using (3.5) and (3.2), we deduce that $w_d(s) = 0$ if s is odd (as is otherwise obvious) and if $s = 2r$ is even we have

$$w_d(2r) = \frac{d(8d-4)^{r+1}}{4\pi}\int_0^{\pi}\frac{\cos^{2r}\theta\, d\theta}{d^2+(d-1)^2\cot^2\theta}. \tag{13.1}$$

A polynomial in d for $w_d(2r)$ is provided by (5.1c):

$$w_d(2r) = \binom{2r}{r}(2d-1)^r - 2(d-1)\sum_{k=0}^{r-1}\binom{2r}{k}(2d-1)^k \tag{13.2}$$

and another polynomial, obtained from (8.6), (3.4) is

$$w_d(2r) = (2d)^{2r} - \sum_{k=1}^{r}\frac{d}{2k-1}\binom{2k}{k}(2d-1)^k(2d)^{2r-2k}. \tag{13.3}$$

This expression (13.3) is the published solution to [**4**].

It would be interesting to give a direct proof of the fact that the two polynomials (13.2), (13.3) are equivalent and, in particular, that (13.3) is a polynomial in d of degree r.

At the computational level, (13.2) is clearly much superior to (13.1) and (13.3), but the simplest expression of all (I believe) is obtained from (7.3). The result is

$$w_d(2r) = \sum_{k=0}^{r}\binom{2r}{k}\frac{2r+1-2k}{2r+1-k}(2d-1)^k. \tag{13.4}$$

A table of $w_d(2r)$

	$r=$ 1	2	3	4	5	6	7
$d=$ 1 :	2	6	20	70	252	924	3432
2 :	4	28	232	2092	19864	195352	1970896
3 :	6	66	876	12786	197796	3183156	52718616
4 :	8	120	2192	44248	949488	21237168	489517344
5 :	10	190	4420	113950	3128140	89608780	2647358920

14. Appendix. A proof of (7.1). Write $w = u^2 + t - 1$ and suppose that $s - m - n = 2r + 1$ is odd. Then, from (4.6b), (6.3), and (6.5) we have

$$t^{-1}C^{(s)}_{m,n}(u;t) = \sum_{\lambda=0}^{r}\binom{s}{r-\lambda}(d_{2\lambda+2} - d_{2\lambda})$$

$$= \sum_{\lambda=0}^{\infty}\binom{s}{r-\lambda} \tag{14.1}$$

$$\times\left(\sum_{j=0}^{\lambda}\binom{2\lambda+2}{2j+1}u^{2(\lambda-j)+1}w^j - \sum_{j=0}^{\lambda-1}\binom{2\lambda}{2j+1}u^{2(\lambda-j)-1}w^j\right).$$

Switching the order of summation and ignoring terms with a zero binomial coefficient as factor, we write $\lambda = j + i$ in the positive sum and $\lambda = j + i + 1$ in the negative sum, and we also write (7.1a) $k = r - i - j$. This gives

$$t^{-1}C^{(s)}_{m,n}(u;t) = \sum_{j=0}^{\infty} w^j \sum_{i=0}^{\infty} u^{2i+1}\binom{2i+2j+2}{2j+1}\left(\binom{s}{k} - \binom{s}{k-1}\right), \tag{14.2}$$

immediately yielding (7.1b).

The proof of (7.1c) is entirely similar.

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UNSOLVED PROBLEMS

EDITED BY RICHARD K. GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

Further Problems on Partitions

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The main question we wish to consider is this:

For what sets of positive integers S and T is $p(S, n) = p(T, n-1)$ for all $n \geqq 1$ (where $p(S, n)$ is the number of partitions of n into elements of S)?

Intuitively I expected that apart from the trivial case $S = T = \{1\}$ there would be no solutions. I was quite surprised to discover that two identities of Gessel and Stanton [**4**; p. 196, eqs. (7.13) and (7.15)] imply that a solution is

$$\begin{aligned} S &= \{n \mid n \text{ odd or } n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}\} \\ T &= \{n \mid n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}\}. \end{aligned} \tag{1}$$

An examination of the Gessel-Stanton identities in [**2**] provides a method that can easily be adapted to show that

$$\begin{aligned} T = \{n \mid n \equiv {} & \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \\ & \pm 14, \pm 15, \pm 16, \pm 17, \pm 19 \pmod{40}\} \\ S = \{n \mid n \equiv {} & \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \\ & \pm 11, \pm 13, \pm 15, \pm 16, \pm 19 \pmod{40}\} \end{aligned} \tag{2}$$

is also a solution of our problem.

Question 1. Are there other pairs S and T besides those already listed such that $p(S, n) = p(T, n-1)$ for all $N \geqq 1$?

Of course we may consider more generally the identity $p(S, n) = p(T, n-a)$ for $n \geqq a$ with a fixed. For $a = 2$ there are at least two nontrivial solutions: First

$$\begin{aligned} S &= \{n \mid n \equiv \pm 2 \pmod 5 \quad \text{or} \quad n \equiv \pm 11 \pmod{55}\} \\ T &= \{n \mid n \equiv \pm 1 \pmod 5 \quad \text{or} \quad n \equiv \pm 22 \pmod{55}\}; \end{aligned} \tag{3}$$

*Partially supported by National Science Foundation Grant DMS 86503324.

and

$$S = \{n \mid n \equiv \pm 2, \pm 3, \pm 5, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13, \pm 15, \pm 17 \pmod{40}\}$$
$$T = \{n \mid n \equiv \pm 1, \pm 3, \pm 5, \pm 7, \pm 8, \pm 10, \pm 12, \pm 13, \pm 15, \pm 17, \pm 18, \pm 19 \pmod{40}\}. \tag{4}$$

The pair in (3) was given by D. Bressoud in his Ph.D. thesis [3; p. 178] as a corollary of an identity of Ramanujan.

Since (1), (2), and (4) can all be derived from one master identity we sketch their proof. Let

$$F_{M,a} = \prod_{n=1}^{\infty} \frac{(1 + q^{Mn-a})(1 + q^{M(n-1)+a})}{(1 - q^{Mn-a})(1 - q^{M(n-1)+a})}.$$

Then by Jacobi's triple product identity [1; p. 21]

$$F_{M,a} \pm 1 = \frac{\sum_{\lambda=-\infty}^{\infty} q^{M\binom{\lambda}{2}+a\lambda} \pm \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{M\binom{\lambda}{2}+a\lambda}}{\prod_{n=1}^{\infty} (1 - q^{Mn})(1 - q^{Mn-a})(1 - q^{M(n-1)+a})}.$$

Consequently,

$$\begin{aligned} \frac{F_{M,a} + 1}{2} &= \frac{\sum_{\lambda=-\infty}^{\infty} q^{M\lambda(2\lambda-1)+2a\lambda}}{\prod_{n=1}^{\infty} (1 - q^{Mn})(1 - q^{Mn-a})(1 - q^{M(n-1)+a})} \\ &= \prod_{n=1}^{\infty} \frac{(1 + q^{4M(n-1)+M+2a})(1 + q^{4Mn-M-2a})(1 - q^{4Mn})}{(1 - q^{Mn})(1 - q^{Mn-a})(1 - q^{M(n-1)+a})}, \\ \frac{F_{M,a} - 1}{2} &= \frac{\sum_{\lambda=-\infty}^{\infty} q^{M\binom{2\lambda+1}{2}+a(2\lambda+1)}}{\prod_{n=1}^{\infty} (1 - q^{Mn})(1 - q^{Mn-a})(1 - q^{M(n-1)+a})} \\ &= q^a \prod_{n=1}^{\infty} \frac{(1 + q^{4M(n-1)+M-2a})(1 + q^{4Mn-M+2a})(1 - q^{4Mn})}{(1 - q^{Mn})(1 - q^{Mn-a})(1 - q^{M(n-1)+a})}. \end{aligned}$$

Pair (1) is the case $M = 4$, $a = 1$; pair (2) is the case $M = 5$, $a = 1$, and pair (4) is the case $M = 5$, $a = 2$. In each instance we have after algebraic simplification

$$\frac{F_{M,a} - 1}{2} = q^a \prod_{n \in T} \frac{1}{1 - q^n},$$

and

$$\frac{F_{M,a}+1}{2} = \prod_{n \in S} \frac{1}{1-q^n}.$$

Hence in each case

$$\prod_{n \in S} \frac{1}{1-q^n} = 1 + q^a \prod_{n \in T} \frac{1}{1-q^n}, \tag{5}$$

and so $p(S, n) = p(T, n-a)$ follows immediately. The pair in (3) is harder to handle and was first treated by L. J. Rogers [5].

Question 2. Apart from the examples just given, for what pairs of sets of positive integers S and T is it true that $p(S, n) = p(T, n-a)$ for all $n \geqq a$ (where a is fixed)?

The "improbability" of equation (5) suggests that it is always some sort of identity involving modular forms. This naturally suggests:

Question 3. Are there any instances of (5) in which the infinite products appearing there are not essentially modular forms?

Question 4. For each pair S and T which answers Question 2 (or 1) can a bijection be found between the partitions of n into elements of S and the partitions of $n-a$ into elements of T?

It should be pointed out that every solution of (5) can be trivially extended to an infinite family of solutions by the substitution $q \to q^N$. Thus the answers to Questions 1 and 2 that we would like are other than these trivial dilations of known answers.

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The Mathematician's Dictionary

What mathematical concept can be defined as:

Contribute money to an organization that promotes the health of the mind?

(See bottom of page 474)

NOTES

EDITED BY CAROL G. CRAWFORD, RICHARD LIBERA, AND ANITA E. SOLOW

All the Way With Wirtinger: a Short Proof of Bonnesen's Inequality

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Wirtinger's inequality (given below) is a useful tool in the analytic treatment of plane convex bodies (ovals). The venerable isoperimetric inequality, for example, is an easy consequence (see [**4**]). Wirtinger's inequality can be used to derive the more general (planar) Brunn-Minkowski inequality (see [**1**], p. 115). Below, we shall see that Bonnesen's refinement of the Brunn-Minkowski inequality also follows easily from Wirtinger's inequality. For the small amount of mathematical ammunition needed, I recommend the lovely paper [**2**], which also contains an eminently readable integral-geometric proof of Bonnesen's result.

Let f be 2π-periodic and absolutely continuous with $f' \in L^2[0, 2\pi]$. Wirtinger's inequality says that if $\int_0^{2\pi} f(\theta)\, d\theta = 0$, then

$$\int_0^{2\pi} f^2(\theta)\, d\theta \leqslant \int_0^{2\pi} (f'(\theta))^2\, d\theta,$$

with equality exactly when $f(\theta) = A\cos(\theta - \theta_0)$. By far the quickest proof proceeds via the following Fourier series argument. If a_n are the (complex) Fourier coefficients of f, then those of f' are ina_n. Applying Parseval's identity to f' and remembering that

$$a_0 = (1/2\pi)\int_0^{2\pi} f = 0,$$

we get

$$(1/2\pi)\int_0^{2\pi} (f')^2 = \sum_{-\infty}^{\infty} n^2 |a_n|^2 \geqslant \sum_{-\infty}^{\infty} |a_n|^2 = (1/2\pi)\int_0^{2\pi} f^2.$$

The case of equality is clear.

Let K be an oval. Recall that the support function p of K is defined as follows: the line $x\cos\theta + y\sin\theta = p(\theta)$ is a supporting line for K and K is contained in the half-plane $x\cos\theta + y\sin\theta \leqslant p(\theta)$. The quadratic form

$$Q(p) = (1/2)\int_0^{2\pi} p^2 - (p')^2$$

gives the area of K. The associated bilinear form

$$B(p_1, p_2) = (1/2)\int_0^{2\pi} p_1 p_2 - p_1' p_2' = A_{12}$$

is the so-called mixed area of K_1 and K_2, where p_i is the support function of K_i (all

of this may be found in [2]). The quantity A_{12} plays a crucial role in the metric theory of ovals. For example, if K_2 is the unit disc centered at 0, $p_2(\theta) \equiv 1$, and

$$B(p_1, p_2) = (1/2)\int_0^{2\pi} p_1 = L/2,$$

where L is the length of the boundary of K_1 (Cauchy's formula). The famous Brunn-Minkowski Theorem (see [2] or [3]) says simply that

$$\Delta = A_{12}^2 - A_1 A_2 \geqslant 0,$$

where A_i is the area of K_i. Equality holds precisely when $K_1 = cK_2 + \vec{v}$, $c > 0$, $\vec{v} \in R^2$. For the case above,

$$\Delta = \left(L^2/4\right) - \pi A \quad (A \text{ is the area of } K_1)$$

so that $\Delta \geqslant 0$ is just the classical isoperimetric inequality.

Let

$$R = \min\{t > 0\text{: some translate of } tK_1 \text{ contains } K_2\},$$

and

$$r = \max\{t > 0\text{: some translate of } tK_1 \text{ is contained in } K_2\}.$$

Bonnesen's refinement of the Brunn-Minkowski inequality gives a lower bound for Δ in terms of the intuitive geometric parameters R and r.

BONNESEN'S THEOREM. $\sqrt{\Delta} \geqslant \dfrac{A_1}{2}(R - r)$.

We will assume that each supporting line to K_i intersects ∂K_i (the boundary) in just one point. The general case follows from a standard approximation argument. We will need a geometric lemma whose proof will be immediate to the reader as soon as he draws a picture.

LEMMA. *Suppose L_1 and L_2 are ovals with $L_2 \subseteq L_1$ and each supporting line to L_1 intersects ∂L_1 just once. Assume also that for no $t < 1$ does any translate of tL_1 contain L_2. Let a parallel pair of supporting lines to L_1 separate ∂L_1 into the open arcs α_1 and α_2. Then $\partial L_1 \cap \partial L_2$ cannot be wholly contained in just one of the α's.*

Proof of Bonnesen's Theorem. Define

$$q(\xi) = Q(\xi p_1 - p_2) = \xi^2 Q(p_1) - 2\xi B(p_1, p_2) + Q(p_2),$$

and let the two positive roots of q be R_1 and r_1, so that

$$\sqrt{\Delta} = \frac{Q(p_1)}{2}(R_1 - r_1).$$

Note that $\xi \in [r_1, R_1]$ if and only if $q(\xi) \leqslant 0$. Consequently we need only prove

$R \leqslant R_1$ and $r \geqslant r_1$. We prove only the first inequality, the second following upon interchanging K_1 with K_2. By translation invariance of everything in sight, we may assume $RK_1 \supseteq K_2$, so that

$$f(\theta) = Rp_1(\theta) - p_2(\theta) \geqslant 0.$$

If we now apply the lemma with $L_1 = RK_1$ and $L_2 = K_2$, we see that every closed interval of length π must contain a zero of f.

We need only show that $Q(f) \leqslant 0$. Supposing for convenience that $f(0) = f(2\pi) = 0$ (by rotating), let θ_1 be the largest zero of f for which $\theta_1 \leqslant \pi$ and θ_2 be the smallest zero of f for which $\theta_2 \geqslant \pi$. Then $0 \leqslant \theta_2 - \theta_1 \leqslant \pi$. Now it is enough to show that for each interval

$$I_1 = [0, \theta_1], \quad I_2 = [\theta_1, \theta_2], \quad \text{and} \quad I_3 = [\theta_2, 2\pi],$$

we have

$$\int_{I_j} f^2 - (f')^2 \leqslant 0.$$

Consider, for example I_1. Define $g(\theta)$ by

$$g(\theta) = \begin{cases} f(\theta), & \text{if } 0 \leqslant \theta \leqslant \theta_1, \\ 0, & \text{if } \theta_1 \leqslant \theta \leqslant 2\pi - \theta_1, \\ -f(2\pi - \theta), & \text{if } 2\pi - \theta_1 \leqslant \theta \leqslant 2\pi. \end{cases}$$

Then $\int_0^{2\pi} g = 0$, and Wirtinger's inequality gives

$$0 \geqslant \int_0^{2\pi} g^2 - (g')^2 = 2\int_{I_1} f^2 - (f')^2.$$

The argument for I_2 and I_3 is identical except for notation.

Finally, the reader wishing to pursue the subject further is encouraged to see **[5]**, which contains many inequalities of Bonnesen type as well as generalizations and some history.

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When is a Free Union of Regular Measures Regular?

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Let X be a Hausdorff topological space. By a *regular Borel measure* on X we mean a measure μ defined on the Borel subsets $\mathscr{B}$ of X with these properties:

(1) $\mu(K) < \infty$ for each compact set K;

(2) $\mu(V) = \sup\{\mu(K)\colon K \subset V, K \text{ compact}\}$ for each open set V;

(3) $\mu(B) = \inf\{\mu(V)\colon V \supset B, V \text{ open}\}$ for each $B \in \mathscr{B}$.

Regular Borel measures play a fundamental role in analysis, for by the Riesz representation theorem, they represent the positive linear functionals on the continuous real functions with compact support, defined on a locally compact space (see, e.g. **[1]**, **[2]**, or **[3]**). Now if $\{X_i\colon i \in I\}$ is a family of disjoint Hausdorff spaces, we can make their union X a Hausdorff space by declaring $V \subset X$ to be open provided that for each $i \in I$, $V \cap X_i$ is open in X_i. Notice that this topology, called the *free union* of $\{X_i\colon i \in I\}$, makes each X_i clopen in X. Suppose for each $i \in I$, μ_i is a measure on X_i. A natural way to try to define a measure on X is as follows:

$$\mu(A) = \sup\left\{\sum_{i \in F} \mu_i(A \cap X_i)\colon F \subset I,\ F \text{ finite}\right\}.$$

We abbreviate this supremum by $\sum_{i \in I}\mu_i(A \cap X_i)$ in the sequel. Since the set of subsets of X whose intersection with each X_i is a Borel set in X_i contains the open subsets of X and forms a sigma algebra, μ is at least defined for each Borel subset of X. As luck would have it, μ actually is a measure on the Borel subsets of X (show μ is finitely additive and continuous from below). The measure μ is called the *free union* of $\{\mu_i\colon i \in I\}$.

Unfortunately, the regularity of each μ_i does not guarantee regularity for μ. For example, if the family $\{X_i\colon i \in I\}$ consists of uncountably many copies of the line, each equipped with Lebesgue measure, the free union measure fails to be regular. In particular, property (3) fails. To see this, pick for each $i \in I$ a point x_i in X_i. Since points in X_i have μ_i-measure zero, the μ-measure of the closed set $\{x_i\colon i \in I\}$ must be zero. However, if V is an open superset of $\{x_i\colon i \in I\}$, then for each i $\mu_i(V \cap X_i) > 0$. Since I is uncountable, there exists $\varepsilon > 0$ for which $\mu_i(V \cap X_i) > \varepsilon$ for infinitely many i, and we obtain $\mu(V) = \infty$.

The purpose of this note is to display necessary and sufficient conditions for the regularity of a free union of regular Borel measures. But first, we need to recall the notion of support of a Borel measure.

whenever $i \notin M$. Finally, let V be the union of these two subsets of X:

$$\bigcup\{V_i\colon i \in M\} \quad \text{and} \quad \bigcup\{X_i - \operatorname{supp}\mu_i\colon i \notin M\}.$$

Evidently, V is an open superset of B. Since the complement of a support has measure zero, we have

$$\begin{aligned}\mu(V) &= \mu\Big(V \cap \bigcup_{i\in M} X_i\Big) + \mu\Big(V \cap \bigcup_{i\notin M} X_i\Big)\\ &= \sum_{i\in M} \mu_i(V \cap X_i) + \sum_{i\notin M} \mu_i(X_i - \operatorname{supp}\mu_i)\\ &\le \sum_{i\in M} \mu_i(B \cap X_i) + \varepsilon 2^{-i} + 0\\ &\le \mu(B) + \varepsilon,\end{aligned}$$

completing the proof.

It is obvious that the free union X of a family of locally compact Hausdorff spaces $\{X_i\colon i \in I\}$ is again a locally compact Hausdorff space. Thus, if μ_i is a regular Borel measure on X_i for each $i \in I$, then the positive linear functional on the continuous functions on X with compact support determined by the induced free union measure is also often represented by an additional (unique) regular measure. As pointed out to us by A. van Roij and R. Simeonov, it is easy to see that this regular measure assigns infinity to any set A where $\{i \in I\colon A \cap \operatorname{supp}\mu_i \neq 0\}$ is uncountable, and agrees with the free union measure otherwise.

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Real Division Algebras of Dimension > 1 Contain $\mathbb{C}$

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1. Introduction. A division algebra is a real, finite-dimensional vector space K with a bilinear multiplication $\mu\colon K \times K \to K$ such that

(1) there are no zero divisors (v, w nonzero implies $\mu(v, w) \neq 0$),

(2) there is a two-sided identity for μ, i.e., there is $e \in K$ such that $\mu(e, v) = \mu(v, e) = v$ for all $v \in K$.

whenever $i \notin M$. Finally, let V be the union of these two subsets of X:

$$\bigcup\{V_i: i \in M\} \quad \text{and} \quad \bigcup\{X_i - \operatorname{supp}\mu_i: i \notin M\}.$$

Evidently, V is an open superset of B. Since the complement of a support has measure zero, we have

$$\begin{aligned}\mu(V) &= \mu\Big(V \cap \bigcup_{i \in M} X_i\Big) + \mu\Big(V \cap \bigcup_{i \notin M} X_i\Big)\\ &= \sum_{i \in M} \mu_i(V \cap X_i) + \sum_{i \notin M} \mu_i(X_i - \operatorname{supp}\mu_i)\\ &\le \sum_{i \in M} \mu_i(B \cap X_i) + \varepsilon 2^{-i} + 0\\ &\le \mu(B) + \varepsilon,\end{aligned}$$

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Real Division Algebras of Dimension > 1 Contain C

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(1) there are no zero divisors (v, w nonzero implies $\mu(v, w) \neq 0$),

(2) there is a two-sided identity for μ, i.e., there is $e \in K$ such that $\mu(e, v) = \mu(v, e) = v$ for all $v \in K$.

We say K has dimension n if K is a real n-dimensional vector space.

The purpose of this note is to give a simple topological proof of the following assertion.

THEOREM. *If* $\dim K > 1$, *then* K *has a subalgebra isomorphic to* $\mathbb{C}$.

The only external result we use is that every continuous map $g\colon S^{2n} \to S^{2n}$, $n = 0, 1, \ldots$ (S^{2n} the unit sphere in R^{2n+1}) has either a fixed point or sends some point to its antipode [3]. As an immediate corollary it will follow that the only 2-dimensional real division algebra is $\mathbb{C}$.

The first proof of this result that we are aware of is by Albert [**1**]. For a lovely topological proof using homology groups of spheres and the Künneth formula see Yang, Theorem 1, [**4**].

In all that follows we shall suppose K is identified with R^n in such a way that $e \to e_1$ where $\{e_1, \ldots, e_n\}$ is the usual orthonormal basis for R^n and we view μ as a bilinear multiplication on R^n. Also, when not needed for emphasis, we suppress the μ and write simply $v \cdot w$ for $\mu(v, w)$. Note that because of the linearity and absence of zero divisors, for $0 \neq v \in R^n$, $\mu(v, \ )$ and $\mu(\ , v)$ are both elements of $\mathrm{GL}(n, R)$, the non-singular $n \times n$ matrices over R. We let l denote the association $\mu(v, \ )$ with an element of $\mathrm{GL}(n, R) \cup \{[0]\}$ where $\mu(0, \ )$ is the zero matrix, [0], i.e., $l(v)(w) = \mu(v, w)$. Observe that linearity of the multiplication in the first entry implies that

$$l\colon R^n \to \mathrm{GL}(n, R) \cup \{[0]\} \subseteq R^{n^2}$$

is linear and consequently continuous.

Finally note that to prove the theorem it suffices to find a $v \in R^n$ such that $v \cdot v = -e_1$, for then the map

$$\tau\colon \mathbb{C} \to R^n, \qquad \tau(a + bi) = ae_1 + bv$$

is easily confirmed to be the required algebra isomorphism. So we prove the following statement.

LEMMA. *If* $n > 1$ *there exists* $v \in R^n$ *such that* $v \cdot v = -e_1$.

2. Proof of Lemma.

SUBLEMMA 1. *If* $n > 1$, *then* n *is even.*

Proof. $l(\pm e_1) = \pm I_n$, where I_n is the $n \times n$ identity matrix. If n is odd, then $\det(\pm I_n) = \pm 1$, so taking a path $\phi\colon [-1, 1] \to R^n$ with $\phi(\pm 1) = \pm e_1$ and $\phi(t) \neq 0 \in R^n$ for all $t \in [-1, 1]$, we have

$$\det(l(\phi(\pm 1))) = \pm 1 \quad \text{and} \quad \det(l(\phi(t))) \neq 0 \quad \text{for all } t \in [-1, 1].$$

But the Intermediate Value Theorem applied to the continuous function $\det(l(\phi(\ )))\colon [-1, 1] \to R$ says there must be $t \in (-1, 1)$ with $\det(l(\phi(t))) = 0$. Contradiction! QED

Given $v \in R^n - \{0\}$, since $l(v) \in \mathrm{GL}(n, R)$ there exists a unique $v' \in R^n$ with

$$e_1 = l(v)(v') = \mu(v, v').$$

So the association $v \to v'$ is a well defined function which we denote by f, $\mu(v, f(v)) = e_1$ for all $v \in R^n - \{0\}$. If $V = \{R^n - te_1 \colon t \in R\}$, then it is a simple matter to confirm that f is a bijection from V to itself. In all that follows f will denote this restricted function from V to V.

SUBLEMMA 2. *f is continuous.*

Proof. We've shown above that

$$l \colon R^n \to \mathrm{GL}(n, R) \cup \{[0]\} \subseteq R^{n^2}$$

is continuous. Now it is well known that the inversion map on $\mathrm{GL}(n, R)$, $A \to A^{-1}$, is continuous and so is the map from $\mathrm{GL}(n, R) \to R^n$, $A \to Ae_1$. It remains to observe that for $v \in V$, $f(v) = l(v)^{-1}(e_1)$ and to conclude that

$$v \to l(v) \to l(v)^{-1} \to l(v)^{-1}(e_1) = f(v)$$

as a composition of continuous maps is continuous. QED.

Now observe that f takes rays in V to rays in V, since, for $t > 0$ in R, $f(tu) = (1/t)f(u)$. So we restrict our attention to the action of f on S^{n-1} minus the points $\pm e_1$, i.e., define

$$\tilde{f} \colon S^{n-1} - \{\pm e_1\} \to S^{n-1} - \{\pm e_1\}$$

via: $\tilde{f}(v) =$ that point on S^{n-1} cut out by the ray $tf(v)$, $0 < t \in R$. Finally we further restrict our domain to the $n-2$ sphere, $S^{n-2} = S^{n-1} \cap e_1^{\perp}$ and we define $g \colon S^{n-2} \to S^{n-2}$ via: $g(v) =$ that point in S^{n-2} cut out by the great semi-circle arc through $\pm e_1$ and $\tilde{f}(v)$. (See Fig. 1.)

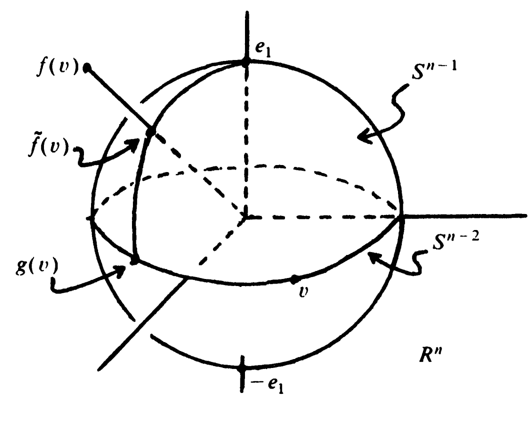

FIG. 1

As f and the two projections (from V to $S^{n-1} - \{\pm e_1\}$ and from $S^{n-1} - \{\pm e_1\}$ to S^{n-2}) are all continuous, g is continuous. But n is even hence $n - 2$ is even, so we conclude there is a $v \in S^{n-2}$ such that $g(v) = \pm v$ (here we call on our external result). This implies that $f(v)$ is an element of the two-dimensional subspace P spanned by v and e_1,

$$f(v) = \alpha v + \beta e_1, \quad 0 \neq \alpha, \quad \beta \in R.$$

Hence,

$$e_1 = v \cdot (\alpha v + \beta e_1) = \alpha v \cdot v + \beta v \quad \text{so} \quad v \cdot v = -\beta\alpha^{-1}v + \alpha^{-1}e_1 \in P$$

and we conclude $\mu|_P$ is a closed operation. It is now easy to verify that $\mu|_P$ is in fact associative, commutative and endows P with a field structure. Multiplicative inverses exist for all $0 \neq w \in P$ since $\mu|_p(w, \ ): P \to P$ is a nonsingular linear map hence surjective. (Note that if n was 2 to begin with, all of the above is true but unnecessary.)

Now let $C = \{\cos\theta e_1 + \sin\theta v : 0 \leqslant \theta \leqslant \pi\}$ and $h: C \to P - \{0\}$ be defined by $h(w) = w \cdot w = w^2$ for $w \in C$. Since (i) h is continuous (easy to see if h is viewed as a function of θ), (ii) $h(\pm e_1) = e_1$, and (iii) $a^2 = b^2$ implies $(a - b)(a + b) = 0$, which implies $a = \pm b$ for $a, b \in P$, we conclude that Image h is a simple closed curve in $P - \{0\}$. In fact we can say more however, i.e., h is injective as a map from $C - \{e_1\}$ to rays from the origin in P, for if $h(w_1) = t_1 x$ and $h(w_2) = t_2 x$, where x is a unit vector in P and t_1, t_2 are positive real numbers, then

$$\left(\frac{1}{\sqrt{t_1}} w_1\right)^2 = \left(\frac{1}{\sqrt{t_2}} w_2\right)^2$$

and consequently

$$\frac{1}{\sqrt{t_1}} w_1 = \pm \frac{1}{\sqrt{t_2}} w_2;$$

but this is impossible for distinct w_1 and w_2 in $C - \{e_1\}$. Therefore we conclude that

$$\tilde{h}: C - \{e_1\} \to S^1 \subseteq P, \quad \tilde{h}(w) = \frac{w^2}{\|w^2\|},$$

is bijective and there is a $w \in C$ such that $w^2 = -te_1$ where $t = \|w^2\| > 0$. So

$$\frac{1}{\sqrt{t}} w \in P \subseteq R^n$$

satisfies

$$\left(\frac{1}{\sqrt{t}} w\right)^2 = \frac{1}{t} w^2 = -e_1.$$

EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

The Place of $\ln x$ Among the Powers of x

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Many calculus texts introduce $\ln x$ by means of the definition

$$\int_1^x \frac{1}{t}\,dt = \ln x \tag{1}$$

which, they observe, fills the gap (k cannot be zero) in the set of formulas

$$\int t^{k-1}\,dt = \frac{t^k}{k} + C. \tag{2}$$

It is perhaps worth making explicit the observation that $\ln x$ is not quite so isolated from the power functions x^k/k as might at first sight seem to be the case. For the selection of a specific set of antiderivatives in (2) yields

$$\int_1^x t^{k-1}\,dt = \frac{x^k - 1}{k}. \tag{3}$$

One would guess from (1) and (3), and verify by l'Hopital's rule that

$$\lim_{k \to 0} \frac{x^k - 1}{k} = \ln x. \tag{4}$$

Sketches of a few graphs of the functions $f_k(x) = (x^k - 1)/k$ along with that of $\ln x$ show $\ln x$ fitting in nicely among these power functions.

A Unified Treatment of Various Theorems in Elementary Analysis

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The purpose of this paper is to provide a unified approach to the proofs of a number of theorems in elementary analysis. In addition the proofs are often simplified by the method suggested. Our technique depends upon the following definition and lemma from [**1**].

THE TEACHING OF MATHEMATICS

EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

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The purpose of this paper is to provide a unified approach to the proofs of a number of theorems in elementary analysis. In addition the proofs are often simplified by the method suggested. Our technique depends upon the following definition and lemma from [1].

DEFINITION. A collection C of closed subintervals of $[a, b]$ is a *full cover* of $[a, b]$ if to each $x \in [a, b]$ there corresponds a number $\delta(x) > 0$ such that every closed subinterval of $[a, b]$ that contains x and has length less than $\delta(x)$ belongs to C.

LEMMA. *If C is a full cover of $[a, b]$, then C contains a partition of $[a, b]$, i.e., there exist $a = x_0, x_1, x_2, \dots, x_n = b$ such that $x_{k-1} < x_k$ and $I_k = [x_{k-1}, x_k]$ is in C for each k.*

Proof. (The proof is given in [**1**, p. 79] and is repeated here for convenience.) Suppose C contains no partition of $[a, b]$. Then by repeated bisection of $[a, b]$, there must be a sequence $\{J_n\}$ of closed subintervals of $[a, b]$ such that $J_n \supseteq J_{n+1}$ for each n, $|J_n| \to 0$, and C contains no partition of any J_n. ($|J_n|$ denotes the length of J_n.) By the Nested Interval Theorem there exists an x in the intersection of the sequence $\{J_n\}$. Consider $\delta(x)$ as is given by the above Definition. Since $|J_n| \to 0$ there exists a positive integer N so large that $|J_N| < \delta(x)$. Therefore, J_N belongs to C whence C trivially contains a partition of J_N, a contradiction.

We now use this lemma to prove three well-known theorems for continuous functions.

THEOREM 1. *If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Let

$$C = \{I\colon I \text{ is a closed subinterval of } [a, b] \text{ and } f \text{ is bounded on } I\}.$$

We claim that C is a full cover of $[a, b]$. To see this let $x \in [a, b]$. Since f is continuous at x, there exists $\delta(x) > 0$ such that f is bounded on $(x - \delta(x), x + \delta(x))$. Obviously this interval must be slightly modified if x is either a or b. Now let I be any closed subinterval of $[a, b]$ with $x \in I$ and $|I| < \delta(x)$. Then $I \subseteq (x - \delta(x), x + \delta(x))$ so that f is bounded on I and I is in C. Thus C is a full cover of $[a, b]$ and the lemma implies that C contains a partition of $[a, b]$. Since f is bounded on each set in the partition, f is bounded on $[a, b]$.

THEOREM 2 (Intermediate Value Theorem). *If f is continuous on $[a, b]$ with $f(a)f(b) < 0$, then there exists $x_0 \in (a, b)$ such that $f(x_0) = 0$.*

Proof. Suppose f is never 0 on $[a, b]$ and let

$$C = \{I\colon I \text{ is a closed subinterval of } [a, b] \text{ and } f \text{ has one sign on } I\}.$$

To show that C is a full cover of $[a, b]$ let x be in $[a, b]$. Since f is continuous at x and $f(x) \neq 0$, there exists $\delta(x) > 0$ such that f has the same sign as $f(x)$ on $(x - \delta(x), x + \delta(x))$. If I is any closed subinterval of $[a, b]$ with x in I and $|I| < \delta(x)$, then $I \subseteq (x - \delta(x), x + \delta(x))$. Thus f has one sign on I and I is in C. Since C is a full cover of $[a, b]$, C contains a partition $I_1, I_2, \dots, I_n$ of $[a, b]$. Assuming that these intervals are listed in increasing order, we see that f must have one sign on $[a, b]$, which is a contradiction.

THEOREM 3. *If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. Let

$C = \{I$: I is a closed subinterval of $[a, b]$ and $|f(y) - f(z)| < \epsilon/2$ for any y and z in $I\}$.

Using the continuity of f on $[a, b]$, it is easy to show that C is a full cover of $[a, b]$. Therefore, C contains a partition $I_1, I_2, \dots, I_n$ of $[a, b]$. Let

$$\delta = \min\{|I_k| : k = 1, 2, \dots, n\}.$$

If x and y belong to $[a, b]$ with $|x - y| < \delta$ then either x and y belong to the same subinterval of the partition or to two adjacent subintervals of the partition. In either case $|f(x) - f(y)| < \epsilon$ which completes the proof.

We complete this paper with the proofs of the Heine-Borel and Bolzano-Weierstrass Theorems.

THEOREM 4 (Heine-Borel Theorem). *Any open cover of $[a, b]$ has a finite subcover.*

Proof. Let G be any collection of open sets covering $[a, b]$. Let

$C = \{I$: I is a closed subinterval of $[a, b]$ and I is a subset of a set in $G\}$.

Clearly C is a full cover of $[a, b]$ and therefore C contains a partition of $[a, b]$. Since each subinterval of the partition is a subset of a set in G, $[a, b]$ can be covered by a finite number of such sets, and the proof is complete.

THEOREM 5 (Bolzano-Weierstrass Theorem). *If S is a bounded infinite set of real numbers, then S has an accumulation point.*

Proof. Since S is bounded there exists $[a, b]$ such that $S \subseteq [a, b]$. Suppose S has no accumulation points. Let

$$C = \{I\colon I \text{ is a closed subinterval of } [a, b] \text{ and } I \cap S \text{ is finite}\}.$$

Since C is a full cover of $[a, b]$, C contains a partition $I_1, I_2, \dots, I_n$ of $[a, b]$. Now

$$S = S \cap [a, b] = S \cap \cup_k I_k = \cup_k (S \cap I_k),$$

which implies that S is finite. This contradiction completes the proof of the theorem.

Obviously there are additional theorems in analysis that could be proved by the method of this paper. As an example one could prove without using the Mean Value Theorem that a function with derivative identically 0 on a closed interval is constant on that interval. The search for such theorems could be a valuable exercise for the student.

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A Simplification of Taylor's Theorem

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Taylor's formula with remainder is an essential part of the presentation of representation of functions by power series in a calculus course. While there are several proofs available, none is very satisfying to students. Perhaps for this reason, the treatment of Taylor's formula in calculus texts varies greatly. An informal survey of nine recent texts designed for "students of business or the social or biological sciences" revealed that one-third give a statement of Taylor's theorem without proof while the remainder managed to pass from Taylor polynomials to Taylor series without mention of remainders. A parallel survey of 18 recent texts designed for "the standard three semester sequence" showed that all gave a statement and proof of Taylor's theorem; three essentially different proofs were used.

Approximately three-fourths of the texts surveyed used either a proof based on Rolle's theorem (e.g., **[1]**) or a proof involving an application of the Cauchy form of the mean value theorem (e.g., **[2]**). All of these proofs make use of auxiliary functions whose choice cannot be motivated; their appearance can only reinforce students' belief that mathematical proofs are arbitrary and illogical creations. Four of the books examined used a proof based on repeated integration by parts (e.g., **[3]**). This proof is better motivated, but it still tends to lose students in technical details because integration by parts is usually not well understood by elementary students and because the factors must be chosen cleverly to make the integrated terms vanish. It is probably not at all helpful to point out that repeated integration by parts can be replaced by repeated interchange of order of integration in an iterated integral.

In most applications, the explicit form of the remainder in Taylor's formula is irrelevant; what matters is that a function can be approximated by a Taylor polynomial with an error term which can be estimated. The purpose of this note is to give an error estimate directly without going by way of an explicit representation for the remainder. The proof depends only on the representation of a function as the integral of its derivative and the fact that a definite integral is bounded by the product of the maximum of the absolute value of the integrand and the length of the interval of integration.

THEOREM. *Suppose that for some integer $n \geqslant 0$ the functions $f, f', \dots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists and satisfies $|f^{(n+1)}(t)| \leqslant M$ on (a, b). Then on $[a, b]$,*

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \end{aligned} \tag{1}$$

with

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}. \tag{2}$$

Proof. We give the proof for the case $a < b$; the modification to treat the case $a > b$ is easy. The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is chosen to have the properties

$$f(a) = P_n(a), \qquad f'(a) = P_n'(a), \qquad \ldots, \qquad f^{(n)}(a) = P_n^{(n)}(a). \tag{3}$$

Because P_n is a polynomial of degree n,

$$P_n^{(n+1)}(x) \equiv 0, \qquad a \leqslant x \leqslant b. \tag{4}$$

The remainder $R_n(x)$ is defined by (1), so that

$$R_n(x) = f(x) - P_n(x).$$

Because of (3),

$$R_n(a) = R_n'(a) = \cdots = R_n^{(n)}(a) = 0, \tag{5}$$

and because of (4),

$$f^{(n+1)}(t) = R_n^{(n+1)}(t), \qquad a < t < x. \tag{6}$$

Integration of (6) with the aid of (5) gives

$$R_n^{(n)}(x) = R_n^{(n)}(a) + \int_a^x R_n^{(n+1)}(t)\,dt = \int_a^x f_n^{(n+1)}(t)\,dt,$$

and the bound $|f^{(n+1)}(t)| \leqslant M$ gives

$$|R_n^{(n)}(x)| \leqslant \int_a^x |f^{(n+1)}(t)|\,dt \leqslant M(x-a).$$

A second integration gives

$$R_n^{(n-1)}(x) = R_n^{(n-1)}(a) + \int_a^x R_n^{(n)}(t)\,dt = \int_a^x R_n^{(n)}(t)\,dt,$$

$$|R_n^{(n-1)}(x)| \leqslant \int_a^x |R_n^{(n)}(t)|\,dt \leqslant \int_a^x M(t-a)\,dt = \frac{M}{2}(x-a)^2.$$

We may continue in this way, or prove by induction that for $j = 0, 1, \ldots, n$

$$|R_n^{(n-j)}(x)| \leqslant \frac{M}{(j+1)!}(x-a)^{j+1}, \qquad a \leqslant x \leqslant b. \tag{7}$$

The desired estimate (2) is the case $j = n$ of (7), and this completes the proof.

We suggest that this approach gives a way of treating Taylor series clearly and comprehensibly for students of elementary calculus.

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1. R. Ellis and D. Gulick, Calculus with Analytic Geometry, 3rd ed., Harcourt Brace Jovanovich, 1986, pp. 557–558.
2. L. Leithold, The Calculus with Analytic Geometry, 5th ed., Harper & Row, 1986. pp. 738–740.
3. J. Marsden and A. Weinstein, Calculus, Benjamin-Cummings, 1980, pp. 574–576.

(i) Show that the four segments can be so placed that the endpoints determine a rectangle containing P, and show that this rectangle may have any specified area between 0 and some maximum value $M(a,b,c,d)$.

(ii) Find $M(a,b,c,d)$.

E 3209. *Proposed by Bruce A. Reznick, University of Illinois at Urbana-Champaign.*

Let $C_0, C_1, C_2, \ldots$ be the sequence of circles in the Cartesian plane defined as follows:

(i) C_0 is the circle $x^2 + y^2 = 1$,

(ii) for $n = 0,1,2,\ldots$ the circle C_{n+1} lies in the upper half-plane and is tangent to C_n as well as to both branches of the hyperbola $x^2 - y^2 = 1$.

Let r_n be the radius of C_n. Show that r_n is an integer and give a formula for r_n.

E 3210. *Proposed by Paul A. Smith (student), University of Würzburg.*

Determine all pairs (m, n) of integers such that

$$1 \leq m \leq n, \qquad m^2 \equiv -1 \pmod{n}, \qquad n^2 \equiv -1 \pmod{m}.$$

E 3211. *Proposed by László Tóth, Satu Mare, Romania.*

Let $\omega(k)$ denote the number of distinct prime factors of the positive integer k and let (i, n) denote the greatest common divisor of the positive integers i and n. Express

$$\sum_{i=1}^{n} 2^{\omega((i,n))}$$

in terms of the prime factorization of n.

E 3212. *Proposed by J. Zhu, Pembroke College, Cambridge, England.*

Answer the following question mentioned by Paul Erdös in his article "Ulam, the Man and the Mathematician" [*Journal of Graph Theory*, 9 (1985) 445–449]: Is it true that if n is sufficiently large and $a_1, a_2, \ldots, a_n$ is any permutation of $1, 2, \ldots, n$, then there is an arithmetic progression $i, i+d, i+2d$ with $1 \leq i < i+d < i+2d \leq n$ such that a_i, a_{i+d}, a_{i+2d} also forms an arithmetic progression?

SOLUTIONS OF ELEMENTARY PROBLEMS

Two Exponential Diophantine Equations

E 3019 [1983, 644]. *Proposed by Chen-Te Yen, Chung-Yuan Christian University, Taiwan.*

Find all solutions to the equations (i) $1 + 3^a = 7^b + 3^c$ and (ii) $1 + 5^a = 7^b + 5^c$ in integers a, b, c.

Solution by T. S. Bolis, University of Ioannina, Greece. (i) Obviously, if $b = 0$, then $a = c$. If $b < 0$, then $0 < 7^b = 1 + 3^a - 3^c < 1$, which implies $a < c < 0$. This leads to the contradiction $1 < 1 + 3^a = 7^b + 3^c < 1$. For $b = 1$, we obtain the solution $a = 2$, $c = 1$. Let $b > 1$. Then $a > 3$ and $c \geqq 1$. By reducing (1) mod 27, we obtain $b \equiv 4, 3, 0 \bmod 9$ if $c = 1, 2, \geqq 3$, respectively. The first two cases are excluded by reducing mod 37 while in the third case the same modulus gives $a \equiv c \bmod 18$, which is proven impossible by looking at (1) mod 7. Thus the only solutions are $(a, 0, a)$ and $(2, 1, 1)$.

(ii) In a similar manner we get the obvious solutions $b = 0$, $a = c$, and that for $b < 0$ there are no solutions. Let $b > 0$. By working mod 5 we get $b \equiv 0 \bmod 4$. Then, reduction mod 21 yields $a \equiv 1$ and $c \equiv 3$ or $a \equiv 0$ and $c \equiv 4 \bmod 6$, while reduction mod 9 yields $b \equiv 1 \bmod 3$. Finally, working mod 13, we get that there are no solutions other than $b = 0$, $a =$ c.

Also solved by M. Bencze (Romania), A. Bondesen (Denmark), N. Robbins, I. J. Schoenberg, D. Tyler, N. Tzanakis (Greece), and the proposer. L. Foster and J. Suck (Germany) pointed out that similar equations are treated in J. L. Brenner and L. L. Foster, Exponential Diophantine Equations, Pacific Journal of Mathematics, 101 (1982) 263–301. Partially solved by J. S. Frame and D. Wilson.

An Often-Summed Sum

E 3027 [1983, 706]. *Proposed by Tran Minh Crung, Skogn, Norway.*

Evaluate in closed form $B(r, m, n) = \sum_{k \equiv r (\mathrm{mod}\ m)} \binom{n}{k}$.

Solution by T. S. Bolis, University of Ioannina, Greece. Let ζ be a primitive mth root of unity. Since

$$\sum_{j=0}^{m-1} \zeta^{sj} = \begin{cases} m & \text{if } s \equiv 0 \ (\mathrm{mod}\ m) \\ 0 & \text{if } s \not\equiv 0 \ (\mathrm{mod}\ m), \end{cases}$$

we have

$$\begin{aligned}\sum_{j=0}^{m-1} (1 + \zeta^j)^n \zeta^{-rj} &= \sum_{j=0}^{m-1} \sum_{k=0}^{n} \binom{n}{k} \zeta^{(k-r)j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{m-1} \zeta^{(k-r)j} \\ &= mB(r, m, n).\end{aligned}$$

Editor's note. The sum can also be expressed in the forms

$$\frac{2^n}{m} \sum_{i=0}^{m-1} \left(\cos(n - 2r)\frac{i\pi}{m}\right)\left(\cos\frac{i\pi}{m}\right)^n$$

Find all solutions to the equations (i) $1 + 3^a = 7^b + 3^c$ and (ii) $1 + 5^a = 7^b + 5^c$ in integers a, b, c.

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(ii) In a similar manner we get the obvious solutions $b = 0$, $a = c$, and that for $b < 0$ there are no solutions. Let $b > 0$. By working mod 5 we get $b \equiv 0 \bmod 4$. Then, reduction mod 21 yields $a \equiv 1$ and $c \equiv 3$ or $a \equiv 0$ and $c \equiv 4 \bmod 6$, while reduction mod 9 yields $b \equiv 1 \bmod 3$. Finally, working mod 13, we get that there are no solutions other than $b = 0$, $a = c$.

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we have

$$\sum_{j=0}^{m-1} (1 + \zeta^j)^n \zeta^{-rj} = \sum_{j=0}^{m-1} \sum_{k=0}^{n} \binom{n}{k} \zeta^{(k-r)j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{m-1} \zeta^{(k-r)j}$$
$$= mB(r, m, n).$$

Editor's note. The sum can also be expressed in the forms

$$\frac{2^n}{m} \sum_{i=0}^{m-1} \left(\cos(n - 2r)\frac{i\pi}{m}\right)\left(\cos\frac{i\pi}{m}\right)^n$$

and

$$\frac{1}{m}\sum_{i=0}^{n}\binom{n}{i}\sum_{j=1}^{m}\cos\frac{2j(n-r-i)\pi}{m}.$$

The problem was located in *The Art of Computer Programming*, vol. 1 (1968), by D. E. Knuth, who attributes it to C. Ramus (1834), as an exercise on page 70 with a later solution; in *Combinatorial Analysis* (1958), by John Riordan; in *Combinatorial Enumeration* (1983), by I. P. Goulden and D. M. Jackson; as a problem in *Elemente der Mathematik* (vol. 13 (1958), solved in vol. 14 (1959)); and as a consequence of a formula in *Combinatorial Identities* (1972), by H. W. Gould.

Also solved by A. Bager (Norway), M. Bencze (Romania), A. Bondesen (Denmark), M. Elia (Italy), L. Kuipers (Switzerland), O. P. Lossers (The Netherlands), W. A. Newcomb, H. Prodinger (Austria), A. J. Schwenk, C. Toll, P. Y. Wu (Taiwan), and the proposer.

Inequalities for an Inscribed n-gon

E 3059 [1984, 580]. *Proposed by Meigu Guan, Weixuan Li, University of Waterloo, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Let H denote the regular n-gon with side length equal to one, $n \geqslant 4$. Show that if K is any n-gon inscribed in H with side-length x_i $(i = 1, 2, \ldots, n)$, then

$$n(1-\cos\theta)/2 \leqslant \sum_{i=1}^{n} x_i^2 \leqslant n(1-\cos\theta),$$

where θ denotes the internal angle of H. Discuss the cases when either equality holds.

Editor's note. K should have been required to be convex. Moreover, each side of H is assumed to contain a vertex of K. (Otherwise the minimum of Σx_i^2 is zero.)

Solution by Hong Oh Kim, Korea Advanced Institute of Science and Technology, Seoul, Korea.

Proof. The vertices of H will be denoted by A_i and those of K by B_i, arranged so that B_i lies on the side A_iA_{i+1} $(i = 1, 2, \ldots, n;\ A_{n+1} = A_1)$. Let x_i denote the length of $B_{i-1}B_i$ and y_i the length of $B_{i-1}A_i$ $(i = 1, 2, \ldots, n;\ B_0 = B_n)$. By the law of cosines applied to the triangle $\triangle B_{i-1}A_iB_i$,

$$x_i^2 = y_i^2 + (1-y_{i+1})^2 - 2y_i(1-y_{i+1})\cos\theta \tag{1}$$

$[x, x+\Delta x] \times [g(x)-1, g(x)+1]$ or a subset of this rectangle. Then we obtain

$$\left|\int_{g(x)}^{g(x+\Delta x)} f(x+\Delta x, y)\, dy\right| \geqslant \varepsilon |g(x+\Delta x) - g(x)|,$$

so that $g(x+\Delta x) - g(x) = o(\Delta x)$ as $\Delta x \to 0$, in other words g is constant.

Also solved by L. Kuipers (Switzerland), B. Margolis (France), J.-M. Monier (France), D. Rawsthorne, B. L. Schwartz, and the proposer.

Irreducible Polynomials—Divisibility Results

E 3082 [1985, 215–216]. *Proposed by F. W. Dodd and L. E. Mattics, University of South Alabama.*

Let $g(x)$ be a polynomial with positive degree and integer coefficients which is irreducible over the rationals. Prove the following: (a) there exists an integer n such that $g(n)$ is not a square; (b) for every positive integer r there exists a positive integer n such that $p \| g(n)$ for at least $r+1$ primes p. (Here $p \| g(n)$ means $p | g(n)$ but $p^2 \nmid g(n)$.)

Solution by S. V. Kanetkar, Old Dominion University, Norfolk, Virginia. The first statement follows from the second since $p \| g(n)$ implies $g(n)$ is not a perfect square. The second statement is a generalization of Problem 112, part VIII, page 131 of [1]. We follow [1] closely. Since $g(x)$ is irreducible, there exist polynomials $a(x)$ and $b(x)$ such that

$$a(x)g(x) + b(x)g'(x) = m \tag{1}$$

for some integer m. Let r be any positive integer. Let $p_1, p_2, \ldots, p_{2r+1}$ be distinct primes such that $p_i \nmid m$ but $p_i | g(n_i)$ for integers n_i, $1 \leqslant i \leqslant 2r+1$ (this is guaranteed by Problem 108, ibid.). By the Chinese Remainder Theorem, there is an integer N such that $N \equiv n_i \pmod{p_i}$, $1 \leqslant i \leqslant 2r+1$. It follows that $g(N) \equiv 0 \bmod (p_1 p_2 \cdots p_{2r+1})$. Note that from (1) it follows that $p_i \nmid g'(N)$, $1 \leqslant i \leqslant 2r+1$. Now

$$g(N + p_1 p_2 \cdots p_{2r+1}) = g(N) + (p_1 p_2 \cdots p_{2r+1}) g'(N) + \cdots .$$

Hence

$$g(N + p_1 p_2 \cdots p_{2r+1}) - g(N) \equiv p_1 p_2 \cdots p_{2r+1} g'(N) \bmod (p_1^2 \cdots p_{2r+1}^2).$$

This shows that for each i one of $g(N + p_1 p_2 \cdots p_{2r+1})$ and $g(N)$ is divisible by p_i but not by p_i^2. Hence there are $r+1$ primes, say, $p_1, p_2, \ldots, p_{r+1}$ such that $g(N)$ or $g(N + p_1 p_2 \cdots p_{2r+1})$, say $g(N)$, is divisible by exactly the first power of those primes.

Remark. It can be noted that (1) holds if $g(x)$ has distinct roots. Hence the result still holds if irreducibility is replaced by the weaker assumption that $g(x)$ has distinct roots.

REFERENCE

1. G. Polya and G. Szego, Problems and Theorems in Analysis, Vol. II, Springer-Verlag, New York, 1976.

Also solved by N. Elkies, R. Gilmer and B. Nashier, S. Marivani, and the proposers.

A Raabe-type Test for Asymptotic Functional Behavior

E 3090 [1985, 359; 1985, 507]. *Proposed by Alfonso Villani, Seminario Matematico, Catania, Italy.*

Let $f:[0,+\infty)\to(0,\infty)$ be a continuous function. Let $h>0$ be given and let $g:[0,+\infty)\to R$ be defined by

$$g(x)=x\left(\frac{f(x)}{f(x+h)}-1\right).$$

(a) Show that if $\liminf_{x\to+\infty} g(x)>0$, then $\lim_{x\to+\infty} f(x)=0$.

(b) Show that if $\limsup_{x\to+\infty} g(x)<0$, then $\lim_{x\to+\infty} f(x)=+\infty$.

(c) Assume that $\lim_{x\to+\infty} g(x)=0$. Is it possible for f to satisfy both $\liminf_{x\to+\infty} f(x)=0$ and $\limsup_{x\to+\infty} f(x)=+\infty$?

(d) How, in (a) and (b), can the assumption "f is continuous" be weakened? Can this assumption simply be omitted?

Editor's Note. The statement of this problem contained misprints when it originally appeared in the *Monthly*. Instead of the printed definition for $g(x)$ as

$$g(x)=\frac{f(x)}{f(x+h)-1},$$

the correct definition should have been given as

$$g(x)=x\left(\frac{f(x)}{f(x+h)}-1\right),$$

as in the statement of the problem given above.

William P. Wardlaw provided counterexamples to the problem as it was originally printed. For example, a counterexample for part (a) is:

Remark. It can be noted that (1) holds if $g(x)$ has distinct roots. Hence the result still holds if irreducibility is replaced by the weaker assumption that $g(x)$ has distinct roots.

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(a) Show that if $\liminf_{x\to+\infty} g(x) > 0$, then $\lim_{x\to+\infty} f(x) = 0$.

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The corrected version of the problem was also solved by V. Pambuccian (Romania), and the proposer. N. Sivakuman (student, Canada), W. P. Wardlaw, and M. Wiegner (West Germany) all noted that the problem was incorrectly stated as originally printed.

An Inscribed Equilateral Triangle Inequality

E 3091 [1985, 360]. *Proposed by Calin P. Popescu, student, Bucharest, Romania.*

Equilateral triangle ABC is inscribed in triangle XYZ, with A between Y and Z, B between Z and X, and C between X and Y. Show that $XA + YB + ZC < XY + YZ + ZX$. Is equality possible?

Solution by Jiro Fukuta, Japan. The given problem is a special case of the following: If triangles XBC, YCA, and ZAB are drawn on the sides of equilateral triangle ABC, then

$$XA + YB + ZC \leqslant BX + XC + CY + YA + AZ + ZB.$$

Proof. Ptolemy's theorem (or inequality; see H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, p. 42) applied to quadrilateral $ABXC$ yields

$$AX \cdot BC \leqslant BX \cdot CA + XC \cdot AB,$$

so $AX \leqslant BX + XC$, since triangle ABC is equilateral. Similarly we have

$$BY \leqslant CY + YA \quad \text{and} \quad CZ \leqslant AZ + ZB.$$

Addition gives the required inequality. Equality holds if and only if X, Y, Z are in the circumcircle of triangle ABC. The given problem is the special case when $\measuredangle ZBX$, $\measuredangle XCY$ and $\measuredangle YAZ$ are equal to π, and in this case equality holds if and only if triangle XYZ coincides with triangle ABC.

Also solved by S. Arslanagić and D. Milošević (Yugoslavia), N. Bejlegaard (Norway), L. Kuipers (Switzerland) and P. Szüsz, O. P. Lossers (The Netherlands), M. Vowe (Switzerland), and the proposer.

A Path Bijection

E 3096 [1985, 428]. *Proposed by the editors* (*cf.* E 2903 [1981, 619; 1983; 483; 1985, 430]).

Consider paths in the northeast quadrant which start at $(0,0)$, such that each step is either one unit east or one unit north, and end at $(2n, 2n)$. A path is called a Whitworth path if it does not go above the diagonal $y = x$. A path is called a Shapiro path if it avoids the points $(1,1), (3,3), \ldots, (2n-1, 2n-1)$, i.e., those on the diagonal with odd coordinates. Exhibit, for fixed n, a one-to-one correspondence between the set of Whitworth paths and the set of Shapiro paths.

Solution by Warren Nichols, Florida State University. We represent a path as a sequence of E's and N's. We consider only paths with the same number of E's and N's. We reserve the letter D for paths which have positive length and which have endpoints on the diagonal and other points below the diagonal. A path D is called even or odd according to the number of E's. The path obtained from a path X by interchanging the symbols E and N is denoted X'. The path obtained from a path D by deleting the leading E and the trailing N is denoted D^*. The null path is denoted $\varnothing$.

We define bijections between Whitworth paths and Shapiro paths recursively, by splitting off the portion of the path up to the first return to the diagonal. Define ψ mapping Shapiro paths to Whitworth paths by $\psi(\varnothing) = \varnothing$, $\psi(DX) = D\psi(X)$, $\psi(D'X) = E\psi(X)ND^*$. Define ϕ mapping Whitworth paths to Shapiro paths by $\phi(\varnothing) = \varnothing$, $\phi(DY) = D\phi(Y)$ (if D is even), $\phi(DY) = NY'E\phi(D^*)$ (if D is odd). To prove these are bijections, we show that $\phi\psi$ and $\psi\phi$ are the identity, by induction on the length of the argument sequence. We have $\phi\psi(\varnothing) = \varnothing$, also $\phi\psi(DX) = \phi(D\psi(X)) = D\phi\psi(X) = DX$, and $\phi\psi(D'X) = \phi(E\psi(X)ND^*) = N(D^*)'E\phi((E\psi(X)N)^*) = D'\phi\psi(X) = D'X$. Similarly $\psi\phi(\varnothing) = \varnothing$, also $\psi\phi(DY) = \psi(D\phi(Y)) = D\psi\phi(Y) = DY$ (if D is even), and $\psi\phi(DY) = \phi(NY'E\phi(D^*)) = E\psi\phi(D^*)NY = ED^*NY = DY$ (if D is odd). Hence, ψ and ϕ are bijections.

Let us use W to denote a Whitworth path, possibly of zero length. Each Shapiro path can be written uniquely as $S = W_1D_1'W_2D_2' \cdots W_rD_r'W_{r+1}$. The mapping of S to $W_1EW_2E \cdots EW_rW_{r+1}ND_r^*ND_{r-1}^* \cdots ND_1^*$ satisfies the recursive definition, and thus it is the map ψ.

Editorial Comment. It is also possible to give a nonbijective proof that the two sets of paths have the same size. It is well known that the Whitworth paths are counted by the Catalan numbers C_{2n}, where $C_n = \binom{2n}{n}/(n+1)$. The Shapiro paths can be counted using a deep formula of Gessel and Viennot for counting lattice paths (see Stanley, *Enumerative Combinatorics*, Vol I, p. 82–84, for an exposition). In particular, the lattice paths from one point to another that avoid a specified set of points are counted by a determinant in which all entries are binomial coefficients; for the case of lattice paths from $(0,0)$ to $(2n, 2n)$ avoiding all odd diagonal points, an $n+1$ by $n+1$ determinant results. On the other hand, evaluation of this determinant is sufficiently difficult that the above bijection can be regarded as a short bijective proof that its value is C_{2n}.

Also solved by O. Matouš (Czechoslovakia) and D. Wolfe.

An Old Sum Reappears

E 3103 [1985, 507]. *Proposed by Douglas B. Tyler, Fullerton, California.*

Solution by Warren Nichols, Florida State University. We represent a path as a sequence of E's and N's. We consider only paths with the same number of E's and N's. We reserve the letter D for paths which have positive length and which have endpoints on the diagonal and other points below the diagonal. A path D is called even or odd according to the number of E's. The path obtained from a path X by interchanging the symbols E and N is denoted X'. The path obtained from a path D by deleting the leading E and the trailing N is denoted D^*. The null path is denoted $\varnothing$.

We define bijections between Whitworth paths and Shapiro paths recursively, by splitting off the portion of the path up to the first return to the diagonal. Define ψ mapping Shapiro paths to Whitworth paths by $\psi(\varnothing) = \varnothing$, $\psi(DX) = D\psi(X)$, $\psi(D'X) = E\psi(X)ND^*$. Define ϕ mapping Whitworth paths to Shapiro paths by $\phi(\varnothing) = \varnothing$, $\phi(DY) = D\phi(Y)$ (if D is even), $\phi(DY) = NY'E\phi(D^*)$ (if D is odd). To prove these are bijections, we show that $\phi\psi$ and $\psi\phi$ are the identity, by induction on the length of the argument sequence. We have $\phi\psi(\varnothing) = \varnothing$, also $\phi\psi(DX) = \phi(D\psi(X)) = D\phi\psi(X) = DX$, and $\phi\psi(D'X) = \phi(E\psi(X)ND^*) = N(D^*)'E\phi((E\psi(X)N)^*) = D'\phi\psi(X) = D'X$. Similarly $\psi\phi(\varnothing) = \varnothing$, also $\psi\phi(DY) = \psi(D\phi(Y)) = D\psi\phi(Y) = DY$ (if D is even), and $\psi\phi(DY) = \phi(NY'E\phi(D^*)) = E\psi\phi(D^*)NY = ED^*NY = DY$ (if D is odd). Hence, ψ and ϕ are bijections.

Let us use W to denote a Whitworth path, possibly of zero length. Each Shapiro path can be written uniquely as $S = W_1D_1'W_2D_2' \cdots W_rD_r'W_{r+1}$. The mapping of S to $W_1EW_2E \cdots EW_rW_{r+1}ND_r^*ND_{r-1}^* \cdots ND_1^*$ satisfies the recursive definition, and thus it is the map ψ.

Editorial Comment. It is also possible to give a nonbijective proof that the two sets of paths have the same size. It is well known that the Whitworth paths are counted by the Catalan numbers C_{2n}, where $C_n = \binom{2n}{n}/(n+1)$. The Shapiro paths can be counted using a deep formula of Gessel and Viennot for counting lattice paths (see Stanley, *Enumerative Combinatorics*, Vol I, p. 82–84, for an exposition). In particular, the lattice paths from one point to another that avoid a specified set of points are counted by a determinant in which all entries are binomial coefficients; for the case of lattice paths from $(0,0)$ to $(2n, 2n)$ avoiding all odd diagonal points, an $n+1$ by $n+1$ determinant results. On the other hand, evaluation of this determinant is sufficiently difficult that the above bijection can be regarded as a short bijective proof that its value is C_{2n}.

Also solved by O. Matouš (Czechoslovakia) and D. Wolfe.

An Old Sum Reappears

E 3103 [1985, 507]. *Proposed by Douglas B. Tyler, Fullerton, California.*

But $\sum_{k=1}^{\infty} x^{2k}/\{k(2k+1)\}$ is the Maclaurin series for the function

$$f(x) = 2 - \ln(1 - x^2) + \frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)$$

in $|x| < 1$ (with $f(0) = 0$), so that

$$f\left(\frac{1}{2n}\right) = 2 + \ln\frac{(2n)^2(2n-1)^{2n-1}}{(2n+1)^{2n+1}}.$$

Thus,

$$S = \lim_{N\to\infty} \sum_{n=1}^{N} f\left(\frac{1}{2n}\right) = \lim_{N\to\infty} \ln\left[e^{2N}\frac{2^{2N}(N!)^2}{(2N+1)^{2N+1}}\right]$$

$$= \ln \lim_{N\to\infty} \frac{1}{2}\left(\frac{N!e^N}{N^{N+1/2}}\right)^2\left(\frac{2N}{2N+1}\right)\left(1 + \frac{1}{2N}\right)^{-2N}.$$

But Stirling's formula yields

$$\lim_{N\to\infty} \frac{N!e^N}{N^{N+1/2}} = \sqrt{2\pi},$$

so that

$$S = \ln\tfrac{1}{2}\cdot(\sqrt{2\pi})^2\cdot 1\cdot e^{-1} = \ln\pi - 1,$$

as claimed.

Also solved by 28 other readers and the proposer. A. J. Krishna (student) pointed out that this problem appeared as E 1801 almost exactly 20 years earlier (problem: E 1801 [1965, 666]; solution [1967, 80]).

$(xy)^n = yx$ Implies Commutativity

E 3109 [1985, 591]. *Proposed by Desmond MacHale and Mícheál Ó Searcóid, University College, Cork, Ireland.*

Let R be an associative ring such that $(xy)^n = yx$, for all $x, y \in R$, for some fixed natural number n. Must R be commutative?

Editorial Comment. Many solvers pointed out that the answer is yes for arbitrary semigroups R since

$$xy = (yx)^n = [(xy)^n]^n = [(xy)(xy)^{n-1}]^n = (xy)^{n-1}(xy) = (xy)^n = yx$$

for any $n > 1$, and the result is clearly true if $n = 1$.

Solution and generalization by Charles Lanski. Let R be an associative ring such that, for all x, y in R, there is an $n = n(x, y) > 1$ with $xy = (yx)^n$.

Observe first that $xy = 0$ if and only if $yx = 0$, so that the left annihilator of R and the right annihilator of R coincide. Denote this ideal of R by A. Next suppose that $a \in R$ and $a^2 = 0$. Then, for all $y \in R$, $0 = a^2y = a(ay)$; since $xy = 0$ if and only if $yx = 0$, it follows that $(ay)a = 0$. Since $n = n(a, y) > 1$, we have $ay = (ya)^n = y(aya)(ya)^{n-2} = 0$, forcing $a \in A$. But for any $x \in R$, taking $y = x$ gives $x^2 = x^{2n}$, for $n > 1$, so $(x^{2n-1} - x)^2 = 0$, which yields $x^{2n-1} - x \in A$. Using the well-known theorem of N. Jacobson [Structure theory for algebraic algebras of bounded degree, *Annals of Math.*, 46 (1945) 695–707] one has that R/A is commutative. If $x, y \in R$, then $xy = yx$ lies in A, which is a two-sided ideal (by our initial remark), so that $(xy - yx)z = 0 = z(xy - yx)$ for all $x, y, z \in R$. Hence, $(xy)z = x(yz) = xz(y) = (xz)y = (zx)y = z(xy)$. Consequently, if $xy = (yx)^n$ and $yx = (xy)^m$, then $xy = (yx)^n = (xy)^{mn} = x(yx)^{mn-1}y = ((yx)^{mn-1}y)x = (yx)^{mn} = (xy)^m = yx$, showing that R is commutative.

Also solved by 29 other readers and the proposer. The above generalization was also given by P. Boisen.

ADVANCED PROBLEMS

6545. *Proposed by L.A. Rubel, University of Illinois at Urbana-Champaign.*

Suppose $p_n(z)$ is a polynomial over $\mathbb{C}$ for $n = 0, 1, 2, \ldots$ and suppose $\sum_{n=0}^{\infty} p_n(z)w^n$ converges for all complex z and w. Must $\sum_{n=0}^{\infty} p_n'(z)w^n$ converge for all complex z and w? (Here $p_n'(z)$ is the ordinary derivative of $p_n(z)$.)

6546. *Proposed by Juan Arias-de-Reyna, Universidad de Sevilla, Spain.*

Suppose E is a separable infinite-dimensional, normed linear space. Construct a sequence $x_1, x_2, x_3, \ldots$ in E such that

(a) $\lim_{n \to \infty} \|x_n\| = +\infty$.

(b) Every x^{**} in E^{**} is a weak* cluster point of the sequence $\{x_n\}$.

6547. *Proposed by P. L. Butzer and E. L. Stark*†*, Rheinisch-Westfalische Technischen Hochschule, Aachen, West Germany.*

† Professor Stark died on April 23, 1986.

In discussing the nondifferentiability of the famous "Riemann function" $\sum n^{-2}\cos(n^2x)$, Christoffel in a letter to E. Prym of June 18, 1965 states that

$$\text{“}\sum_{n=1}^{\infty}\frac{\cos(n^2x)}{n^2}=\frac{\pi^2}{6}-\frac{1}{2}\left(\frac{\pi x}{2}\right)^{1/2}+\frac{\pi^{3/2}}{2^{1/2}}\sum_{n=1}^{\infty}n\int_{n^2\pi^2/x}^{\infty}\frac{\sin t-\cos t}{t^{3/2}}\,dt.\text{”}$$

Show that, except for the coefficient of the term in $x^{1/2}$, this formula is correct. The text of Christoffel's letter is given in the article by the proposers, "*Riemann's example*" *of a continuous nondifferentiable function in the light of two letters* (1865) *of Christoffel to Prym*, to be published in: Professor Guy Hirsch Birthday Volume: Bull. Soc. Math. Belg., 38A (1986).

SOLUTIONS OF ADVANCED PROBLEMS

6502. *Proposed by Klaus Schürger, University of Bonn, Federal Republic of Germany.*

Each of the following conditions is clearly necessary for the convergence of a sequence $\{S_n\}$:

(A) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}S_k$ exists; (B) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}S_{m2^k}$ exists for every integer $m\geqslant 1$.

If $\{S_n\}$ is a real sequence which is bounded above and $\{nS_n\}$ is nondecreasing, is either condition sufficient for the convergence of $\{S_n\}$?

Solution by Ellen Hertz, Volt Information Sciences, Herndon, VA. To see that (B) is not sufficient, let kS_k be the greatest power of 2 that does not exceed k. Then kS_k is nondecreasing, S_k is bounded above by 1, and $S_{2^k}=1$ while $S_{2^k-1}\to 1/2$ as $k\to\infty$. On the other hand, define $r=r(m)\geq 0$ by

$$2^r\leq m<2^{r+1}.$$

Then

$$m2^kS_{m2^k}=2^{r+k},$$

and the summands of (B) are independent of k.

To see that (A) is sufficient, we observe that the nondecreasing property of $\{nS_n\}$ implies that S_n is bounded below; without loss of generality we may assume that S_n is positive for all n (since we may add a positive constant to S_n if necessary). Set

$$T_n=\frac{1}{n}\sum_{k=1}^{n}S_k,$$

so that identically

$$T_{n+r}=nT_n/(n+r)+\left(\sum_{j=n+1}^{n+r}S_j\right)/(n+r).$$

For $n + 1 \le j \le n + r$ we have $nS_n \le jS_j \le (n + r)S_{n+r}$ and hence

$$\frac{n}{n+r}S_n \le S_j \le \frac{n+r}{n}S_{n+r}.$$

It follows from the identity that

$$\frac{nr}{(n+r)^2}S_n \le T_{n+r} - \frac{nT_n}{n+r} \le \frac{r}{n}S_{n+r}.$$

Now assume that $T_n \to L$, and let $B > 0$ be arbitrary. If we let n and r approach infinity in such a way that $n/r \to B$, we discover from the left side of the above inequality that

$$\frac{B}{(B+1)^2}\limsup S_n \le \frac{L}{B+1}$$

and hence

$$\limsup S_n \le \frac{(B+1)}{B}L.$$

Similarly, the right side yields

$$\frac{B}{(B+1)}L \le \liminf S_n.$$

Upon letting $B \to \infty$, we see that the sequence $\{S_n\}$ converges.

Editorial comment. Note that the condition that $\{S_n\}$ is bounded above is not used in the above proof of the sufficiency of (A). Indeed, the existence of an upper bound is established in the course of the proof, as a consequence of (A) and the nondecreasing property of $\{n S_n\}$.

The sufficiency of (A) could also be deduced from the strong Tauberian theorem of Hardy and Landau: *If* $\{S_n\}$ *is a real sequence such that* $n(S_{n+1} - S_n)$ *is bounded below, then* (A) *implies the convergence of* $\{S_n\}$. (cf. T. J. I.'A. Bromwich, *An Introduction to the Theory of Infinite Series*, London, 1926, p. 423.) Note that in our application $n(S_{n+1} - S_n) \ge -S_{n+1}$, by the nondecreasing property of $\{n S_n\}$.

The proposer remarks that some related but rather more intricate Tauberian theorems may be found in D. Hochbaum and J. M. Steele, Steinhaus's geometric location problem for random samples in the plane, Adv. Appl. Prob. 14 (1982), 56–67 (see esp. Lemma 3.4).

Also solved by The San Bernardino Problem Solving Group, L. E. Clarke (England), O. P. Lossers (The Netherlands), Howard Morris, Daniel Ullman, and the proposer.

6503. *Proposed by Gérard Letac, Université Paul-Sabatier, Toulouse, France.*

The Markov chain $(X_n)_{n=0}^{\infty}$ on the interval $(-1, 1)$ is such that, given X_n, X_{n+1} is uniformly distributed on $(X_n, 1)$ if $X_n \leqslant 0$, and on $(-1, X_n)$ if $X_n > 0$. Find the stationary distribution.

For $n + 1 \leq j \leq n + r$ we have $nS_n \leq jS_j \leq (n + r)S_{n+r}$ and hence

$$\frac{n}{n+r}S_n \leq S_j \leq \frac{n+r}{n}S_{n+r}.$$

It follows from the identity that

$$\frac{nr}{(n+r)^2}S_n \leq T_{n+r} - \frac{nT_n}{n+r} \leq \frac{r}{n}S_{n+r}.$$

Now assume that $T_n \to L$, and let $B > 0$ be arbitrary. If we let n and r approach infinity in such a way that $n/r \to B$, we discover from the left side of the above inequality that

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and

$$D = \left\{\log\left[b^{-1}a^{-a/b}(a+b)^{(a+b)/b}\right]\right\}^{-1};$$

these are clearly positive constants.

For $a = b = 1$, we obtain for $|x| < 1$ that

$$f(x) = C(1+|x|)^{-1},$$

with $C = (2\log 2)^{-1}$.

Also solved by L. E. Clarke (England), A. A. Jagers (The Netherlands), Ole Jørsboe (Denmark), O. P. Lossers (The Netherlands), Daniel Neuenschwander (Switzerland), Nicholas Passell, Mark Pinsky, Eugene Salamin, Daniel Ullman, and the proposer.

6504. *Proposed by Michael S. Perkins, Stanford Univerity.*

Find each function $f(z)$ which is analytic at $z = 0$ and for which $nf_n(z/n)$ converges uniformly to $f(z)$ in some neighborhood of $z = 0$, where $f_1 = f$ and f_n is the composition of f with itself n times.

Solution by Roger Cooke, University of Vermont, Burlington. Either $f(z) = 0$ for all z, or $f(z) = z/(1 - az)$ for some complex number a. The key to the present proof is the formula

$$f(z) = (k+1)f\left(\frac{1}{k}f\left(\frac{kz}{k+1}\right)\right), \qquad k = 1, 2, \dots .$$

This is deduced from

$$f(z) = \lim_{n\to\infty} n(k+1)f_{n(k+1)}\left(\frac{z}{n(k+1)}\right)$$

upon writing the expression on the right as $k+1$ times the limit of $nf_n(w_n/n)$, where

$$w_n = \frac{1}{k}\cdot kn\, f_{kn}\left(\frac{kz/(k+1)}{kn}\right) \to \frac{1}{k}f\left(\frac{kz}{k+1}\right)$$

as $n \to \infty$. The formula now follows by uniform convergence.

Upon dividing both sides of the formula by $k+1$ and letting k go to infinity, we find that $f(0) = 0$. By differentiating it, we find that

$$f'(z) = f'\left(\frac{1}{kf}\left(\frac{kz}{k+1}\right)\right)f'\left(\frac{kz}{k+1}\right);$$

by letting k go to infinity we obtain $f'(z) = f'(0)f'(z)$. Thus, upon excluding the identically zero function, we have $f'(0) = 1$.

and

$$D = \left\{\log\left[b^{-1}a^{-a/b}(a+b)^{(a+b)/b}\right]\right\}^{-1};$$

these are clearly positive constants.

For $a = b = 1$, we obtain for $|x| < 1$ that

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upon writing the expression on the right as $k+1$ times the limit of $nf_n(w_n/n)$, where

$$w_n = \frac{1}{k}\cdot kn\, f_{kn}\left(\frac{kz/(k+1)}{kn}\right) \to \frac{1}{k}f\left(\frac{kz}{k+1}\right)$$

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by letting k go to infinity we obtain $f'(z) = f'(0)f'(z)$. Thus, upon excluding the identically zero function, we have $f'(0) = 1$.

By rewriting various differences in terms of difference quotients, and using the above formula for $f'(z)$, we find that

$$
\begin{aligned}
f''(0)f(z) &= \lim_{k\to\infty} k\left[f'\left(\frac{1}{k}f\left(\frac{kz}{k+1}\right)\right) - f'(0)\right] \\
&= \lim_{k\to\infty} k\left[\frac{f'(z) - f'\left(\frac{kz}{k+1}\right)}{f'\left(\frac{kz}{k+1}\right)}\right] \\
&= \frac{zf''(z)}{f'(z)},
\end{aligned}
$$

or

$$\left[a(f(z))^2 + f(z)\right]' = [zf'(z)]'$$

where $2a = f''(0)$. Hence

$$zf'(z) = a(f(z))^2 + f(z) + C,$$

and by setting $z = 0$ we find that $C = 0$. Therefore,

$$\left(\frac{z}{f(z)} + az\right)' = 0,$$

and (note that $f(z)/z \to 1$ as $z \to 0$) the result follows. [Note: The editor introduced the unmotivated $f''(0)f(z)$ to avoid calculations; Cooke's more natural approach was simply to differentiate the key formula twice, "rearrange," let $k \to \infty$, and then solve the resulting *third*-order differential equation.]

Robert B. Israel employed similar functional considerations; all other solutions involved explicit considerations of the coefficients of the power series expansion of $f(z)$. For pointwise rather than uniform convergence, M. J. Pelling has a counterexample.

Also solved by S. J. Bernau, Robert B. Israel (Canada), O. P. Lossers (The Netherlands), William A. Newcomb, T. S. Norfolk, Eugene Salamin, and the proposer.

Answer to "The Mathematician's Dictionary", page 439:

Fund a mental group.

(Contributions to this department are invited; acknowledgement of all of them will not be possible.)

EDITED BY ALLAN EDMONDS AND JOHN EWING

General Topology. By Jacques Dixmier. Spring-Verlag, New York, 1984. x + 140 pp.
General Topology and Homotopy Theory. By I. M. James. Springer-Verlag, New York, 1984. iv + 248 pp.
Topology. By Klaus Jänich. Springer-Verlag, New York, 1984. ix + 192 pp.

ROBERT F. BROWN
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Here are three textbooks, all apparently on the same subject, all produced by the same publisher in the same year. That doesn't even happen in the "College Algebra market." Is general topology the new growth industry of the academic world?

The facts are likely to be rather less dramatic. The precise timing is probably an accident. The books of Dixmier and Jänich were published earlier (1981 and 1980, respectively) in the languages of their authors and it may be that the translations happened to be ready just as James was coming out. Still, someone at Springer decided to publish all three. Here's where a bit of economics may have come into play. Springer prices these books at nearly 15 cents a page; that's two to four times the cost per page of a college algebra or calculus text. So it seems reasonable to conclude that a pretty modest sales figure will allow the publisher to break even and questions of competition, of the type that haunt publishers of college algebra texts, simply don't arise.

Furthermore, these three books are not all aimed at the same audience. James assumes that his readers will have some background in topology, although he does include a fairly complete development of those rather specialized topics from general topology that he needs for the homotopy theory part of the book. Thus James is a second-level text whereas Dixmier and Jänich are written for the student's first course in topology.

In order to understand better just what a first course in topology means in this country, I took a decidedly unscientific sample of 25 good to excellent American colleges and universities. A look at their catalogues uncovered the unsurprising fact that all offer an undergraduate topology course. Where the contents were specified, the general area was usually identified as "point-set topology" followed by phrases like: topological and metric spaces, compactness, connectedness, and the like. Since the course is not generally prerequisite to any other, I suspect the actual contents of these courses vary from year to year, depending on the choice of text and prejudices of the instructor. About a third of the courses listed advanced calculus as prerequisite while, of the rest, most required only lower-division calculus. The differences in the level of preparation of the students are probably much less than the

catalogues suggest because upper-division math majors, who constitute the majority of the audience for such a course, would likely take it rather late in their undergraduate careers, and thus after advanced calculus in any case. Undergraduate math majors take the course because in many places they are encouraged to do so. In fact two of the schools require the topology course as part of the math major. Another eight present the students with a shopping list of courses, including topology, from which a major has to be constructed by taking most, but not all, of them. In the remaining institutions, topology is an available option for the elective courses, sometimes with a note recommending it to students headed for graduate school. In my experience, the undergraduate topology course is perceived as a good opportunity to instill in serious mathematics students that ill-defined but essential virtue of "mathematical sophistication" that is often the main prerequisite of a graduate course. The parallel with the traditional euclidean geometry course in high school is quite striking; neither course is tied too closely to the student's previous background in mathematics yet both have the potential for offering the student significant insights into what mathematics is really about.

Thus we can imagine the typical student in an undergraduate topology course as a junior or, more likely, senior math major who has taken modern algebra, survived advanced calculus, and sampled some other branches of mathematics at the post-calculus level. Quite likely the student is pretty good at math, likes it (most of the time), and is thinking about the possibility of getting some graduate training, if not immediately then somewhere down the line. In recognition of the fact that, at the undergraduate level at least, mathematics is no longer predominantly male, I'll give my hypothetical student the name of Melissa and see how Melissa-the-math-major reacts when I confront her with the texts of Dixmier and of Jänich.

A quick look at Dixmier's table of contents brought immediately to my mind a sentence from the Preface of John Kelley's pioneering (1955) text for this very course [3]: "I have, with difficulty, been prevented by my friends from labeling it: What Every Young Analyst Should Know." Putting the tables of contents of Dixmier and Kelley side by side, it is even possible to line up roughly equivalent chapters. This is not to suggest that Dixmier's book is a new incarnation of Kelley, but rather that any text that teaches topology with an eye towards more advanced analysis is bound to look superficially like any other one.

When Melissa picks up Dixmier's text, she won't realize that Chapter VIII ("Normed Spaces") is intended to place her foot firmly on the first step of a staircase headed for the heights of Functional Analysis. She will however notice that the text looks a bit different from those she is familiar with. Each paragraph or two is headed with a boldface number such as 6.2.9, often followed by a familiar word; usually "Theorem" but sometimes "Corollary" and, less often, "Example," "Definition," or "Remark." What she won't see is the indication "Proof" after any of those theorems nor an end-of-proof symbol. The reason is that Dixmier doesn't need any of that apparatus because the paragraphs immediately after the statement of the

theorem and up to the next boldface number invariably present the proof. But where, the experienced instructor may ask, are the motivating remarks, special cases, and other ruffles and flourishes with which so many authors like to decorate their arguments, especially the difficult ones? It seems there aren't any. The proofs are neat, formal, and boy are they condensed! They fit just right on the page, between those boldface numbers. Melissa may well admire the fortitude of her French counterparts. She'll find the exercises with some difficulty since they are collected, by chapter, at the end of the book, and she'll notice that they aren't plentiful. Are four really enough to cover 13 tightly written pages on limits and continuity? On the other hand, she should like Dixmier's innovation of labeling particularly important theorems with a "triangle" symbol. That seems a most useful guidepost to the student through this forest of otherwise undifferentiated fact. Perhaps anticipating some criticism, Dixmier notes in his Preface that the choice of these triangled results "entails a large amount of arbitrariness," but it was nice of him to offer the student the benefit of his well-informed judgement in this way.

Melissa thinks Dixmier's text looks a lot like a catalogue from an auction of government-surplus property. On the other hand, as a reader of the Scientific American since junior high, she's pleased that Jänich looks just like what she thought all topology books were like. For instance, there are 13 illustrations in all of Dixmier; Jänich has that many by page 15. A visit to the index may even remind Melissa of some of the words she's seen in the Scientific American from time to time like bordism group, category and functor, Fourier series, Fréchet space, Morse theory, space form, Thom space, and vector bundle.

If Dixmier's is a forward-looking text, in the sense that it focuses on a sort of analytic horizon, Jänich's poses dangers of whiplash as it careens among functional analysis (Chaps. II and IV), algebraic topology (Chaps. V, VII, IX) and point-set topology (the rest). Melissa may wonder how she's going to be able to learn all this trendy mathematics in a book of well under 200 pages, much of it occupied with pictures, which doesn't assume any prerequisite knowledge on the part of the reader. The answer of course is that she isn't going to "learn" all this fashionable, advanced mathematics, rather she's going to "learn about it" which isn't the same thing at all. Bordism groups get a page or so of discussion, Morse theory gets a bit more, but Clifford-Klein forms rather less. As Jänich himself says on page 76, "Those things are admittedly way above what can be done with the tools and on the level of this book, and a critical observer may find it outrageous to talk of them here."

But there are some topics that do get a lot of attention from Jänich. Normal spaces and their basic properties occupy some ten pages, with 14 figures; three-and-a-half pages devoted just to the proof of Urysohn's Lemma. Dixmier's presentation of normality is collected into Theorem 7.6.1, which says that four statements are equivalent: statement (i) is the usual definition of normality (separating disjoint closed sets), (iii) is Urysohn's Lemma and (iv) is Tietze's Extension Theorem. The proof of all this is one of the longest in Dixmier: two-and-a-half pages. In Jänich,

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Why Math? By R. P. Driver. Springer-Verlag, New York, 1984. xiv + 233 pp.

UNDERWOOD DUDLEY
Department of Mathematics and Computer Science, DePauw University, Greencastle, IN 46135

Colleagues, it is a good question. The actual question is, "Why should students who are going to be neither mathematicians, scientists, nor engineers study some topics in precalculus mathematics?", but that would never do as a title for a book. In this essay, I will indicate how the question has been answered in the past, how it is answered now, how it is answered by Professor Driver, and at the end, just as in the textbooks we all use, I will give the *right* answer.

Previous ages have had answers which are no longer given. The Pythagorean Brotherhood (c. -500) held that the study of mathematics aided spiritual development. It is too bad that the Pythagoreans have died out, since their answer should be satisfying to us. It is especially satisfying to me, since what the Pythagoreans were studying was number theory, from which it follows that those of us skilled in elementary number theory are at the peak of human spirituality. But the Brotherhood's time has gone, as has the time (c. 650) when Brahmagupta could write

> As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them.

Ah, had those days survived! Quadratic equations in the House of Representatives! "Can the honorable gentleman from Nebraska find a number which is three more than its reciprocal?" However, those days are gone, probably forever, as are the 1800's, when mathematics was an aid to morality:

> The mathematics are friends to religion, inasmuch as they charm the passions, restrain the impetuosity of the imagination, and purge the mind from error and prejudice.

That was John Arbuthnot, writing on the *Usefulness of Mathematical Learning*. Arbuthnot had a point: in my calculus students I notice very little passion and hardly any imagination, but even so no one makes such claims now.

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1. R. Geoghegan, Review of *General Topology and Homotopy Theory* by I. M. James, Math. Reviews 86d: 55001.
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The turn of the century was probably the time when the most popular reason for studying mathematics was the exercise it gave to the brain:

> Our future lawyers, clergy, and statesmen are expected at the University to learn a good deal about curves, and angles, and numbers and proportions; not because these subjects have the smallest relation to the needs of their lives, but because in the very act of learning them they are likely to acquire that habit of steadfast and accurate thinking, which is indispensible to success in all the pursuits of life.

(J. C. Fitch, *Lectures on Teaching*, 1906.) That *sounds* reasonable, but it is not heard much today because there is no "scientific" evidence that graphing hyperbolas or solving quadratics increases clarity of thought. (It is difficult to think of how such evidence could be collected, short of teaching one identical twin everything up to Green's Theorem while keeping the other ignorant of anything beyond the area of a circle.) During the Second World War, the Navy found that officers who had completed a course in calculus were better officers than those who had not, but that can be explained by saying that, in general, people who can complete a semester of calculus are smarter than people who cannot. Nevertheless, the belief that mathematics is good for the mind persists, notwithstanding the lack of evidence. When I asked a graduate of my school (not a mathematics major) if he ever used the mathematics he had learned, he answered, "I use it every day." Amazed, since his job was running a television station in Tennessee, I pressed him: it turned out that he thought that he used the mathematical method of thought every day. I would have been impolite to ask for examples, so I did not. Though one example does not prove a theorem, I am convinced that his attitude is common among educated people, and common beliefs—the immortality of the soul, the perfectability of humanity and so on—should not be dismissed out of hand.

However, few professional mathematicians justify mathematics on those grounds. Those who bother to justify it at all usually do so on the ground that it is beautiful, or that it is useful. There is no denying either assertion. Mathematics is so useful that there could be no civilization without it, and it is so beautiful that some theorems and their proofs—those which cause us to gasp, or to laugh out loud with delight—should be hanging in museums. During the 1960's beauty was in the saddle: Hardy's *A Mathematician's Apology*, which I will resist the temptation to quote from at length, was reprinted in 1967. Now, things have changed, and in the realm of the elementary textbook, utility is all. We are back to Lancelot Hogben's definition of mathematics in *Mathematics in the Making*:

> Mathematics is the technique of discovering and conveying in the most economical possible way useful rules of reliable reasoning about calculation, measurement and shape.

Rules. *Useful* rules. That is mathematics. Professor Driver is of a similar mind:

> This text aims to show that mathematics is useful to virtually everyone. I hope that users will complete the course with greater confidence in their ability to solve *practical* (my emphasis) problems.

Utility, or beauty? For which reason should students study mathematics? For neither one, I say. Certainly not for utility. The vast majority of the human race, and the vast majority of the college-educated human race never need any mathematics beyond arithmetic to survive successfully. I know arithmetic has been enough for me; has it not been enough for you? To pretend otherwise is to be more than a little fraudulent. There follow some "practical" problems. All my examples are taken from Professor Driver's text, but this is not meant to imply that it is a bad or worthless book. Whatever Springer-Verlag sets its hand to, it does well, and if you must (shudder) teach compulsory General Mathematics to college freshmen, you could probably find very few texts which are superior. Any elementary text with "applied" or "with applications" in its title would serve as well to provide as many examples as anyone would want.

> A motorist drives from city A to city B in 2 hours maintaining a constant speed. On the return trip he increases his speed by five miles per hour and reduces the travel time to 1 hour 45 minutes. How far apart are A and B, and what were the motorist's speeds?

The motorist (a quaint term, surviving only in practical problems) *knows* how far apart A and B are. He knows already. He can look at the odometer, or at the road map. He *knows*.

> A cup of coffee in a room where the temperature is 70° F cools from 190° to 130° in 6 minutes. How much longer will it take to cool down to 85° F?

Has anyone, outside of a textbook, *ever* had to answer this question? Will anyone, ever? Don't we instead take a sip to see if the coffee is cool enough?

> A zero coupon CD costing $703.25 to buy will mature to $1000 in 4 years. Find the effective annual rate of interest if compounded annually.

I *know* the rate. There is a *law*, passed by the Congress of the United States requiring that the rate of interest be stated explicitly. No one *ever* has to do this problem.

> An investment club decided to make a certain stock purchase for a total of $9000 with each member buying an equal share of the price. But, before the transaction was completed, two members left the club and the remaining members had to pay an extra $50 each in order to still produce the $9000. How many members were in the club initially?

If I am a member of the club, I know the answer. If I am not a member of the club, I will ask a member. The member will then either tell me the answer or say that it is none of my business. If the member replies with the conundrum above, he should be beaten about the head until he promises to behave in a civilized manner.

I could go on, and on, and on, producing more, and more, and more examples of the utterly useless and impractical "useful" and "practical" problems. You may think that I am selecting especially objectionable examples, but I am not. Examples are as thick on the ground as dandelions in the spring. The how-far-does-the-bouncing-ball-travel problem is here, the bottle-and-cork-together-cost-$1.10-the-bottle-

mind for his material. Here, he said implicitly, it *is*; make of it what you will. Some of you will make use of it, some of you will go further and see beauty, some of you may have the quality of your thought improved, but for the rest of you (as I read his mind), here is Problem 27 and hundreds like it; their value is exactly the same as the value of any other game.

It is a game. School mathematics is a game. There are definite rules. There are clearly defined outcomes. If you do Problem 27 and get -1, you win. If you get something else, you lose. Like other games, it can be repeated, indefinitely. As in other games, a player can get better and better, up to the limits imposed by talent and application, and feel the improvement. Like other games, it can have beneficial side effects. Human beings like games. They like their clear definitions and hard edges, their definite beginnings and endings, their pleasant rewards of winning. They do not mind too much the unreward of losing since it is, after all, only a game. Children especially like games, which is partly why they put up with mathematics in the schools, every year after every year. When our former students tell us how they liked their mathematics (except they call it "math"), do they not mean that they remember with a glow of pleasure their wins, wins playing against themselves, wins playing against the answer in the back of the book, and wins playing against their classmates, when they were able to do problems others could not, or do them in a better way? Yes, colleagues, that is exactly what they mean.

Colleagues, it is time to stop making excuses for mathematics. It is time to stop writing textbooks asserting that one-variable polynomial calculus has important everyday applications. It is time to stop pretending that geometric series are studied so that we can find how far balls bounce. It is time to retire the bottle-and-cork problem. It is not time to begin writing textbooks asserting that trigonometry presents vistas of breathtaking beauty. Nor is it time to claim that mathematics firms the mind or improves morals. It is time to present mathematics for the sake of mathematics. *We* know that it is useful, *we* know that it is beautiful, and we may even suspect that it makes us clearer thinkers and better people, but for most students it will never be more than a game. We do not need to tell them that, but we should not mislead them otherwise. Let us have the confidence to say that reducing rational expressions to their lowest terms is important, and therefore the student should do the following ten or twenty problems. Let us present mathematics, as mathematics. Why math? *Because*.

Introductory Graph Theory. By Gary Chartrand.

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Graph theory is filled with interesting problems and results that could be explained to someone who has very little mathematical sophistication. Gary

mind for his material. Here, he said implicitly, it *is*; make of it what you will. Some of you will make use of it, some of you will go further and see beauty, some of you may have the quality of your thought improved, but for the rest of you (as I read his mind), here is Problem 27 and hundreds like it; their value is exactly the same as the value of any other game.

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Graph theory is filled with interesting problems and results that could be explained to someone who has very little mathematical sophistication. Gary

Chartrand uses basic concepts of graph theory in various models, many of which are fascinating, and most of which are accessible to anyone. His book is written for college undergraduates, but its appeal is not limited to them.

There is an interesting section in which the game of Instant Insanity (Parker Brothers) is discussed. Four cubes are given, and each of their faces is assigned one of four colors. The problem is to stack the cubes vertically, so that on each side of the stack, all four faces have different color. By trial and error, the solution can be tricky. However, Chartrand explains how a graph can be constructed that can be used to find a solution efficiently. A solution exists if the graph has two subgraphs of a certain type, and a satisfactory stacking is easily determined by these two subgraphs. What is interesting is that the graphical representation elegantly reduces the stacking problem to a simple inspection of the graph.

Some basic theoretical results are presented, and the proofs are quite readable. Some of the theoretical results are motivated by well-known puzzles. For example, consider the problem of the three houses and three utilities (water, gas, electricity): can the utilities be connected to the houses by distinct lines, such that no line crosses another? (They can, if one of the lines passes directly underneath one of the houses, but that possibility is implicitly not allowed.) Represent the utilities and the houses as vertices of a graph, where each utility-vertex is joined by an edge to each house-vertex. The resulting graph (with $p = 6$ vertices and $q = 9$ edges) is called $K_{3,3}$. Chartrand proves Euler's polyhedron formula ($q + 2 = p + r$, for a planar graph G with p vertices, q noncrossing edges, and r regions), and then uses this to show that $K_{3,3}$ is not planar (i.e., cannot be drawn in the plane without crossings). The graph $G = K_5$, consisting of 5 vertices, each pair of them joined by an edge, is shown to be nonplanar, too. Then Kuratowski's famous theorem is stated without proof: an arbitrary graph G is planar if and only if some subgraph of G is topologically equivalent to K_5 or $K_{3,3}$. Chartrand presents these results clearly, without assuming that the reader ever heard of topology. He continues in another chapter by asking the reader to draw K_5, K_6, and K_7 on the torus, without allowing crossing edges.

A final chapter is an elementary discussion of the automorphism group of a graph. For certain students, chapters such as these could be omitted, but more advanced students would likely prefer to read this chapter.

An instructor using this book as a text would occasionally want to add a few observations as problems, or state conjectures that are missing. For example, there is an extensive discussion of trees (both theory and applications), but there is no statement that a tree has some vertices that are each incident with just one edge. This observation could be included as a problem.

There are few open questions presented as conjectures in this book. This is unfortunate, because many students are attracted to mathematics by the lure of unsolved problems (such as Fermat's Last Theorem in number theory). Graph theory is full of open problems that can be presented at a level that a beginner could understand. For instance, there are conjectured conditions in the literature for a

graph to be hamiltonian (i.e., to have a cycle containing every vertex of the graph). Another famous problem is the Reconstruction Conjecture: let G be a graph on n vertices $\{u_1, u_2, \ldots, u_n\}$ and let H be a graph on n vertices $\{v_1, \ldots, v_n\}$. If, for each i, the subgraphs $G - u_i$ and $H - v_i$ are isomorphic, then show that G and H are isomorphic. The last conjecture has been called a "disease," because it has affected so many people.

One open problem Chartrand does present would certainly whet many students' interest, and that concerns Ramsey numbers. The complete graph K_n is a graph on n vertices and $\binom{n}{2}$ lines, where every pair of vertices is joined by a line. A red-blue coloring of K_n is an assignment of the colors red and blue to the lines of K_n, so that each line has one color. For a given integer t, what is the smallest integer $n = n(t)$ (the Ramsey number of t), such that every red-blue coloring of K_n contains a K_t whose edges are either all red or all blue? For example, $n(2) = 2$, $n(3) = 6$, and $n(4) = 18$. No general formula for Ramsey numbers is known.

There is a good quantity of problems throughout the text. Since they vary in difficulty, one can choose those problems that are at the appropriate level. Furthermore, there are answers to selected excercises.

The index is, fortunately, more than just an index of definitions. This can be helpful to a reader, too. However, there are a few omissions: Euler is credited on page 196 for his famous polyhedron formula, $q + 2 = p + r$ (proved on pages 195–196), but the Euler references in the index point only to the Bridges of Königsberg Problem and Eulerian graphs. Kuratowski's characterization of planar graphs is stated without any mention of his name (page 199).

At the end of each chapter there are helpful suggestions for further reading. Most of the references are to pre-1977 books and papers. *Introductory Graph Theory* was first published in 1977 under the title *Graphs in Mathematical Models*, and in this updated version, very few recent references were added. One that ought to have been added in my opinion, is Bondy and Murty's *Graph Theory with Applications* (Elsevier North Holland), which is written for a more sophisticated audience but is, nevertheless, very readable. The actual contents of the various chapters of *Introductory Graph Theory* are, however, up-to-date.

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatabilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, S*, L*. *Littlewood's Miscellany.* Ed: Béla Bollobás. Cambridge U Pr, 1986, 200 pp, \$32.50; \$11.95 (P). [ISBN: 0-521-33058-0; 0-521-33702-X] An edited reprinting of Littlewood's famous *A Mathematician's Miscellany* (1953, Methuen), together with *The Mathematician's Art of Work* (1967, Rockefeller U Pr). Includes several new sections containing a miscellany of Littlewood's observations on people and on academic life, as well as a lengthy foreward by editor Bollobás. Witty, pithy, unremittingly inventive. LAS

General, L*.** *Mathematical Models from the Collections of Universities and Museums.* Ed: Gerd Fischer. Friedr Vieweg & Sohn, 1986, xii + 129 pp, DM 118 [ISBN: 3-528-08991-1]; *Commentary,* viii + 83 pp, DM 118. A boxed set of two anniversary volumes, celebrating 200 years of Vieweg-Verlag, containing 132 photos and commentary on the collection of models of mathematical surfaces created in the latter part of the nineteenth century in the leading European research institutes, most notably Göttingen. Grouped by mathematical type: analytic geometry, algebraic surfaces, differential geometry, bodies of constant width, regular star-polyhedra, real projective plane, complex functions. Companion volume provides crisp mathematical explanations for each surfaces' features. Four reproductions of plaster models are also available from the publisher. LAS

General, P, L**.** *The Joy of TEX: A Gourmet Guide to Typesetting with the AMS-TEX Macro Package.* M.D. Spivak. AMS, 1986, xviii + 290 pp, \$32 (P). [ISBN: 0-8218-2999-8] The definitive guide to AMS-TEX, the macro package for the TEX typesetting system designed by the American Mathematical Society as the standard for their journals and books. Written with the same mix of wit and clever examples as Knuth's *The TEXbook* (TR, October 1984; August-September 1986), this volume is actually a better introduction to TEX for novices since it does not attempt to explain everything. Opening sections deal with regular text editing and error responses; later sections deal with mathematics and with AMS typesetting standards for theorems, displayed equations, references, etc. Appendices deal with conflicts between AMS-TEX and plain TEX (there are some), and with extension fonts yet to come from AMS. For some the joy of *Joy* will be diminished by clever yet unrelenting texual innuendo densely embedded in headings, text, exercises, and puns. LAS

General, C, P.** *MicroTEX* (Version 1.5A1). MS-DOS or PC-DOS. Donald Knuth, David Fuchs. Addison-Wesley, 1986. A true implementation of Version 2-ϵ of TEX82 which works well on standard MS-DOS microcomputers (with at least two disk drives, 512K memory, and Version 2 or 3 of MS-DOS). Works much better with a hard disk. Installation requires care, especially since the instruction manual does not clearly describe the directory trees assumed by the software. Operation is quick and effective: MicroTEX produces ".dvi" (device independent) output files from properly prepared TEX input files that can either be sent through a device driver to a printer or transferred to any other computer environment running TEX . (Warning: This release of MicroTEX comes with "plain" macros that use the earlier "am" (almost modern) font tables, whereas most timesharing installations of TEX use "cm" (computer modern) font tables. However, MicroTEX permits

substitutions in case one wants to switch.) Comes with utilities to compile customized versions of TeX (e.g., AMS-TeX) from external macro packages, to convert ".dvi" files to text form as a debugging aid, as well as tools to convert the encoded font metric (".tfm") files to and from a human-readable form. Includes a copy of *The TeXbook* by Knuth (TR, August-September 1986), and a barely adequate 56 page system guide giving details of how MicroTeX interacts with MS-DOS. Extra cost options include macro files for LaTeX, various device drivers to enable the ".dvi" files to drive a real printer, and a screen previewer (i.e., a "device driver" for a CRT) for PCs with graphics capability. LAS

General, C*, P. *TeX Preview.* MS-DOS or PC-DOS. Textset, Inc. (416 Fourth St., PO Box 7993, Ann Arbor, MI 48107; (313)996-3566). A program that will read any ".dvi" file prepared by the typesetting system TeX82 and display it on the screen of a graphics-equipped IBM PC-compatible computer. Will display ".dvi" files produced by TeX on either microcomputers or timesharing systems (but beware of font incompatibilities caused by mixing the "am" pixel files as supplied with the "cm" standard for timesharing installations of TeX). Requires a hard disk for efficient use; the associated ".pxl" pixel files occupy nearly 3 megabytes. Operates at three levels of magnification: in medium, only about half a page fits on the screen; in small, the entire page fits but characters can barely be read; in large, details of subscripts become clear. A subset of standard medium resolution pixel files can be configured for a two drive system. Simple commands allow the user to see different parts of a page or different pages in the file. Includes a *very* cryptic user manual. LAS

Precalculus, T???(13: 1). *Mathematics: A Second Start.* S.G. Page. Math. & Its Applic. Halsted Pr, 1986, 409 pp, $22.95 (P). [ISBN: 0-470-20752-3] Attempts to cover algebra, some trigonometry and beginning calculus. Generally unappealing with unattractive format: print which is difficult to read and choppy prose. JNC

Precalculus, T. *Algebra and Trigonometry with Analytic Geometry, Sixth Edition.* Earl W. Swokowski. Prindle, Weber & Schmidt, 1986, xi + 659 pp. [ISBN: 0-87150-878-8] Major revision: more emphasis on graphs; additional exercises and applications to diverse disciplines; discussions somewhat less formal; more attention to natural exponentiation and logarithmic functions; rearrangement of certain topics. (*Fourth Edition,* TR, December 1978.) LCL

Education, S(17), P. *Chalking It Up: Advice to a New TA.* Bruce A. Reznick. Random House/Birkhauser, 1985, 14 pp, (P). "Try to present yourself as an alpine guide...rather than as part of the mountain." Sensible, witty, realistic hints for new teaching assistants—on preparing, on lecturing, on grading, on coping. Available from the publisher for distribution to TAs. LAS

Education, S, P. *How to Enrich Geometry Using String Designs.* Victoria Pohl. NCTM, 1986, iv + 68 pp, $5 (P). [ISBN: 0-87353-227-9] Contains directions for creating string designs on polygons and polyhedra, created for grades 6-10. JNC

Education, S, P. *How to...Teach Perimeter, Area, and Volume.* Vern Beaumont, Roberta Curtis, James Smart. NCTM, 1986, iii + 67 pp, $5 (P). [ISBN: 0-87353-232-5] One of the NCTM "How To..." series, this booklet contains suggestions and resource pages for use by pre-service and in-service teachers of elementary and middle school mathematics. JNC

Education, S, P. *Activities for Implementing Curricular Themes from the Agenda for Action.* Ed: Christian R. Hirsch. NCTM, 1986, iii + 204 pp, $9 (P). [ISBN: 0-87353-229-5] A collection of 30 activities for grades 7-12 from the *Mathematics Teacher;* themes are problem solving, basic skills, calculator and computer activities, and manipulatives. JNC

History, L*. *Abrégé d'histoire des mathématiques 1700-1900.* Jean Dieudonné. Hermann, 1986, x + 517 pp, 210F (P). [ISBN: 2-7056-6024-0] New revised edition of the 1978 work in one volume (*First Edition,* TR, August-September 1979). Describes the development of several important parts of mathematics from around 1700 until 1900. Attempts to help understand present day mathematics by placing its most elementary notions into their historical contexts and by connecting them to the applications from which they came. Begins with remarks on mathematicians and how they work. Chapters on analysis in the 18th century, algebra and geometry up to 1840, algebra since 1840, analytic functions, number theory, foundations of analysis, elliptic functions, abelian integrals, functional analysis, differential analysis, topology, logic and the axiomatic method. Chapter bibliographies. Historical index. Terminology index. RJA

History, S*, P, L*. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass.* Umberto Bottazzini. Springer-Verlag, 1986, 332 pp, $39. [ISBN: 0-387-96302-2] A well-written historical account of nineteenth century analysis, emphasizing more than have other volumes on the same theme the foundations of complex analysis. Contains many helpful quotations suggesting how famous mathematicians of the past were thinking about their mathematics. Written in a style and level suitable for undergraduate students. LAS

Combinatorics, P. *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* George E. Andrews. CBMS Reg. Conf. Ser. in Math., No.

66. AMS, 1986, xii + 130 pp, (P). [ISBN: 0-8218-0716-1] A very readable work on q-series, which underlie the generating functions of diverse combinatorial and analytic objects. Applications have recently appeared to Lie algebras and statistical mechanics (which are not so distantly related!). Based on expository lectures for a CBMS conference at Arizona State University in 1985. BC

Number Theory, P. *Lecture Notes in Mathematics-1219: Stabile Modulformen und Eisensteinreihen.* Rainer Weissauer. Springer-Verlag, 1986, iii + 147 pp, $12.80 (P). [ISBN: 0-387-17181-9] A detailed study of Eisenstein series and Hecke summation (to cope with convergence) for Siegel modular forms, using the Siegel operator. Applications to stable forms and theta functions. BC

Number Theory, P. *Distribution Theorems of L-functions.* David Joyner. Res. Notes in Math. Ser., V. 142. Longman Scientific & Technical (US Distr: Wiley), 1986, 247 pp, $44.95 (P). [ISBN: 0-470-20373-0] Expository chapters on L-functions and distribution theorems for the Riemann zeta function (mostly work of Atle Selberg), followed by a generalization to "motivic" and Langland's L-functions. Some formidable formulas (par for the course in analytic number theory), but the writing style is informal and informative. BC

Number Theory, P. *Lectures on Siegel Modular Forms and Representation by Quadratic Forms.* Y. Kitaoka. Springer-Verlag, 1986, 227 pp, $11 (P). [ISBN: 0-387-16472-3] Representations by quadratic forms are studied both analytically, via Fourier coefficients of Siegel modular forms, and arithmetically. Self-contained lectures with lots and lots of detail. BC

Number Theory, S(16), P. *Fibonacci Numbers and Their Applications.* Ed: Andreas N. Philippou, Gerald E. Bergum, Alwyn F. Horadam. Math. & Its Applic. D Reidel, 1986, xxiv + 304 pp, $59. [ISBN: 90-277-2234-X] The proceedings of the first international conference on Fibonacci numbers and their applications which was held August 27-31, 1984 at the University of Patras, Greece. CEC

Linear Algebra, S(13-14). *Geometry and Linear Algebra: An Introduction in Two and Three Dimensions.* Gillian Thornley, Michael Hendy. Dunmore Pr, 1986, 365 pp, $49.95 (P). [ISBN: 0-86469-053-3] Aim is to teach techniques of linear algebra used in applications and to prepare for further study in mathematics and statistics. Approach is to study certain topics in two- and three-dimensional geometry. Start with solution of systems of linear equations and, throughout text, use modifications of the technique given here. Next, introduce vectors to study lines and planes; this leads to matrices and linear transformations. Conclude with eigenvalues, eigenvectors, and quadratic equations. Many exercises. Solutions to exercises. Examples. Index. RJA

Linear Algebra, S(15). *An Introduction to Matrices, Sets and Groups for Science Students.* G. Stephenson. Dover, 1986, xi + 164 pp, $5 (P). [ISBN: 0-486-65077-4] An unabridged and slightly corrected republication of the '74 edition published by Longman Group; contains a wealth of information, but systems of linear equations are solved only with Cramer's rule: still a best buy. JNC

Algebra, P. *Field Arithmetic.* Michael D. Fried, Moshe Jarden. Ser. of Modern Surv. in Math., B. 11. Springer-Verlag, 1986, xvii + 458 pp, $94.50. [ISBN: 0-387-16640-8] A thoroughly engaging exposition of the elementary (in the sense of first order logic) theory of fields. The focus is on arithmetic, algebraic, and algorithmic properties of pseudo-algebraically closed fields. The treatment is relatively self-contained with background chapters on algebra (profinite groups), number theory (zeta functions, density theorems), algebraic geometry (Riemann-Roch, varieties), and logic (ultraproducts, decision procedures). SG

Algebra, P. *Lecture Notes in Mathematics-1220: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin.* Ed: M.-P. Malliavin. Springer-Verlag, 1986, iv + 200 pp, $19.40 (P). [ISBN: 0-387-17185-1] Seminar proceedings from Paris, 1985. LAS

Complex Analysis, P. *Lecture Notes in Mathematics-1198: Séminaire d'Analyse P. Lelong—P. Dolbeault—H. Skoda.* Ed: P. Lelong, P. Dolbeault, H. Skoda. Springer-Verlag, 1986, x + 260 pp, $21.30 (P). [ISBN: 0-387-16762-5] Papers presented at the 1983-84 seminar; a sequel to *LNM-919* and *LNM-1028.* LAS

Differential Equations, T*(15: 1), S, L. *Fundamentals of Differential Equations.* R. Kent Nagle, Edward B. Saff. Benjamin/Cummings, 1986, xiv + 653 pp, $31.95. [ISBN: 0-8053-6810-8] A standard beginning course which keeps the student in mind. Includes such features as projects, numerical algorithms, historical footnotes, motivating problems and modern, as well as classical, applications. Well written with lots of worked examples and exercises. CEC

Differential Equations, T(14-15: 1), S, L. *A First Look at Perturbation Theory.* James G. Simmonds, James E. Mann, Jr. Robert E Krieger, 1986, vii + 136 pp, $12.50. [ISBN: 0-89874-816-X] From the preface: "Perturbation theory is fun, useful, and, we believe, accessible to undergraduates in engineering and the physical sciences." Assumes only calculus and an introduction to ordinary differential equations. Roots of polynomials, Poincaré's method, the two-scale method, the $WKB(J)$ approximation,

transition point problems, boundary layer theory. Applications, examples, exercises. DFA

Differential Equations, P. *Lecture Notes in Mathematics-1186: Lyapunov Exponents.* Ed: L. Arnold, V. Wihstutz. Springer-Verlag, 1986, vi + 374 pp, $28.80 (P). [ISBN: 0-387-16458-8] Proceedings of a November 1984 workshop held in Bremen, Germany. Opens with a survey of Lyapunov exponents by L. Arnold and V. Wihstutz. LAS

Differential Equations, S(18). *Infinite Dimensional Morse Theory and Its Applications.* Kung-Ching Chang. Pr U Montreal, 1985, 154 pp, $19 (P). [ISBN: 2-7606-0734-8] Lectures using isolated critical point theory of Gromoll-Meyer within a variational frame. Applications to elliptic boundary value problems and periodic solutions to Hamiltonian systems. Simple proof of Arnold's conjecture on the number of fixed points of symplectic maps. MR

Functional Analysis, P. *Lecture Notes in Mathematics-1200: Asymptotic Theory of Finite Dimensional Normed Spaces.* Vitali D. Milman, Gideon Schechtman. Springer-Verlag, 1986, viii + 156 pp, $15 (P). [ISBN: 0-387-16769-2] A brief monograph introducing results, problems, and methods in the theory of normed spaces. "Asymptotic" refers to behavior as the (finite) dimension of a space grows to infinity. Five brief appendices cover foundational material. PZ

Functional Analysis, T(18: 1, 2), S, P. *An Invitation to von Neumann Algebras.* V.S. Sunder. Universitext. Springer-Verlag, 1986, xiv + 171 pp, $29.80 (P). [ISBN: 0-387-96356-1] Theory of von Neumann algebras is used in ergodic theory, mathematical physics, differential geometry, K-theory, and elsewhere. This is a short, selective, enthusiastic introduction to the subject's basics, accessible to analytically well-prepared graduate students. There are numerous exercises, some with hints; solutions are used freely in the sequel. PZ

Analysis, P. *Ten Papers in Analysis.* V.A. Zmorovich, et al. AMS Transl., Ser. 2, V. 131. AMS, 1986, vii + 120 pp. [ISBN: 0-8218-3106-2] Translated from the Russian; primarily in the area of complex analysis. BH

Analysis, P. *Function Estimates.* Ed: J.S. Marron. Contemp. Math., V. 59. AMS, 1985, ix + 178 pp, (P). [ISBN: 0-8218-5062-8] A report of a conference held at Humboldt State University in 1985; sixteen papers deal with spline estimations and convolution problems. AWR

Analysis, S?(18). *The Laplace Transform.* Richard E. Bellman, Robert S. Roth. World Science, 1984, xv + 158 pp, $21. [ISBN: 9971-966-73-5] Looks at diverse applications of Laplace transform techniques to linear ordinary differential equations, differential-difference equations, partial differential equations, and partial differential equations from an applied point of view. Last chapter discusses numerical inversion of the Laplace transform. BH

Analysis, T(17), S*. *General Theory of Functions and Integration.* Angus E. Taylor. Dover, 1985, x + 437 pp, $10 (P). [ISBN: 0-486-64988-1] A reprint of the second, corrected printing of a classic book first published by Blaisdell in 1966. AWR

Analysis, P, L*.** *An Atlas of Functions.* Jerome Spanier, Keith B. Oldham. Hemisphere Pub, 1987, ix + 700 pp, $149.50. [ISBN: 0-89116-573-8] A stunning reconceptualization of the old-fashioned handbook of mathematical tables. Detailed information on 2^8 families of functions (from quadratic polynomials and Legendre polynomials to complete and incomplete elliptic integrals), including definitions, expansions, "intrarelationships" (e.g., identities and recurrence relationships), multicolor computer-generated graphs, algorithms for calculating numerical values to eight significant digits (on a scientific calculator or simple computer), approximations, complex arguments, and much more. Establishes a new genre in presentation of mathematical information. An essential reference tool for every user of applied calculus. LAS

Algebraic Geometry, P. *The Lefschetz Centennial Conference, Part I: Proceedings on Algebraic Geometry.* Ed: D. Sundararaman. Contemp. Math., V. 58. AMS, 1986, ix + 275 pp, $29 (P). [ISBN: 0-8218-5061-X] A collection of 21 papers by many eminent mathematicians. Included are Lefschetz's own autobiographical article from the *Bulletin,* and Hodge's mathematical obituary to Lefschetz. SG

Algebraic Geometry, P. *Lecture Notes in Mathematics-1196: The Enumerative Theory of Conics after Halphen.* Eduardo Casas-Alvero, Sebastian Xambó-Descamps. Springer-Verlag, 1986, ix + 130 pp, $11.60 (P). [ISBN: 0-387-16495-2] Exposition of modern proofs and generalizations of classical theorems of Halphen on the number of projective complex conics satisfying certain conditions. SG

Algebraic Geometry, P. *Geometry and Arithmetic: Around Euler Partial Differential Equations.* Rolf-Peter Holzapfel. Math. & Its Applic. D Reidel, 1986, 184 pp, $34.50. [ISBN: 90-277-1827-X] A study of systems of Euler-Picard partial differential equations in arbitrary dimensions with application to the classification of certain algebraic surfaces. SG

Geometry, S(15-16), L*. *The Adventures of Archibald Higgins: Here's Looking at Euclid (And Not Looking at Euclid).* Jean-Pierre Petit. Transl: Ian Stewart. William Kaufmann, 1985, 63 pp, $7.95 (P). [ISBN: 0-86576-092-6] Translation of *Le Geometricon,* one in a series of French comic book interpretations of serious science: the adventures of one

Archibald Higgins in the land of intrinsic geometry, geodesics and hyperspheres. Sophisticated geometery in an attractive, informal package. Written by a physicist-artist in Aix-en-Provence. LAS

Algebraic Topology, P. *Elliptic Structures on 3-Manifolds.* C.B. Thomas. London Math. Soc. Lect. Note Ser., V. 104. Cambridge U Pr, 1986, 122 pp, $16.95 (P). [ISBN: 0-521-31576-X] The question considered is: If a finite group G acts freely on S^3, then is the resulting orbit space homeomorphic to an elliptic manifold, i.e., a manifold of constant positive curvature? The principal result is a reduction theorem (to cyclic subgroups) in case G is solvable. The case $G = SL(2, F_5)$ is also considered as are the actions of certain groups on certain CW-complexes. SG

Operations Research, T(18: 1), S, P. *Regeneration and Networks of Queues.* Gerald S. Shedler. Appl. Prob. Springer-Verlag, 1987, viii + 224 pp, $24.80. [ISBN: 0-387-96425-8] Discrete-event and regenerative simulation methods applied to Markovian and non-Markovian networks of queues. Written for queueing specialists and students; assumes familiarity with stochastic processes. BC

Operations Research, S, P. *Lecture Notes in Economics and Mathematical Systems-260: Degeneracy Graphs and the Neighbourhood Problem.* H.-J. Kruse. Springer-Verlag, 1986, viii + 128 pp, $14.60 (P). [ISBN: 0-387-16049-3] An introduction to the theory of degeneracy graphs which consists of the terminology and definitions needed to understand this new area of operations research. The author then gives an algorithm to solve the neighborhood problem. CEC

Optimization, T(16-17). *Mathematical Vector Optimization in Partially Ordered Linear Spaces.* Johannes Jahn. Methoden und Verfahren der math. Physik, B. 31. Verlag Peter Lang, 1986, 310 pp, Sfr. 65,00 (P). [ISBN: 3-8204-8940-1] Three main sections: 1) Convex analysis: including standard existence theorems and Fréchet derivatives; 2) Theory of vector optimization: scalarization, generalized multiplier rules, and duality; 3) Mathematical applications: simultaneous approximation, Chebyshev vector approximation, n player differential games. Attractive, but no exercises. AWR

Optimization, T(17-18: 2), P. *Nondifferentiable Optimization.* Vladimir F. Dem'yanov, Leonid V. Vasil'ev. Springer-Verlag, 1985, xvii + 452 pp, $72. [ISBN: 0-387-90951-6] A systematic exposition of the theory of optimization of non-differentiable functions of several variables. Covers convex analysis, subdifferentials, constrained optimization and quasidifferential functions. SM

Dynamical Systems, P. *Advances in Nonlinear Dynamics and Stochastic Processes.* Ed: R. Livi, A. Politi. World Scientific, 1985, viii + 221 pp, $26. [ISBN: 9971-50-018-3] Proceedings of a meeting held in Florence, Italy on January 7-8, 1985. Contains 19 papers of interest to those working in mathematical physics. SM

Control Theory, T(16-17: 1), L. *Introduction to Mathematical Control Theory, Second Edition.* S. Barnett, R.G. Cameron. Appl. Math. & Comput. Sci. Ser. Clarendon Pr, 1985, xi + 404 pp, $29.95. [ISBN: 0-19-859640-5] A good yet inexpensive text for a first course in control theory at the advanced undergraduate or graduate level. Brief introduction to necessary matrix theory. Multivariable systems; stability; optimal control. Numerous exercises. (*First Edition,* TR, February 1976.) SM

Probability, S, L*. *The Craft of Probabilistic Modelling: A Collection of Personal Accounts.* Ed: J. Gani. Appl. Prob. Springer-Verlag, 1986, xiv + 313 pp, $42.50. [ISBN: 0-387-96277-8] Nineteen essays by pioneers in the art of mathematical modelling based on probabilistic methods: a collage of motivation, brief biographies, blind alleys, and successful strategies ranging from genetics to quantum mechanics, from early exploration to mature models. LAS

Probability, P, L. *Extremal Graph Theory with Emphasis on Probabilistic Methods.* Béla Bollobás. CBMS Reg. Conf. Ser. in Math., No. 62. AMS, 1986, vii + 64 pp, $12 (P). [ISBN: 0-8218-0712-9] Lectures given at the NSF-CBMS Regional Conference held in June 1984 at Emory University, Atlanta. One of the aims of the lectures was to update the author's book *Extremal Graph Theory* (TR, January 1979). This monograph focuses on a close look at a few of the deeper results, e.g., the use of probabilistic ideas (such as random graphs) in tackling main line extremal problems. LCL

Statistics, P. *Proceedings of the Thirty-First Conference on the Design of Experiments.* US Army Research Office (PO Box 12211, Research Triangle Park, NC 27709). Report No. 86-2. 1986, xi + 280 pp, (P). Proceedings of a conference held in Madison, Wisconsin, in October 1985. Technical sessions deal with time series and multivariate analysis, consistency analysis, experimental design, statistical modeling, data analysis, and reliability and quality control. Also includes invited talks on other statistical topics of current interest to Army personnel. RSK

Statistics, P*. *Robustness of Statistical Methods and Nonparametric Statistics.* Ed: Dieter Rasch, Moti Lal Tiku. Theory & Decision Lib., Ser. B. D Reidel, 1985, 172 pp, $54. [ISBN: 90-277-2076-2] Proceedings of a conference held in Schwerin, GDR, May 29-June 4, 1983. Contains most of the invited and contributed papers. Aim of the conference was "mainly to present new results regarding the robustness of classical statistical methods and to compare their performance with nonparametric procedures." RSK

Statistics, P. *Proceedings of the Fourth Pannonian Symposium on Mathematical Statistics.* D Reidel, 1985. *Volume A: Probability and Statistical Decision Theory.* Ed: F. Konecny, et al. xi + 340 pp, $56 [ISBN: 90-277-2089-4]; *Volume B: Mathematical Statistics and Applications.* Ed: Wilfried Grossmann, et al. viii + 315 pp, $53. [ISBN: 90-277-2088-6] Contains 45 of the 92 papers presented at this symposium, held in Bad Tatzmannsdorf, Austria, in September 1983. In *Volume A* the probability papers concentrate on stochastic processes and invariance principles, while the statistics papers deal primarily with decision theory and nonparametric estimation. *Volume B* contains papers in three main categories: applications of probability theory, mathematical statistics for models of real data, and aspects of computational statistics and the application of Monte Carlo methods. RSK

Statistics, P. *Lecture Notes in Statistics-39: Optimal Unbiased Estimation of Variance Components.* James D. Malley. Springer-Verlag, 1986, ix + 146 pp, $18.30 (P). [ISBN: 0-387-96449-5]

Statistics, S(15-18), P, L. *The Statistical Analysis of Compositional Data.* J. Aitchison. Mono. on Stat. & Appl. Prob. Chapman & Hall, 1986, xv + 416 pp, $49.95. [ISBN: 0-412-28060-4] Presents a unified approach to the statistical methodology for compositional data (where the components of a composition are subject to a unit-sum constraint). A diskette is available which supplies programs for the method discussed. FLW

Statistics, S*(13-18), P*, L*. *Drawing Inferences from Self-Selected Samples.* Ed: Howard Wainer. Springer-Verlag, 1986, xiii + 163 pp, $18.50. [ISBN: 0-387-96379-0] Four papers and discussions from a conference on the topic, one that is pertinent to judging educational progress by means of SAT scores. FLW

Statistics, S(14-18), P, L. *Model Selection.* H. Linhart, W. Zucchini. Ser. in Prob. & Math. Stat. Wiley, 1986, xii + 301 pp, $34.95. [ISBN: 0-471-83722-9] Considers the general problem of the selection of the appropriate probability model in various classical statistical situations. FLW

Statistics, P. *Lecture Notes in Mathematics-1215: Lectures in Probability and Statistics.* Ed: G. del Pino, R. Rebolledo. Springer-Verlag, 1986, 491 pp, $40.50 (P). [ISBN: 0-387-16822-2] Papers on robustness (in Spanish), stochastic particle systems, time series, and martingales. FLW

Statistics, T(15-18: 1), S, P. *Sequential Methods in Statistics, Third Edition.* G. Barrie Wetherill, Kevin D. Glazebrook. Mono. on Stat. & Appl. Prob. Chapman & Hall, 1986, xi + 264 pp, $33. [ISBN: 0-412-28150-3] This edition contains new material on sequential resource application, sequential estimation, and sequential decision theory. (*First Edition,* TR, May 1967; *Second Edition,* TR, May 1976.) FLW

Computer Science, P. *Lecture Notes in Computer Science-225: Third International Conference on Logic Programming.* Ed: Ehud Shapiro. Springer-Verlag, 1986, ix + 720 pp, $39 (P). [ISBN: 0-387-16492-8] Proceedings of a conference that took place in London, July 14-18, 1986. Contains invited talks and papers by leading researchers on parallel applications and implementations, theory and higher-order functions, program analysis, applications and teaching, implementations and data bases, theory and negation, compilation, models of computation and implementation. Contains a wealth of references. RJA

Computer Science, P*. *Non-Uniform Random Variate Generation.* Luc Devroye. Springer-Verlag, 1986, xvi + 843 pp, $68. [ISBN: 0-387-96305-7] An immense work on computer generation of random variate. Does not treat Monte Carlo methods, but does cover nearly everything else you would ever want to know, with a useful annotated bibliography. TAV

Computer Science, P. *Studies in Complexity Theory.* Ed: Ronald V. Book, et al. Res. Notes in Theoret. Comput. Sci. Wiley, 1986, 226 pp, $22.95 (P). [ISBN: 0-470-20293-9] Papers based on three expository talks at a conference at the University of California, Santa Barbara in 1983. Papers show relationships between the $P = ?NP$ problem and related areas: the use of discrete complexity theory for the study of numerical computation, sparse sets and structure of complexity classes, problems in logic related to models of arithmetic. RM

Computer Science, P. *Lecture Notes in Computer Science-224: Current Trends in Concurrency: Overviews and Tutorials.* Ed: J.W. de Bakker, W.-P. de Roever, G. Rozenberg. Springer-Verlag, 1986, xii + 716 pp, $46.50 (P). [ISBN: 0-387-16488-X] Proceedings from a workshop in concurrency held by the Dutch Concurrency Project in June 1985. Eleven readable papers on current trends in concurrency with emphasis on syntactic, semantic, and proof-theoretic issues. Topics include logic programming, communicating sequential processes, and Petri nets. RM

Computer Science, T*(14: 1), L. *Computer Organization.* Michael Andrews. Prin. of Comput. Sci. Ser. Computer Science Pr, 1987, xvi + 556 pp, $37.95. [ISBN: 0-88175-114-6] Designed for use in a first course on computer organization and assembly language programming. Presents hierarchical views of computer organization from both hardware and software perspectives. Similar in spirit to Tanen-

baum's *Structured Computer Organization.* Merits serious course adoption consideration. AO

Computer Science, P. *Computational Complexity.* K. Wagner, G. Wechsung. Math. & Its Applic. D Reidel, 1985, 551 pp, $79. [ISBN: 90-277-2146-7] This book deals with computational complexity of algorithms having the full power of Turing machines (as opposed to restricted notions such as automata, straight line programs, or Boolean networks). To this extent, the book aspires to cover all the disciplines of this theory which have emerged up to 1984: main features of each discipline, and additional results and references to the literature. Requires a familiarity with effective computability, recursive functions, and formal languages. LCL

Computer Science, P. *Lecture Notes in Computer Science-222: Advances in Petri Nets 1985.* Ed: G. Rozenberg. Springer-Verlag, 1986, vi + 498 pp, $30 (P). [ISBN: 0-387-16480-4] Contains a tutorial and some papers from the Sixth European Workshop on Applications and Theory of Petri Nets held in Espoo, Finland in June 1985. Also contains some papers submitted directly for inclusion in the volume. RJA

Computer Science, T(16-18: 1), S, P. *Algebraic Approaches to Program Semantics.* Ernest G. Manes, Michael A. Arbib. Texts in Mono. in Comp. Sci. Springer-Verlag, 1986, xiii + 351 pp, $54. [ISBN: 0-387-96324-3] Introduction to denotational semantics of programming languages (the denotation being the function a program computes) in algebraic/category theoretical setting. Scott-Strachey theory, plus the authors' theory of iteration semantics and guarded commands which relates the theory to the logic-based assertion semantics of Floyd, Dijkstra, and Hoare. Control and recursion semantics, data types derived as fixed points of functors. Pure algebraists can find here that they have something to say to computer scientists (and vice versa)! RM

Computer Science, T(13: 2). *Introduction to Computer Mathematics.* Eleanor H. Ninestein. Scott Foresman, 1987, 452 pp, $27.95. [ISBN: 0-673-18205-3] Provides a foundation in mathematics for students planning to study computer-related fields. Topics include: sets, logic, numeration systems, polynomials, function, systems of equations, linear programming, probability and statistics. Algorithms (structured via pseudocode and flow charts) are used throughout. JNC

Computer Science, P. *Lecture Notes in Computer Science-226: Automata, Languages and Programming.* Ed: Laurent Kott. Springer-Verlag, 1986, ix + 474 pp, $27.50 (P). [ISBN: 0-387-16761-7] Proceedings of the 13th International Colloquium on Automata, Languages and Programming, held July 15-19, 1986 in Rennes, France. This is the annual European summer conference on theoretical computer science and is sponsored by the European Association on Theoretical Computer Science. 48 papers. Author index. RJA

Computer Science, P. *Lecture Notes in Computer Science-230: 8th International Conference on Automated Deduction.* Ed: Jörg H. Siekmann. Springer-Verlag, 1986, ix + 708 pp, $50.50 (P). [ISBN: 0-387-16780-3] Proceedings of a conference held in Oxford, England, in July 1986. The conference provided an international forum for presentation of research work on all aspects of automated deduction, computational logic, and the theoretical foundations of artificial intelligence. RJA

Applications, S, P. *U.S. Military Communications: A C^3I Force Multiplier.* Fred J. Ricci, Daniel Schutzer. Elec. Eng. Commun. & Signal Processing. Computer Science Pr, 1986, xvii + 263 pp, $39.95. [ISBN: 0-88175-016-6] Practical and theoretical study of current and future trends in military electronic communications (including artificial intelligence, networks, satellites, fiber optics) that enable communications to serve its C^3I (command, control, communication, intelligence) role as a military force multiplier. RM

Applications, P, L. *Mathematical Modelling in Science and Technology.* Ed: Xavier J.R. Avula, et al. Pergamon Pr, 1987, xv + 812 pp, (P). A special issue of the journal *Mathematical Modelling,* comprising the entire 1987 issue (Volume 8), containing papers from the Fifth International Conference on Mathematical Modelling in Science and Technology held in Berkeley in July 1985. Nearly 200 papers on a dozen different areas, ranging from engineering to morphogenesis, from computer architecture to decision schedules. LAS

Applications, S. *Disequilibrium and Self-Organisation.* Ed: C.W. Kilmister. Math. & Its Applic. D Reidel, 1986, ix + 309 pp, $76. [ISBN: 90-277-2300-1] Papers giving mathematical treatment of dissipative systems and self-organization (spontaneous emergence of coherence or structure) as driving principle in areas such as thermodynamics of irreversible processes, development of nucleotides, stages of emerging life, evolution (as opposed to natural selection), social processes, etc. RM

Applications (Biology), P, L?. *An Introduction to the Mathematics of Neurons.* F.C. Hoppensteadt. Stud. in Math. Biology, V. 6. Cambridge U Pr, 1986, xv + 175 pp, $17.95 (P). [ISBN: 0-521-31574-3] Neurons, basic timers in our bodies, are studied using an electrical circuit model. Prerequisites are said to be differential equations and linear algebra, but there is later acknowledgement that students who used the book recommend an undergraduate major in mathematics and a strong background in the life sciences. Electrical engineering would help too. AWR

Applications (Biology), P. *Templets and the Explanation of Complex Patterns.* Michael J. Katz. Cambridge U Pr, 1986, x + 127 pp, $24.95. [ISBN: 0-521-30673-6] A philosophy of complexity, the "irreducible heterogeneity" appearing especially in the human sciences. Using examples drawn primarily from biology, the author explores the relation between the nature of templates (e.g., DNA code, or algorithms) and the complexity of patterns that they fabricate. LAS

Applications (Cognitive Science), P. *Intelligent Decision Support in Process Environments.* Ed: Erik Hollnagel, Giuseppe Mancini, David D. Woods. NATO ASI Ser. F, V. 21. Springer-Verlag, 1986, xv + 524 pp, $90.50. [ISBN: 0-387-13922-2] Contains papers and discussions from a NATO Advanced Study Institute held September 16-27, 1985, in San Miniato, Italy. Four main areas: decision theory, cognitive engineering, systems engineering, artificial intelligence. References. Abstracts of short papers. Lists of participants. RJA

Applications (Communication Theory), T?(17 18: 1), P. *Recursive Source Coding: A Theory for the Practice of Waveform Coding.* G. Gabor, Z. Györfi. Springer-Verlag, 1986, x + 98 pp, $28. [ISBN: 0-387-96309-X] Mathematical treatment of Fine-McMillan Recursive Quantifier model for waveform source coding. Based on a more complete statistical description of the signal than previous approaches, this general model gives insight into the difficulties with the theoretical, non-constructive data compression methods based on Shannon-style information theory, and the intractible differential predictive algorithms used in practice. RM

Applications (Economics), P. *Lecture Notes in Economics and Mathematical Systems-269: Optimality in Infinite Horizon Economies.* Anders Borglin, Hans Keiding. Springer-Verlag, 1986, vi + 180 pp, $17.50 (P). [ISBN: 0-387-16475-8] Overlapping generations models are presented as one way of coping with lack of compactness when finite planning horizons are removed in economics models. One-good and many-goods models are developed. BC

Applications (Information Theory), P. *Cooperative Interfaces to Information Systems.* Ed: L. Bolc, M. Jarke. Topics in Inform. Syst. Springer-Verlag, 1986, xiv + 328 pp, $39. [ISBN: 0-387-16599-1] Cooperative interfaces are those that use a blend of artificial intelligence, database theory, and programming languages to help with formulation of user requests, useful presentation of results, adaptability, handling of exceptions, and dialogue support. Knowledge-based and semantics-based natural language and graphics tools, internal representations and external presentation, discussion and evaluation of systems. RM

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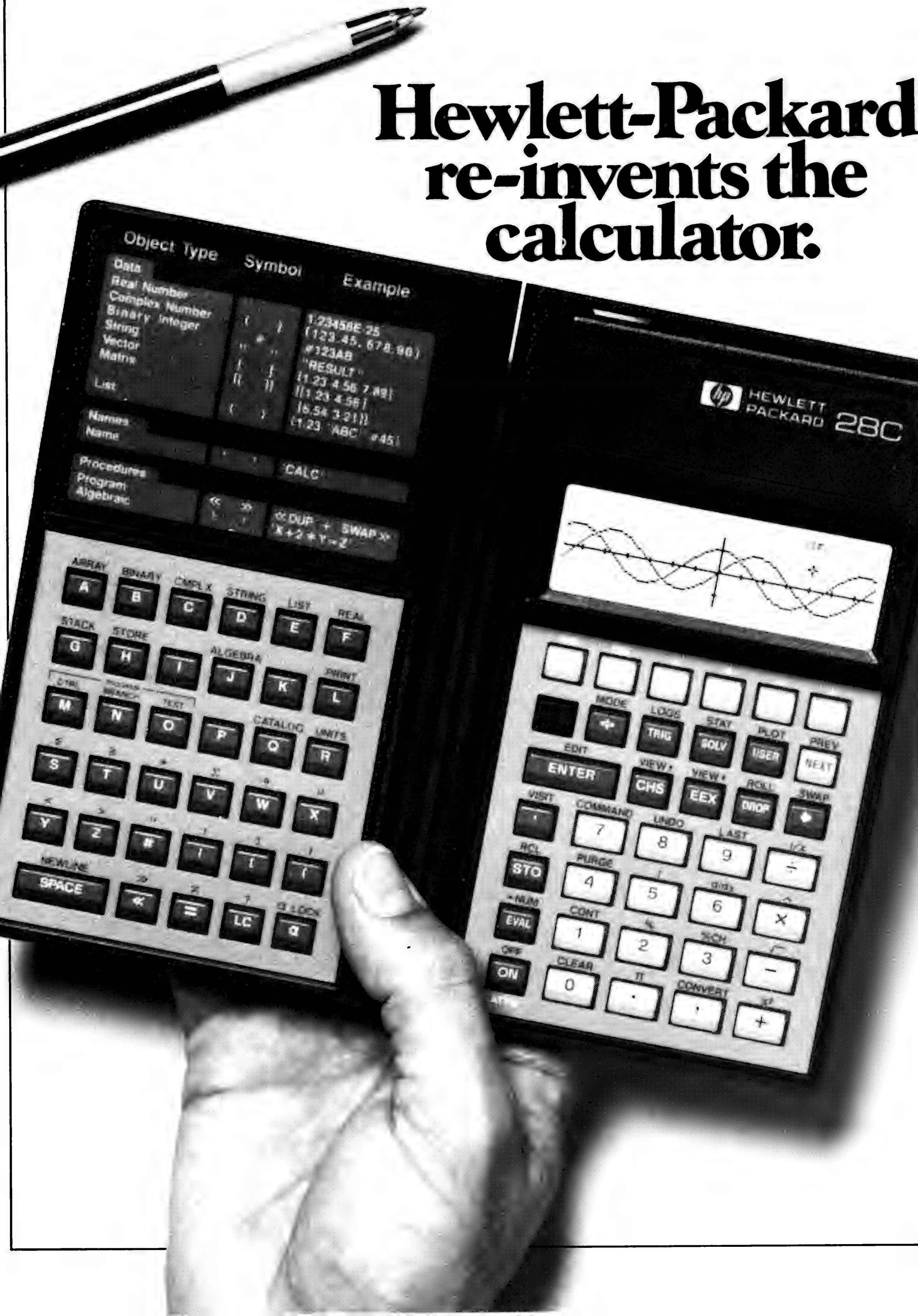

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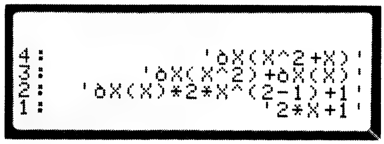

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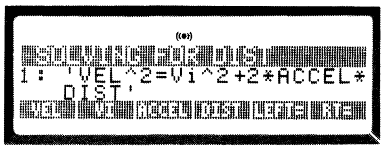

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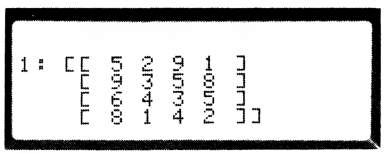

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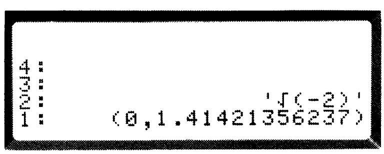

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The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

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The Maximum Order of an Element of a Finite Symmetric Group

William Miller, LeMoyne College

William Miller: I received my doctorate in 1979 from the University of Michigan for work done under the direction of H. L. Montgomery.

Preface. The question, "How large can the order of permutation on n elements be?" is reclusive, eccentric, and charming. It is of common genealogy, the natural offspring of rudimentary concepts from group theory. Yet it shyly declines to appear in modern algebra texts except, occasionally, in the inconspicuous special case where n is small. (See, for example, **[2]**, p. 322; **[5]**, p. 83; **[6]**, p. 158.)

An amusing quirk of the question is its penchant for disguise. It enjoys masquerading in equivalent forms, like the following one.

A deck of n cards is shuffled repeatedly, each shuffle identical to the others. What is the maximum number of shuffles that can be required to restore the deck to its original order? (Here, the term "shuffle" indicates all possible rearrangements of the deck, even those that the best stage magician could not achieve with normal techniques.)

Other known aliases are described in the introduction to **[10]**. (The works cited there are **[16]**, **[18]**, and **[19]**.)

Idiosyncrasies aside, the question possesses a fascinating talent—the uncanny ability to weave seemingly unrelated ideas into a tightly knit and intriguingly artistic fabric. This talent is too delightful for us to leave the question in its present state of obscurity.

1. Introduction. For convenience, let us denote the maximum order of a permutation on n elements by $G(n)$. The goal of this paper is to summarize what is known about $G(n)$ and then present a proof of one of the premier results, namely, that

$$\log G(n) \sim \sqrt{n \log n}\,. \tag{1}$$

(Throughout, "log" denotes the natural logarithm; and, for functions f and g, we write $f(x) \sim g(x)$ and say "f is asymptotic to g" if $\lim_{x\to\infty} f(x)/g(x) = 1$.) For some perspective on this result, recall that the symmetric group on n elements has order $n!$, and that $\log n!$ is asymptotic to $n \cdot \log n$. In this sense, we can regard (1) as quantifying the well-known fact that a symmetric group on more than 2 elements is not cyclic.

The proof of (1) has an aesthetic quality that greatly enhances its appeal: it applies a deep, number-theoretic result to a question from group theory, yet in such

a way that a general reader can appreciate the details. The deep result from number theory is the Prime Number Theorem, which states that if $\pi(x)$ denotes the number of primes not exceeding x, then $\pi(x) \sim x/\log x$. (See [4] for a thorough history of this famous result; a proof is given in [1].)

The remainder of the paper is organized as follows. In Section 2, we review what is known about $G(n)$. Next (Section 3), we discuss some basic notions concerning $G(n)$. Beginning in Section 4, we turn to proving (1). The first step is to make a connection between $G(n)$ and the prime numbers. This connection leads us, in a natural way, to consider a function $F(n)$ that approximates $G(n)$ but is simpler to handle. We then show, in Sections 5 and 6, respectively, that $\log G(n) \sim \log F(n)$ and that $\log F(n) \sim \sqrt{n \log n}$, from which (1) is evident. (The relation $\sim$ is easily seen to be transitive.) Finally, we offer some concluding remarks in Section 7. The arguments of Sections 4 and 5 employ a variety of simple ideas. It is not until Section 6 that we must invoke the Prime Number Theorem.

2. Historical Notes. The papers dealing with $G(n)$ are quite sparse. The first significant information about $G(n)$ was apparently obtained by E. Landau (see [7] and [8], pp. 222–229), who proved (1) in 1903. Thirty-six years later, S. Shah ([15]) refined (1) by providing an estimate for $|\log G(n) - \sqrt{n \log n}|$. In 1980, M. Szalay ([17]) sharpened Shah's estimate somewhat and also gave an estimate for the maximum order of an element of a symmetric semigroup. The estimates of both Shah and Szalay contain noneffective constants. Quite recently, J. Massias ([10]) derived an explicit upper bound for $G(n)$ and determined the value of n at which $(\log G(n) - \sqrt{n \log n})$ attains its maximum.

A few years ago, M. Nathanson ([11]) offered a short, elementary proof showing that $G(n)$ grows more rapidly than any power of n, (a result that is plainly weaker than (1)).

A paper ([12]) of J. Nicolas, which appeared in 1969, exposes a number of interesting properties of $G(n)$. A particularly striking result of that paper is that there are arbitrarily long strings of consecutive integers for which $G(n)$ is stationary. In a second paper ([13]), contemporary with the first, Nicolas described a computer program for calculating $G(n)$.

That very few permutations on n elements have orders as large as $G(n)$ is one result of a 1965 paper of P. Erdös and P. Turán. (See [3].) In fact, "most" (in a sense that Erdös and Turán made precise) permutations on n elements have an order whose logarithm is about $(\log^2 n)/2$.

3. Computing $G(n)$ When n is Given. For a small value of n, it is a routine exercise to calculate $G(n)$. One merely recalls that every finite permutation can be decomposed (uniquely, up to the order in which the factors appear) as a product of disjoint cycles, and that the order of a permutation is the least common multiple of the lengths of its disjoint cycles (see [9], pp. 93–94). Then, using this, one enumerates the possible orders of a permutation on n elements. More explicitly, one considers all distinct representations of n as a sum of positive integers and, for each representation, computes the least common multiple of the integers in the represen-

tation. The largest number thus computed is $G(n)$, and the integers in any representation corresponding to $G(n)$ are the cycle lengths of a permutation on n elements having order $G(n)$. This tedious method can be streamlined, as in [**13**]; but calculating a particular value of $G(n)$ involves substantial trial and error.

The table below displays the values of $G(n)$ for $n < 20$ and gives the corresponding cycle structures of permutations (on n elements) with order $G(n)$.

n	$G(n)$	cycle lengths	n	$G(n)$	cycle lengths
2	2	2	11	30	1, 2, 3, 5 or 5, 6
3	3	3	12	60	3, 4, 5
4	4	4	13	60	1, 3, 4, 5
5	6	2, 3	14	84	3, 4, 7
6	6	1, 2, 3 or 6	15	105	3, 5, 7
7	12	3, 4	16	140	4, 5, 7
8	15	3, 5	17	210	2, 3, 5, 7
9	20	4, 5	18	210	1, 2, 3, 5, 7 or 5, 6, 7
10	30	2, 3, 5	19	420	3, 4, 5, 7

The unruly behavior of $G(n)$ is apparent even in this brief table.

4. The Prime Connection. Let us consider the question of whether, for a given positive integer m, there is a permutation on n elements having order m. As a specific example, we ask, "Is there a permutation on 52 elements having order 51,480?" Now the prime factorization of 51,480 is $2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13$. Therefore, any permutation with exactly five nontrivial (disjoint) cycles whose lengths are $8, 9, 5, 11, 13$ has order 51,480. Moreover, it is possible to construct such a permutation whenever there are at least $8 + 9 + 5 + 11 + 13$ $(= 46)$ distinct elements available for permuting. We see, then, that there *is* a permutation on 52 elements with order 51,480. (This means, by the way, that there are shuffles that have to be performed exactly 51,480 times to restore a standard deck of 52 cards to its original order.)

In general, if we seek a permutation of order m, then we let the prime factorization of m be $\prod_{j=1}^{s} q_j^{e_j}$. Whenever $\sum_{j=1}^{s} q_j^{e_j} \leqq n$ we form a permutation that has $s + (n - \sum_{j=1}^{s} q_j^{e_j})$ disjoint cycles, the first s of lengths $q_1^{e_1}, \dots, q_s^{e_s}$, and the remaining ones of length 1. This permutation is a permutation on n elements (since the cycle lengths sum to n) and has order m (since the least common multiple of the cycle lengths is just their product, which is m). Hence, if $\sum_{j=1}^{s} q_j^{e_j} \leqq n$ (where $m = \prod_{j=1}^{s} q_j^{e_j}$), then there is a permutation on n elements having order m. We show in Corollary 1 below that the converse of this statement also holds.

As the preceeding discussion suggests, it is handy to have an abbreviation for the sum associated with the prime factorization of m.

DEFINITION. The function S is defined on the positive integers by $S(1) = 1$ and $S(m) = \sum_{j=1}^{s} q_j^{e_j}$ for $m > 1$, where $\prod_{j=1}^{s} q_j^{e_j}$ is the prime factorization of m.

We have described a procedure for constructing a permutation of given order m. Our procedure requires that we have at least $S(m)$ distinct elements available for

permuting. We want to know that our procedure is an efficient one, that is, that no other procedure can produce a permutation of order m by using fewer than $S(m)$ distinct elements. The next lemma assures us on this point. In reading the lemma, it is helpful (though not essential) to think of the integers $a_1, \ldots, a_k$ as being the cycle lengths of a permutation of order m.

LEMMA 1. *Let $a_1, \ldots, a_k$ be positive integers and let m be their least common multiple. Then $S(m) \leqq \sum_{i=1}^{k} a_i$.*

Proof. We argue that there are no counterexamples to the lemma. Suppose, instead, that the sequence of positive integers $a_1, \ldots, a_k$ forms a counterexample, and further suppose that the sum of $a_1, \ldots, a_k$ is minimal (among counterexamples). After making a few reductions, we shall arrive at an obvious contradiction.

We first note that all the terms of $a_1, \ldots, a_k$ are greater than 1; if not we could delete one of the terms equal to 1 to get a new sequence that would still refute the lemma, but would have a smaller sum than $a_1, \ldots, a_k$.

We next contend that each term of $a_1, \ldots, a_k$ is a (positive, integral) power of a prime. Otherwise, there would be a term, say, a_i, that could be written as a product of two relatively prime integers, say c and d, both greater than 1. Assuming d to be the larger of c and d, we would have that

$$c + d \leqq c + d(c-1) = cd + (c-d) < cd.$$

Therefore, deleting a_i from the original sequence and inserting c and d would yield a new sequence with smaller sum, yet with the same least common multiple as $a_1, \ldots, a_k$. This would violate the minimality property of $a_1, \ldots, a_k$.

Finally, we observe that the terms of $a_1, \ldots, a_k$ must be powers of *distinct* primes. For, if two of the terms were powers of the same prime, then deleting the term with the smaller power (or deleting either term if the powers were equal) would again yield a new sequence with smaller sum, yet the same least common multiple as $a_1, \ldots, a_k$.

But if $a_1, \ldots, a_k$ are all powers of distinct primes, then the sequence is NOT a counterexample to the lemma. This is because the least common multiple of $a_1, \ldots, a_k$ (which is m) is just their product; moreover, their product is the prime factorization of m, whence $S(m)$ equals the sum of $a_1, \ldots, a_k$. Since the lemma has no counterexample of minimal sum, it must be true.

COROLLARY 1. *There is a permutation on n elements having order m if and only if $S(m) \leqq n$.*

Proof. If $S(m) \leqq n$, then the procedure outlined at the beginning of this section yields a permutation on n elements having order m. Conversely, let $a_1, \ldots, a_k$ be the cycle lengths of a permutation on n elements having order m. Then the sum of the cycle lengths is n and their least common multiple is m. Hence, $S(m) \leqq n$ by Lemma 1.

If we study the table showing values of $G(n)$ for $n < 20$, then we may well anticipate the next corollary.

COROLLARY 2. *Among the permutations on n elements having order $G(n)$, there is at least one whose nontrivial cycles have lengths that are powers of distinct primes.*

Proof. Corollary 1 gives that $S(G(n)) \leqq n$. Therefore, the construction described at the beginning of this section supplies a permutation of order $G(n)$ with the prescribed type of cycle lengths.

We comment that Corollary 2 can be strengthened considerably. It can be shown that if A is a cycle length of a permutation (on n elements) with order $G(n)$, then either A is a power of a prime not dividing any other cycle length, or else $A = 6$. Furthermore, the latter possibility is excluded for all sufficiently large n.

Our final corollary gives a convenient characterization of $G(n)$.

COROLLARY 3. *We have that* $G(n) = \max_{S(m) \leqq n} m$.

Proof. As in the previous proof, $S(G(n)) \leqq n$, so that $G(n)$ cannot exceed the maximum of the m taken over $S(m) \leqq n$. On the other hand, by Corollary 1, if $S(m) \leqq n$, then there is a permutation on n elements having order m. The definition of $G(n)$ thus implies that $m \leqq G(n)$ whenever $S(m) \leqq n$.

5. The Relationship Between $F(n)$ and $G(n)$. The foregoing section tells us that to calculate $G(n)$, we should select powers of distinct primes in such a way that their sum does not exceed n and their product is maximal (subject to the constraint on the sum). One obvious way to select prime powers satisfying the constraint is to choose $2, 3, 5, 7, 11, 13, \ldots,$ continuing until the sum of the primes chosen is as large as possible without exceeding n. For instance, if $n = 52$, we select $2, 3, 5, 7, 11$, and 13. The sum of these primes is 41. We cannot include the next prime, 17, for then the sum would exceed 52. The product of the selected primes is 30,030. Let us call this product $F(52)$. It is plain that $F(52) < G(52)$ because, in our selection process, we can choose 17 instead of 7, or 4 instead of 2, or 9 instead of 3 without violating the constraint. However, since it can be checked that $G(52) = 180,180 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, we see that $F(52)$ and $G(52)$ have some common features. In particular, if we compare $G(52)$ and $F(52)$ on a logarithmic scale (a sensible one to use in dealing with products), we discover that the ratio between $\log G(52)$ and $\log F(52)$ is only about 1.2.

It is easy to use our selection process for arbitrary values of n. We select primes in increasing order until we reach a prime, call it P, such that the sum of all primes less than P is no greater than n, but such that the sum of all primes up to and including P is greater than n. Then we let $F(n)$ be the product of the primes less than P. As already announced, we discover that $\log F(n)$ is asymptotic to $\log G(n)$.

THEOREM 1. *Let P be the largest prime with the property that the sum of the primes less than P does not exceed n, and let $F(n)$ be the product of the primes less than P. Then* $\log F(n) \sim \log G(n)$.

We require two lemmas for the proof of Theorem 1. To understand the purposes of the lemmas, first recall that $F(n) \leqq G(n)$. Hence we need only find a suitable

upper bound for $G(n)$ in terms of $F(n)$. If we compare the prime factorizations of $F(52)$ and $G(52)$ given above, we notice immediately that the same primes divide both numbers. The difference is that the smaller primes appear with higher powers in $G(n)$. Thus, when $n = 52$, the product of the primes dividing $G(n)$ equals $F(n)$. In general, the product of the primes dividing $G(n)$ need not equal $F(n)$. However, Lemma 2 below guarantees that the product is never much larger than $F(n)$.

In light of Lemma 2, we see that the only way for $G(n)$ to be much larger than $F(n)$ is for the prime factorization of $G(n)$ to include primes raised to powers greater than 1. Lemma 3 limits the extent to which this can happen. In fact, Lemma 3 corroborates what is suggested by looking at $G(52)$—that only the smaller primes can appear to higher powers in the factorization of $G(n)$, and that the contribution of such higher-power primes is fairly modest.

LEMMA 2 (Shah). *Let $q_1 < \cdots < q_s$ be all the primes dividing $G(n)$, and let P and $F(n)$ be as in Theorem* 1. *Then*

$$\sum_{j=1}^{s} \log q_j < 2 + \log F(n) + \log P.$$

Proof. For future reference, we observe that $(\log x)/x$ is a decreasing function for $x \geqq 3$ (because its derivative is negative there). Hence, if $3 \leqq a \leqq b$, then $(a/\log a)(\log b) \leqq b$ and $a \leqq (b/\log b)(\log a)$. We also note that P is at least 3 unless $n = 1$, in which case the conclusion of the lemma is clearly true.

Now let $q_1, \ldots, q_{t-1}$ be the primes not exceeding P that divide $G(n)$, and let $p_1, \ldots, p_r$ be the odd primes not exceeding P that do not divide $G(n)$. Thus, the list $p_1, \ldots, p_r, q_1, \ldots, q_{t-1}$ contains every prime not exceeding P exactly once, except that 2 might be omitted. Since

$$\sum_{j=1}^{s} q_j \leqq S(G(n)) \leqq n < \sum_{p \leqq P} p,$$

we find, upon canceling common terms in the above inequality, that

$$\sum_{j=t}^{s} q_j \leqq 2 + \sum_{i=1}^{r} p_i. \tag{2}$$

Moreover, because $3 \leqq P \leqq q_j$ for $t \leqq j \leqq s$ and because $3 \leqq p_i \leqq P$ $(1 \leqq i \leqq r)$, our initial observation implies that $(P/\log P)(\log q_j) \leqq q_j$ and that $p_i \leqq (P/\log P)(\log p_i)$. From this and (2) we infer that

$$\sum_{j=t}^{s} \log q_j \leqq 2(\log P/P) + \sum_{i=1}^{r} \log p_i \leqq 2 + \sum_{i=1}^{r} \log p_i.$$

Adding the terms $\log q_j$ for $1 \leqq j \leqq t-1$ to both sides of this inequality and recalling that $p_1, \ldots, p_r, q_1, \ldots, q_{t-1}$ is just a permuted list of the primes not exceeding P (except that 2 might be omitted), we get the conclusion of the lemma.

LEMMA 3. *Let q be a prime, let e be an integer greater than* 1, *and let P be as in Theorem* 1. *If q^e divides $G(n)$, then $q^e \leqq 2P$ and $q \leqq \sqrt{2P}$.*

Proof. Since $e > 1$, the second assertion is an easy corollary of the first. To prove the first assertion, let Q be the smallest prime not dividing $G(n)$. Now the primes less than Q all divide $G(n)$; hence, their sum is at most $S(G(n))$, which is at most n. On the other hand the sum of the primes not exceeding P is greater than n. It follows that $Q \leqq P$. Therefore, it suffices to show that $q^e \leqq 2Q$.

Suppose, to the contrary, that $q^e > 2Q$, and let N be the positive integer satisfying $q < Q^N < qQ$. (Equality is impossible in the last inequality because q divides $G(n)$ while Q does not.) We put $m = (Q^N/q)G(n)$. Then $m > G(n)$ and

$$S(m) = S(G(n)) + \left(Q^N - q^e + q^{e-1}\right).$$

We claim that the last quantity in parentheses is negative. If $q < Q$, this is true because (by definition) $N = 1$ and (since q^e is supposed greater than $2Q$)

$$-q^e + q^{e-1} \leqq -q^e/2 < -(2Q)/2 = -Q.$$

If $q > Q$, it is true because (since $Q^N < qQ$ and $e > 1$),

$$Q^N - q^e + q^{e-1} < qQ - q(q-1) \leqq qQ - q(Q) = 0.$$

Hence, $S(m) < S(G(n)) \leqq n$. Since $m > G(n)$, this contradicts Corollary 3, thereby establishing the lemma.

The proof of Theorem 1 is now straightforward, save for one detail concerning the relative sizes of $F(n)$ and P.

Proof of Theorem 1. Let $\prod_{j=1}^{s} q_j^{e_j}$ be the prime factorization of $G(n)$. We view $\log G(n)$ as the sum of the terms $\log q_j^{e_j}$ and split this sum into two subsums, the first consisting of the terms for which $e_j = 1$, the second consisting of the terms for which $e_j > 1$. By Lemma 2, the first subsum is at most $2 + \log F(n) + \log P$; by Lemma 3, each term of the second subsum is at most $\log 2P$ and there are at most $\sqrt{2P}$ terms. Combining this information with the fact that $F(n) \leqq G(n)$, we deduce that

$$\log F(n) \leqq \log G(n) \leqq 2 + \log F(n) + \log P + \sqrt{2P}(\log 2P).$$

In the next section, we shall see that there is a positive constant c such that, for all $n > 1, \log F(n) > cP$. Accepting this fact for the present, we obtain Theorem 1 upon dividing the displayed inequality by $\log F(n)$ and letting n approach infinity. (Note that, from its definition, P clearly approaches infinity with n.)

6. The Size of $F(n)$. The goal of this section is to prove that $\log F(n)$ is asymptotic to $\sqrt{n \log n}$. As an instructive prelude to the proof, let us reason heuristically. To compute $\log F(n)$, we first determine the prime P that satisfies the double inequality

$$\sum_{p<P} p \left(= \sum_{p \leqq P-1} p \right) \leqq n < \sum_{p \leqq P} p.$$

(Here and below, p denotes a generic prime.) Then we calculate

$$\log\left(\prod_{p<P} p \right) = \sum_{p<P} \log p.$$

Suppose that we treat P as an independent variable and regard both the sum of $p(p \leqq P)$ and the sum of $\log p(p \leqq P)$ as functions of P. To emphasize this approach, let us replace P by x and put

$$A(x) = \sum_{p \leqq x} p, \qquad \theta(x) = \sum_{p \leqq x} \log p.$$

Now $A(x)$ and $\theta(x)$ are step functions whose values are tedious to determine. Let us ignore this for the present and argue as follows. Since $A(P-1) \leqq n < A(P)$ and $\log F(n) = \theta(P-1)$, we ought to get a good approximation to $\log F(n)$ by solving the equation $A(x) = n$ for x and plugging the solution into $\theta(x)$.

For this program to succeed, we must be able to approximate $A(x)$ and $\theta(x)$ by appropriate functions. Fortunately, thanks to the Prime Number Theorem, we can! The following two consequences (equivalent forms, actually) of the Prime Number Theorem are just what we need.

$$A(x) \sim x^2/(2\log x) \tag{3}$$

$$\theta(x) \sim x \tag{4}$$

According to our heuristic scheme, then, $\log F(n)$ is approximately equal to the value of x that solves the equation $x^2/(2\log x) = n$. Moreover, as is easily verified, $x = \sqrt{n \log n}$ is "almost" a solution to this equation. Thus we suspect that $\log F(n) \sim \sqrt{n \log n}$.

Before we make this plausibility argument rigorous, let us add a few comments about (3) and (4). The derivations of (3) and (4) are applications of a standard technique based on integration by parts. For those unfamiliar with this useful technique, we sketch the derivation of (4). (Recall below that $\pi(x)$ denotes the number of primes not exceeding x and that the Prime Number Theorem states that $\pi(x) \sim x/(\log x)$.)

From the definitions of $\pi(x)$, $\theta(x)$, and the Stieltjes Integral,

$$\theta(x) = \int_1^x (\log t)\, d(\pi(t)).$$

Integration by parts yields that

$$\theta(x) = \pi(x)(\log x) - \int_2^x (\pi(t))/t\, dt.$$

The last integrand is (by the Prime Number Theorem) no more than a constant multiple of $1/(\log t)$. Moreover, the integral of $1/(\log t)$ from 2 to x is bounded by a constant multiple of $x/(\log x)$, as can be seen by splitting the range of integration at $\sqrt{x}$. Hence, we obtain (4) if we divide the last equation by x, let x approach infinity, and invoke the Prime Number Theorem.

It is evident from (4) that there is a positive constant c' such that $\theta(x) > c'x$ for all $x > 2$. Furthermore, by the definitions of $F(n)$ and $\theta(x)$, we have that $\log F(n) = \theta(P-1)$. Thus, the inequality $\log F(n) > cP$, quoted in the proof of Theorem 1, follows from the Prime Number Theorem. However, this inequality also follows from much weaker statements about the distribution of primes. Relatively

simple arguments (see [**14**], p. 217ff), dating to Chebyshef, show that $\pi(x)(\log x)/x$ is bounded above and below by positive constants; and the technique illustrated in the foregoing paragraph yields corresponding upper and lower bounds for $\theta(x)$. Theorem 1 is therefore independent of deep facts about prime distribution.

To verify that $\log F(n) \sim \sqrt{n \log n}$, we first note that since $\log F(n) = \theta(P - 1)$ and since (with P regarded as a function of n) $P \sim P - 1$, (4) implies that $\log F(n) \sim P$. Thus, it suffices to show that $P \sim \sqrt{n \log n}$.

Now $A(P - 1) \leqq n < A(P)$ by the definitions of $A(x)$ and P. Since it is immediate from (3) that $A(x - 1) \sim A(x)$, we infer that

$$P^2/(2 \log P) \sim n. \tag{5}$$

If P is not asymptotic to $\sqrt{n \log n}$, then there is a positive number $\phi e \to \varepsilon$ such that, for infinitely many values of n, one of the following two inequalities holds:

$$P \leqq (1 - \varepsilon)\sqrt{n \log n} \qquad P \geqq (1 + \varepsilon)\sqrt{n \log n}\,. \tag{6}$$

Because $x^2/(\log x)$ is an increasing function for $x > \sqrt{e}$, the first inequality of (6) implies that

$$P^2/(2n \log P) \leqq (1 - \varepsilon)^2(\log n)/(\log n + \operatorname{loglog} n + 2\log(1 - \varepsilon)).$$

As n approaches infinity, the right side of the last inequality approaches $(1 - \varepsilon)^2$, while (by (5)) the left side approaches 1. Hence, the first inequality of (6) cannot hold for infinitely many n. Similarly, the second cannot either; and we conclude that $P \sim \sqrt{n \log n}$. As explained above, this establishes that $\log F(n) \sim \sqrt{n \log n}$.

7. Concluding Remarks. Landau's proof of (1) contains less combinatorial analysis and more frequent use of the Prime Number Theorem than does ours. Our proof is not substantially shorter or simpler than Landau's, but it does furnish a more complete survey of the methods that have been successful in studying $G(n)$. It also illustrates how a weaker version of (1), with $\log G(n)$ bounded above and below by constant multiples of $\sqrt{n \log n}$, can be derived by using Chebyshev's estimates for $\pi(x)$ rather than the more sophisticated Prime Number Theorem.

The kind of combinatorial analysis typified by our proof of Lemma 3 can be employed very effectively to explore the prime factorization of $G(n)$. (See [**12**] for a vivid demonstration of this.) In particular, one can deduce fairly readily that if $\prod_{j=1}^{s} q_j^{e_j}$ is the prime factorization of $G(n)$ and if $q_i < q_j$, then $e_i \geqq e_j - 1$. With more work, one can show that if Q is the largest prime factor of $G(n)$, then "most" (in various senses) primes less than Q divide $G(n)$. This leads to an asymptotic estimate for the number of prime factors of $G(n)$.

The refinements of (1) mentioned in Section 2 are essentially refinements of estimates for $F(n)$. With slightly more care, Theorem 1 can be sharpened to give an estimate for $|\log G(n) - \log F(n)|$ that is commensurate with the estimate for $|\log F(n) - \sqrt{n \log n}\,|$ that follows from the renowned Riemann Hypothesis. Thus, Theorem 1 is adequate to handle any likely improvements in estimates for $F(n)$.

In the preface, we claimed that this paper's seminal question has a remarkable talent for linking apparently disparate ideas. Can anyone who has seen a question

about card-shuffling linked, in a natural way, to the question (Riemann Hypothesis) of where a certain analytic function has its zeroes dispute the claim?

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Playing Games with Games: the Hypergame Paradox

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Bill Zwicker: I received my Ph.D. from M.I.T. in 1975, with a thesis in Set Theory under Gene Kleinberg, and have been at Union College since then. My research has chiefly been into the structure of spaces which support measures that witness large cardinal properties.

A new paradox offers the opportunity to recapture the sense of confusion and uncertainty that faced mathematicians in the early part of the twentieth century, including those common initial feelings that the eventual fix is a dishonest legalism which walks around a key question instead of answering it. Also, it provides a new probe with which to explore the relationship between paradox and proof.

I stumbled across the Hypergame Paradox when I was teaching a section on games and strategies for a "Mathematics for the Liberal Arts Major" course occasionally offered at Union College, and an idea for a bonus test question came to me. Let us define a game G to be *totally finite* if it satisfies the following conditions:

(1) Two players, **I** and **II**, move alternately, **I** going first. Each has complete knowledge of the other's moves.
(2) There is no chance involved.
(3) There are no ties (when a play of G is complete, there is one winner).
(4) Every play ends after finitely many moves.
(5) At any point in a play of G, there are but finitely many legal possibilities for the next move.

The game *Supergame*, inspired by dealer's-choice poker, has the following rules: on the first move, **I** names any totally finite game G (called the subgame). The players then proceed to play G, with **II** playing the role of **I** while G is being played. The winner of the play of the subgame is declared to be the winner of the play of Supergame.

The two questions I put on the test were "Who has the winning strategy for Supergame?" and "Is naming 'Supergame' a legal first move in a play of Supergame?" Let's consider this second question. Supergame is not totally finite because it fails to satisfy condition (5); there are infinitely many totally finite games. However, Supergame comes close in that it satisfies conditions (1)–(4). In particular, it satisfies (4) because a play of Supergame consists of a single game-naming move,

*The author would like to thank J. Baumgartner, J. Henle, A. Taylor, D. Velleman, and S. Wagon for ideas and for helpful comments on the manuscript.

followed by a necessarily finite play of a totally finite subgame. At this point there is a whiff of paradox in the air. Since (5) is the only sore point, we'll delete it and call any game G satisfying (1)–(4) *somewhat finite*.

The game *Hypergame* has the same rules as Supergame, except that **I** may name any somewhat finite game on the first move. The question leading to the Hypergame paradox is, "Is Hypergame somewhat finite?" On the one hand Hypergame clearly satisfies (1)–(4) for the same reasons that Supergame did, so it is somewhat finite. On the other hand, if Hypergame is somewhat finite the following infinite play is a legal one for Hypergame:

I	**II**
"Hypergame"	
	"Hypergame!"
"Hypergame!!"	
	"Hypergame!!!"
⋮	⋮

etc.

So Hypergame is not somewhat finite. Our question has both "Yes" and "No" as answers, and we are forced into a paradox.

I've presented the Hypergame paradox to mixed groups of undergraduates and mathematics faculty. The responses have been interesting. Many see the construction of the infinite play before I present it, and can barely be restrained from jumping from their seats to shout it out. Almost all find the infinite play counterexample much more compelling than the proof that Hypergame is somewhat finite, and a surprisingly large proportion (including some faculty!) feel that the counterexample really settles the matter with no further comment needed. I then have to convince them that mathematicians cannot simply abandon a proof once a counterexample has been found, for if the internal flaw in such a proof cannot be identified then the counterexample threatens the entire edifice of mathematical proof.

Note that the pattern of answers to the paradoxical question for Hypergame is "Yes" (it *is* somewhat finite) and "If Yes, then No" (if it is somewhat finite, then it isn't). This pattern stands in contrast to that of most of the well-known paradoxes, whose key questions usually have the symmetric response "Yes if and only if No". For example, in the famous Russell's Paradox one defines $R = \{x : x \notin x\}$, asks whether $R \in R$ and answers "$R \in R$ if and only if $R \notin R$." Burali-Forti's paradox is the only reasonably well-known asymmetric paradox that I am aware of (see appendix, and [**4**] pp. 104–112 or [**2**]); it is closely related to the Hypergame paradox in other ways as well (appendix).

REMARK. The first three published accounts of the Hypergame paradox ([7], [**1**], and [**3**]) share a mistake. Roughly, their arguments go: "If Hypergame is somewhat finite, then it isn't somewhat finite, because of the infinite play. If Hypergame isn't somewhat finite then, as it is not available as the first move in the construction of an infinite play, no infinite plays exist, and Hypergame is somewhat finite." However, the unavailability of Hypergame says nothing about the existence of closely related games which could yield infinite plays in essentially the same way. I suspect that the

authors assumed the argument would follow the more familiar symmetric pattern, and wrote it to fit.

But what is a paradox? It is an apparent contradiction in mathematics which, in the fullness of time, comes to be properly viewed as a reductio ad absurdum proof that one of the assumptions (frequently implicit and unrecognized) of the argument is incorrect. The assumption contradicted may be deeply held by the mathematical community and the maturation process of changing interpretations is as much sociological and philosophical as it is mathematical.

Quine's excellent paper [**6**] suggests that there are as many as four identifiable stages in the evolution of a paradox:

1) *antinomy*—confusion reigns over a contradiction in basic laws.

2) *veridical paradox*—the paradox has become a reductio ad absurdum proof that some assumption is false.

3) *artificial fix*—the false assumption is replaced by an alternative (there may be several alternatives) that eliminates the flaw, but the new assumption is viewed as a momentarily convenient, artificial, and counterintuitive patch job.

4) *good fix*—some patch job has come to be viewed as more natural than the original, flawed assumption. Quine puts it nicely:

> Russell's paradox is a genuine antinomy because of the fundamental nature of the principle of class existence that it compels us to give up. When in a future century the absurdity of that principle has become a commonplace, and some substitute principle has enjoyed long enough tenure to take on somewhat the air of common sense, perhaps we can begin to see Russell's paradox as no more than a veridical paradox, showing that there is no such class as that of the nonself-members. One man's antinomy can be another man's veridical paradox, and one man's veridical paradox can be another man's platitude.

These attitude shifts seem quite similar to those that take place in the "change of gestalt" which, Thomas Kuhn [**5**] tells us, is at the heart of a scientific revolution. Most of the pieces are the same after a revolution, yet their relationships shift and invert—we put them together into a different world. Usually the process moves in one direction only, and produces a great gulf of understanding within the crossover generation. Those physicists in the early twentieth century, for example, who came to believe that the speed of light is the limiting speed in the universe, lived in a different world from those of their contemporaries who could not believe.

Do scientific revolutions run through the same four stages as mathematical paradoxes? Perhaps the Michaelson-Morley experiment revealed an antinomy, and the acceptance that the experiment proved the absence of an ether for light waves constituted the "veridical paradox" stage. Next would have been the recognition that the light-speed limitation circumvented the problem, and finally would come the *belief* in special relativity. Actually, the Einsteinian revolution did not have such clearcut and sequential stages; things were much more blurred and simultaneous.

We can expect the stages in the evolution of a mathematical paradox to be more distinct, because proof exists in mathematics. As Alan Taylor has pointed out, one can imagine a community of mathematicians who recognize that they have dis-

proved a long-cherished assumption, but who have nothing to offer in its place. Physicists, however, only abandoned ether when they were lured by a more attractive substitute principle. There are no fixed guidelines as to how badly a hitherto fruitful physics model must fail before it is discarded—there is no precise notion of disproof in physics.

As a metaphor for change of gestalt, Kuhn offers us the familiar illusion of stacked cubes. The vertices which appear to be nearest corners of solid cubes can be made to look like furthest corners of hollow cubes, by a flip of mental perspective. With this reversible metaphor, Kuhn encourages us to switch back in other contexts —to put ourselves in the shoes of physicists puzzled by the paradoxical nature of the ether, for example. Similarly, a consideration of our own mental states as we think through the Hypergame paradox can make the development of Russell's Paradox come alive to those of us who arrived too late to catch the early acts.

As Quine said, Russell's Paradox can be viewed as a proof that there does not exist a set R such that $x \in R$ if and only if $x \notin x$; this proof has to survive *any* reasonable attempt to formalize the assumptions of set theory. To contemporary set theorists (whose interests mostly do not lie in this area anyway) the result seems natural. Similarly, the Hypergame paradox is a proof that, under any reasonable definition of "game" (e.g., within any particular axiomatization of game theory), Hypergame is not a game.

It is easy to have a strong gut reaction against this claim. How can a certain game fail to exist when the rules for playing it are clear.* I have actually played Hypergame (or so I thought at the time) so how can it not have existed? These reactions show that achieving the shift in perspective in our gut is not so easy even when the mathematics is transparent.

The reader might wish to sound his or her own heart on this question: does it seem *natural* that Hypergame is not a game? If not, do you feel a little more sympathy with those to whom Russell's Paradox was baffling and important, even after the publication of Zermelo's paper?

As to which of Quine's stages describes the current state of Russell's Paradox, opinions differ. Apparently, many philosophers would not place us as far along the road as would most working set theorists. Russell discovered his paradox in 1901, and communicated his discovery to a Frege in a letter written in 1902. Zermelo put forth an axiom system in 1908 [Z] which limits the general principles of set formation so that Russell's property "$x \notin x$" no longer yields a set as its extension, and this system was strengthened (by apparently independent suggestions of Fraenkel and Skolem in their 1922 papers) to become the system now commonly referred to as "ZF."

Quine's remarks, quoted earlier, imply that he does not believe ZF to contain the "substitute principle" to which he refers. Jean van Heijenoort, in his 1967 introduction ([**4**], p. 199) to Zermelo's original paper, agrees with Quine, describing Zermelo's work as "an immediate answer to the pressing needs of the mathematician" in

*But are they clear?

which "sets are not simply collections; they are objects satisfying certain axiomatic conditions." Yet to modern mathematically oriented set theorists, working at the time Quine and van Heijenoort were writing, *ZF* had already taken on the air of common sense. Set theorists today see no distinction between collections and Zermelo's objects, they usually do not work within an axiom system any more than do any other mathematicians, and do not accidentally reason in ways which could not be formalized within *ZF* (or *ZF* with the Axiom of Choice). On those occasions when set theorists do need more than *ZF* provides, they view the extra assumptions as natural extensions of *ZF* rather than as rival systems. The truths of *ZF* have been internalized in *this* century.

Do set theorists view *ZF* as "The Solution"—the final resolution of Russell's Paradox? For many, this would be going too far, yet most would agree that the intuitive notion of collection has shifted so that the *ZF* axioms are now true statements about the way collections behave. Of course, they may fail to express *all* the truths.

Historically there are other connections between paradox and reductio proofs, since there is an almost automatic way to turn a paradox into a proof; simply limit the scope of the objects discussed, or modify the terms somewhat, so that the hidden incorrect assumption is eliminated. At this point, some other step in the argument must break down (or so believe the faithful), and one has a reductio proof that this step cannot possibly be completed.

Hypergame gives us a chance to look anew at this paradox-to-proof machine. For example, we can call a game *countably finite* if it satisfies (1)–(4) together with (5′): At any point in a play of G there are only countably many legal possibilities for the next move. Define *Countable Hypergame* in the obvious way, requiring that **I** choose a countably finite game. If there were but countably many countably finite games, the proof that Countable Hypergame is countably finite would proceed as in the original paradox, and an infinite play could again be constructed, yielding a contradiction. Thus the original paradox has been transformed into a proof that an uncountable set exists, namely the set of countably finite games. For the sake of completeness, the reader familiar with set theory may wish to check (see appendix) that the hidden incorrect assumption of the Hypergame paradox has been eliminated, i.e., that Countable Hypergame is a game.

Jim Saxe, a computer scientist who was an undergraduate at Union, came up with the following even nicer example of paradox-to-proof. Let us say that the computer program P_G *adjudicates* the game G if a computer loaded with P_G will allow two players to play G by typing their moves on the computer's keyboard. The computer will tell them when their moves are legal and when not, when the game is over, and who wins. A game G will be *computably finite* if it satisfies (1)–(4) and additionally (5″): there is a program P_G that adjudicates G. *Computable Hypergame* is played as follows: on the first move, player **I** names a program P_G for some computably finite game, etc. Is Computable Hypergame computably finite? If it is, one gets the usual contradiction. One can almost carry through the proof that Computable Hypergame is computably finite, but the hang-up, predictably, comes

in verifying (5″). Yet it is clear that one can write a program to adjudicate all moves of Computable Hypergame except for the first one (since one can write programs that simulate other programs fed them as input). Therefore we have proved an undecidability result—there is no way to write a program P that checks other programs to determine whether the games they adjudicate are somewhat finite (satisfy (1)–(4)) or not.

The reader familiar with undecidability may guess that Saxe's proof can be modified to show the undecidability of the halting problem. This well-known theorem is stronger than Saxe's result (see appendix). Will a new proof based on the Hypergame paradox be any different from the standard proof?

We'll use standard notation. Choose any sufficiently powerful computer language, and fix any reasonable way to code programs in the language as natural numbers. (Such a Gödel-numbering is reasonable if the coding and decoding functions can be performed by an appropriately programmed computer.) If f is a natural number then $\{f\}$ will denote the algorithm performed by a computer loaded with the program whose code is f, and $\{f\}(n)$ refers to the output of the computation performed by such a computer when the input is n. If this computation eventually halts we write $\{f\}(n)\downarrow$, while $\{f\}(n)\uparrow$ means that it never halts. We should like to prove that there is no e such that for all n, $\{e\}(n) = 1$ if $\{n\}(n)\downarrow$ and $\{e\}(n) = 0$ if $\{n\}(n)\uparrow$. The standard proof is one by contradiction; if there were such an e it is shown that it can be modified to form an e' such that $\{e'\}(e')\downarrow$ if and only if $\{e'\}(e')\uparrow$. This follows the standard "Yes if and only if No" of the symmetric paradox.

The Hypergame version unfortunately loses all of its game flavor, but it does preserve the basic asymmetric quality of the Hypergame paradox and is different from the standard proof. Once again, assume by way of contradiction that e is as in the previous paragraph. Let $\{e^*\}$ be the modification realizing the following algorithm:

> Given n as input, first apply $\{e\}$ to determine whether or not $\{n\}(n)\downarrow$. If it does, simulate the action of $\{n\}(n)$ (i.e., the second part of the calculation is the same as that performed by $\{n\}(n)$). If it does not, perform no further calculation and halt.

Note that $\{e^*\}(n)\downarrow$ for all n. Thus in particular $\{e^*\}(e^*)\downarrow$ (this is the "Yes" part of the asymmetric pattern). Once we know $\{e^*\}(e^*)\downarrow$, however it is easy to see that the calculation performed by $\{e^*\}(e^*)$ is an infinite descent of simulations within simulations, hence $\{e^*\}(e^*)\uparrow$. This is the "If Yes then No" half, showing that the original e could not have existed.

There are some much better-known examples of paradox-to-proof. Gödel's Incompleteness Theorem is based on the Liar's Paradox, as Gödel himself pointed out. Cantor's proof that the cardinality of the power set of a set A exceeds that of the set A (and in particular that the continuum is uncountable) can be viewed as being based on Russell's paradox, but the dates suggest that Cantor could not have

thought of it this way. Incidentally, there is an alternate proof of Cantor's theorem which is based on the Hypergame paradox. Can the reader find it?

Appendix: Game Trees and a Resolution of the Paradox

Given a game G its game tree T_G is the set of all legal finite sequences of moves in G, partially ordered by extension. Thus, the top node of T_G is the empty sequence, and it splits into as many nodes on the next level as there are legal first moves in G. Each of these nodes in turn splits into as many nodes immediately beneath it as there are legal responses to the opening move it represents, etc. T_G contains all the information in G, and in particular there is a one-to-one correspondence between plays of G and paths through T_G which start at the top and end at a terminal node (at which point the game is over). Terminal nodes must be labeled to say whether **I** or **II** won.

The game G satisfies condition (4) (plays are of finite length) exactly if the tree T_G has no infinite paths. Such trees are said to be *well founded*. The game tree for Hypergame consists of all possible well-founded trees joined together into a single tree with the addition of a single node at the top linked to each of the top nodes of these well-founded trees. Thus the Hypergame question becomes "Is the tree that glues together all well-founded trees itself well-founded?" It must be well-founded, yet if it is then the entire tree contains a copy of itself, and the copy itself must contain a copy, etc. By tracing along the top nodes of these successive copies within copies one constructs an infinite path, showing that the tree is not well-founded and replicating the Hypergame paradox.

Burali-Forti considered well-founded *linear* orderings, called ordinals. Given two ordinals which are not order-isomorphic, one of them must be order-isomorphic to a proper initial segment of the other, so ordinals are linearly ordered by extension. Burali-Forti asked "Is the set of ordinals, as ordered by extension, itself an ordinal?" It must be, but if it is one again gets an infinite descending sequence from copies within copies. Thus, the Hypergame paradox is a disguised variation of Burali-Forti's paradox.

The most convenient way to formalize game theory is to embed it within set theory by identifying a game with its game tree, which is a set. How about the set $T_{\text{Hypergame}}$, or the set of all ordinals? It is a theorem of ZF that there are no such sets. Hypergame does not exist, so don't even think about playing it.

The game tree for a countably-finite game is one for which each node has at most countably many nodes immediately beneath it. Countable Hypergame's game tree, T_{CH}, is the tree that glues together all such "countably-splitting" well-founded trees. It is a theorem of ZF that the set T_{CH} exists (at least, it does once one chooses but a single tree out of each isomorphism type, rather than throwing in all trees of a given type) so that our Countable Hypergame example did succeed in eliminating the hidden, incorrect assumption of the Hypergame paradox.

The reader familiar with recursion theory will now recognize Saxe's result as showing that a certain Π_1^1—complete set (roughly, the set of recursive presentations of well-founded trees on ω) is not recursive. The tight result should be that a certain complete recursively enumerable set is not recursive—in particular that the halting problem is undecidable.

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The Editor's Corner: The Exponential Distribution

HERBERT S. WILF

Random number generators are built into most home computers now, so many more people are getting a chance to use them then ever before. Usually, a 'random number generator' manufactures, on demand, numbers ξ between 0 and 1, produced in such a manner as to be equally likely to reside anywhere in that interval.

More precisely, if $[a, b]$ is a subinterval of $(0,1)$, then the probability that the next random number ξ lies in $[a, b]$ is $b - a$. For an ideal random number generator that statement would be true for every $[a, b] \subset (0,1)$.

Given such a supply of random numbers, it is easy to produce other kinds of distributions by various manipulations. To choose a number X uniformly at random in the interval $(7,13)$, for instance, we would write $X = 7 + 6\xi$, where ξ is a random number. Again, to choose an *integer* L uniformly at random between 0 and n, just set $L = \lfloor \xi(n+1) \rfloor$, in which ξ is a random number and '$\lfloor \cdot \rfloor$' is the greatest integer function.

Sometimes we want numbers that aren't equally likely to turn up everywhere, but instead are more likely to lie in some places than in others. These can be obtained by twisting random numbers out of shape, and some fearsomely ingenious constructions have been suggested, over the years, in order to achieve various kinds of twists. See Chapter 3 of [**3**] for accounts of some of these.

Among nonuniform distributions one of the most important is the exponential distribution, which describes positive real numbers X that are chosen in such a way that for every $t > 0$, the probability that $X < t$ is $1 - e^{-t}$. It is therefore more likely that X lies between 1 and 2 (the chance is $e^{-1} - e^{-2} = .232$), say, than between 8 and 9 (probability $e^{-8} - e^{-9} = .00021$), etc.

The exponential distribution occurs naturally in many situations. Two categories of these are waiting time problems and particle diffusion problems. In waiting line ('queueing') simulations it is frequently the case that it is more likely that we wait a shorter time and less likely a longer time. Hence in a computer simulation it is often desirable to select the delay times from the exponential distribution. Likewise, when a particle bounces around in a diffusing medium, it is more probable that it will travel a shorter distance D between successive collisions with molecules of the medium than a longer distance. Under reasonable assumptions it isn't hard to show that D obeys the exponential distribution.

A question of mathematical interest and beauty, as well as of practical application (a happy combination) is then the following: given a random number generator, how shall we select numbers X that have the exponential distribution?

Here's the quick answer (then we'll look at two other ways). First choose a random number ξ, then calculate $X = -\log \xi$. That's all. We claim that the numbers X that are so produced have the exponential distribution. To see that we need perform only the following calculation ('Prob' means 'the probability of'):

$$\begin{aligned}\operatorname{Prob}\{X < t\} &= \operatorname{Prob}\{-\log \xi < t\}\\ &= \operatorname{Prob}\{\xi > e^{-t}\}\\ &= 1 - e^{-t}.\end{aligned}$$

Even though the process above is straightforward to use and easy to verify, I'd like to show you two other methods, whose elegance more than justifies their presentation. The original sources of these two methods are [**1**], [**2**], respectively. For an excellent survey of the whole subject see chapter 3 of [**3**].

Flipping preloaded coins. In 1971, G. Marsaglia [**1**] observed that *we can choose the digits of X independently of each other*, with pretabulated probabilities that depend only on which digit we're talking about. He further proved that exponential distributions are essentially uniquely determined by this property.

To say this more carefully, define the numbers

$$p_j = \frac{1}{1 + e^{2^{-j}}} \qquad (j = 1, 2, \ldots)$$

and think of them as being pretabulated. In order to construct a number X we will choose its binary digits $\varepsilon_1, \varepsilon_2, \ldots$ independently, and then we will write

$$X = \sum \varepsilon_j 2^{-j}. \tag{1}$$

First, with probability p_1 let $\varepsilon_1 = 0$, else let it be 1. Then with probability p_2 choose $\varepsilon_2 = 0$, otherwise let it be 1, and so forth. Having chosen the ε's in that way, the number X that is shown in equation (1) lies between 0 and 1. We claim that X has the truncated exponential distribution

$$\text{Prob}\{X < t\} = Ke^{-t} \qquad (0 < t < 1;\ K = e/(e-1))$$

and here is the reason.

Consider the instant at which we have already selected exactly $k - 1$ of the bits of X and we are about to choose the kth. Write $u = .\varepsilon_1\varepsilon_2 \ldots \varepsilon_{k-1}$, so u is the portion of the output number X that has already been constructed. What is the probability that the very next bit chosen, ε_k, is 0? We claim that this probability is *independent* of u, and in fact, is exactly p_k.

Indeed, there are two possible events, $\varepsilon_k = 0$ or $\varepsilon_k = 1$. In either case the end result X must lie in the interval I between u and $u + 2 \cdot 2^{-k}$ since it would have the latter value if every future bit turned out to be a 1, and the former if they all happen to be 0's. The event that $\varepsilon_k = 0$ is identical with the assertion that X will lie in the left half of the interval I. Hence the probability that $\varepsilon_k = 0$ is given by

$$\begin{aligned}
\frac{\left\{1 - e^{-(u+2^{-k})}\right\} - \left\{1 - e^{-u}\right\}}{\left\{1 - e^{-(u+2\cdot 2^{-k})}\right\} - \left\{1 - e^{-u}\right\}} &= \frac{e^{-u} - e^{-(u+2^{-k})}}{e^{-u} - e^{-(u+2\cdot 2^{-k})}} \\
&= \frac{1 - e^{-2^{-k}}}{1 - e^{-2\cdot 2^{-k}}} \\
&= \frac{1}{1 + e^{-2^{-k}}} \\
&= p_k
\end{aligned}$$

independent of u.

Now it's time to say what is meant by 'the exponential distribution is essentially unique' in this respect. One of Marsaglia's main results is the following

THEOREM. *Let X be a random variable on the unit interval with independent binary digits, and let $F(t)$ be the distribution function of X. If F has a positive derivative at $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (i.e., if F' exists and is positive at all of these points), then X has a (generalized) exponential distribution, with density*

$$Ke^{\beta x} \qquad (0 < x < 1;\ K = \beta/(e^{\beta} - 1));$$

for some real β.

For example, what would happen if we choose the digits of X independently so that the probability of a 0 is $1/3$ and the probability of a 1 is $2/3$? Consider the subset S of the unit interval that consists of those real numbers x that have the following property: the asymptotic density of 0's among the binary digits of x is $1/3$. Then, according to the strong law of large numbers (see [**4**], particularly exercise 8, p. 420), with probability 1 our random variable X will lie in the set S. Hence the distribution function of X is concentrated on S and certainly does not have a positive derivative at the reciprocal powers of 2, in keeping with the theorem.

von Neumann's method. As surprising as Marsaglia's method (above) may be, an earlier one due to von Neumann [**2**] is perhaps equally unexpected.

Select a random number Y_1, then another one Y_2, etc., continuing as long as the numbers so far chosen form a decreasing sequence, i.e.,

$$Y_1 > Y_2 > \cdots > Y_n < Y_{n+1}$$

As soon as a random number is drawn that exceeds its immediate predecessor, the process halts.

If n is even, forget the entire sequence of random numbers, call this experiment a failure, and repeat from the beginning.

If, however, n is odd then we're ready for output. The number X that will be output is just this: the integer part of X is the number of failures so far seen, and the fractional part of X is Y_1, the largest (and first) member of the last decreasing sequence chosen.

For example, if we choose random numbers .63, .19, .37 we would record a failure and start again. If the second sequence is .44, .91 then we are ready for output, and the number that is selected by these events is $X = 1.44$.

It is possibly not entirely obvious that the numbers X have the exponential distribution.

However, following von Neumann, we argue as follows. Since there are $n!$ possibilities for the sequence of relative sizes of n different numbers, the event

$$E_n = \{Y_1 > Y_2 > \cdots > Y_n\}$$

has probability $1/n!$.

Next, the probability that Y_1, the largest of n independently selected random numbers, is $< t$ is equal to the probability that all n of the numbers selected are $< t$, and that is t^n, for $0 < t < 1$. If we differentiate to obtain the probability

density, we find that the probability that both the event E_n and the event $\{t < Y_1 < t + dt\}$ occur is

$$nt^{n-1}\,dt/n! = t^{n-1}\,dt/(n-1)!.$$

Therefore, the probability that we have simultaneously the three events

$$t < Y_1 < t + dt; \quad E_n; \quad \text{not } E_{n+1}$$

is, for $0 < t < 1$,

$$\left\{\frac{t^{n-1}}{(n-1)!} - \frac{t^n}{n!}\right\} dt.$$

If we sum over odd values of n we find that for those values of t,

$$\text{Prob}\{t < Y_1 < t + dt\} = e^{-t}\,dt,$$

which is the portion of the exponential distribution on $(0, 1)$.

It remains to deal with the 'failures.' Since the probability of successful termination without a failure is clearly

$$\int_0^1 e^{-t}\,dt = 1 - 1/e$$

it must be that the probability of a failure is e^{-1}. Hence the probability density for numbers t that are produced by this method by means of F failures and a success is

$$e^{-F}e^{-\{t-F\}}\,dt = e^{-t}\,dt \qquad (F \leqslant t < F + 1).$$

The failure counter therefore correctly expands the distribution to the whole positive semi-axis, which explains everything—except how von Neumann got the idea in the first place.

As an exercise, the reader might like to work out the average running time of this method, as measured by the average number of random numbers that must be generated before the output X is obtained. The correct answer is in **[2]**.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

The Way of All Means

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We are interested in presenting some problems concerning the limits of mean iterations. For our current purposes, a *mean* is a continuous real-valued function M of two strictly positive real variables a and b so that, for all such a and b

$$\min(a, b) \leq M(a, b) \leq \max(a, b).$$

This inequality, which ensures that $M(a, b)$ lies between a and b, is the absolutely essential part of the definition. We also assume that means are symmetric and positively homogeneous, that is,

$$M(a, b) = M(b, a) \quad \text{and} \quad M(\lambda a, \lambda b) = \lambda M(a, b) \qquad \lambda \geq 0,$$

though often these assumptions are unnecessary. Three particular classes of means are:

HÖLDER'S MEANS.

$$H_p(a, b) := [(a^p + b^p)/2]^{1/p} \qquad p \neq 0$$

$$G(a, b) := H_0(a, b) := \lim_{p \to 0} H_p(a, b) = \sqrt{(ab)}\,.$$

Then $A := H_1$ is the *arithmetic* mean and G is the *geometric* mean. The mean H_{-1} is called the *harmonic* mean. Hölder's means are sometimes called power means.

LEHMER'S MEANS.

$$L_p(a, b) := (a^p + b^p)/(a^{p-1} + b^{p-1}).$$

Note that $L_1 = A$, $L_{1/2} = G$ and $L_0 = H_{-1}$. These are the only means that are both Lehmer means and Hölder means [7].

STOLARSKY'S MEANS.

$$S_p(a, b) := [(a^p - b^p)/(pa - pb)]^{1/(p-1)} \qquad p \neq 0, 1.$$

The limiting cases ($p = 0, 1$) give the *logarithmic* and *identric* means, respectively.

Thus

$$S_0(a, b) := \lim_{p \to 0} S_p(a, b) = (a - b)/(\log a - \log b)$$

and

$$S_1(a, b) := \lim_{p \to 1} S_p(a, b) = e^{-1}\left[a^a/b^b\right]^{1/(a-b)}.$$

Note that $S_2 = A$ and $S_{-1} = G$.

A (Gaussian) *mean iteration* associated with two means M and N is the two-term iteration

$$a_{n+1} := M(a_n, b_n) \quad \text{and} \quad b_{n+1} := N(a_n, b_n)$$

with initial values $a_0 := a$ and $b_0 := b$. The common limit of $\{a_n\}$ and $\{b_n\}$ when it exists, is called the (Gaussian) *compound* of M and N and is denoted by $M \otimes N := M \otimes N(a, b)$. The compound $M \otimes N$ exists under very general assumptions on M and N. For any pair of means from the three classes above $M \otimes N(1, z)$ is an analytic function in a neighborhood of 1 and the underlying convergence is quadratic—in the sense that $|a_{n+1} - b_{n+1}| = O(|a_n - b_n|^2)$, so that n iterations typically give c^n digits of $M \otimes N$ for some $c > 1$. (See **[3]** or **[6]** for the reasons behind these assertions.)

In certain cases it is easy to identify the limit function. For example

$$H_p(a, b) \otimes H_{-p}(a, b) = \sqrt{(ab)},$$

which follows from the fact that

$$\left[(a^p + b^p)/2\right]^{1/p}\left[(a^{-p} + b^{-p})/2\right]^{-1/p} = ab.$$

The *Gaussian arithmetic-geometric mean iteration* (AGM) is given by

$$a_{n+1} := (a_n + b_n)/2 \quad \textit{and} \quad b_{n+1} := \sqrt{(a_n b_n)}.$$

In our notation this is the compounding of A and G and the common limit is $A \otimes G$. The remarkable fact is that this has a closed form in terms of complete elliptic integrals, namely

$$A \otimes G(1, z) = (\pi/2)\Big/\int_0^{\pi/2}\left[1 - (1 - z^2)\sin^2 t\right]^{-1/2} dt.$$

The AGM sits at the heart of the most rapid algorithms for the extended precision algorithms for π and all the elementary functions. (See **[2]**, **[3]**, **[4]**, **[8]**, and **[9]**.) The reason for this is that, up to trivial changes of variable, this is the only known quadratically convergent algebraic iteration with an identifiable nonelementary limit. Thus $H_p \otimes G$ are the only compounds of two means from the above three classes where we know that we get an explicitly identifiable nontrivial limit.

This leads to the first question.

Question 1. Can one identify, in closed form, either

a] $$A(a,b)\otimes H_2(a,b)$$

or

b] $$A(a,b)\otimes L_2(a,b)?$$

These are perhaps the two most tantalizing of the unknown mean iterations and almost any information about them would be of some interest. Case *b*] is analyzed in some detail by Lehmer [7], where various expansions are provided. What constitutes a closed form answer is not always clear. We can be more precise. A function is said to be *hypertranscendental* if it satisfies no algebraic differential equation of any order (the solutions of an equation like $(f\cdot f'''' + 7xf')^2 = x^2f^3$ are thus not hypertranscendental). Virtually all the special functions are not hypertranscendental, precisely because they arise as solutions of D.E.s. An exception is the Gamma function, which was shown to satisfy no algebraic differential equation by Hölder.

Question 2. Are either

a] $$A(a,b)\otimes H_2(a,b)$$

or

b] $$A(a,b)\otimes L_2(a,b).$$

hypertranscendental?

An easier problem might be to determine whether either of the above functions is hypergeometric. We say that a function is hypergeometric if it has a power series expansion (around some point x_0) with coefficients $\{c_n\}$ that satisfy $c_n/c_{n-1} = R(n)$, where R is a fixed rational function. Many familiar transcendental functions are hypergeometric including $\exp(x_0 := 0$ and $R(x) := 1/x)$, $\log(x_0 := 1$ and $R(x) := (1-x)/x)$, and the complete elliptic integrals.

One can show that none of the most elementary of the transcendental functions (exp, log, sin, etc.) can be Gaussian compounds of algebraic means. This is done in [3]. Some of these functions are, nonetheless, limits of two term iterations.

For example

$$a_{n+1} := \left(a_n + \sqrt{(a_nb_n)}\right)/2 \quad \text{and} \quad b_{n+1} := \left(b_n + \sqrt{(a_nb_n)}\right)/2$$

converge to $(b_0 - a_0)/(\log b_0 - \log a_0)$. This, however, isn't a Gaussian mean iteration by our definition (because of the lack of symmetry) and the convergence isn't quadratic.

The class of possible compounds of *rational* means (means which are also rational functions) is probably more structured. We show in [3] that the only algebraic functions in this class are pth roots of rational functions. In fact, $A \otimes L_2$ is not algebraic while $H_1 \otimes H_{-1}$ clearly is. This leads to

Question 3. Characterize (or say something interesting about)

a] $F_R := \{M \otimes N(1,z) |$ where M and N are rational means$\}$.

b] $F_A := \{M \otimes N(1, z) |$ where M and N are algebraic means$\}$.
c] $F_H := \{M \otimes N(1, z) |$ where M and N are analytic means$\}$.

Here rational, algebraic, or analytic means are means which are, respectively, also rational, algebraic, or analytic functions of each variable (in some neighborhood of 1). Thus A and H_{-1} are rational means; G and H_2 are algebraic means; and S_0 is an analytic mean. Of course, A and H_{-1} are also algebraic and analytic means, as is any rational mean.

Question 4. Do F_R or F_A contain any of the special functions other than algebraic functions or complete elliptic integrals (or some essentially trivial variant thereof)?

There are many multidimensional quadratically convergent analogs of compounding due to Borchardt and others. (See **[1]** and **[3]**.) To our knowledge none are known to lead to new explicit transcendental limits.

Question 5. What can be said about multidimensional compounds?

An example of a three-dimensional compound is the following. Let

$$a_{n+1} := M_1(a_n, b_n, c_n) := (a_n + b_n + c_n)/3$$

$$b_{n+1} := M_2(a_n, b_n, c_n) := (a_n^2 + b_n^2 + c_n^2)/(a_n + b_n + c_n)$$

$$c_{n+1} := M_3(a_n, b_n, c_n) := \left[(a_n^2 + b_n^2 + c_n^2)/3\right]^{1/2}.$$

Then, if a_0, b_0, and c_0 are positive, the three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ converge quadratically to a common limit. What can be said about the limit function?

The fourth question may be the most interesting. A positive answer would be of astonishing consequence. A negative answer would explain the central role of the AGM.

Additional related material is developed in **[1]**, **[3]**, **[10]**, and **[11]**.

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NOTES

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On Computing Discriminants

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Only the sign of the function $d_n = \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)$ changes when a transposition is applied to the subscripts. Such a function is called an alternating function and the largest group of permutations of degree n under which it is invariant is the alternating group A_n of order $\frac{1}{2}n!$. The discriminant function $D_n = d_n^2$ is fixed by any transposition and is therefore a symmetric function.

When this function is evaluated at the zeros of a polynomial $f = \sum_{i=0}^{n} a_i x^{n-i}$, $a_0 \neq 0$, it is called the discriminant of f and denoted by $D_n f$ which we shall have occasion to write later as $D_n(a_0, a_1, \ldots, a_n)$. The name reflects its properties; it discriminates for multiple zeros ($D_n f = 0$) and also discriminates the Galois group of the splitting field of the polynomial (over the coefficient field) as a subgroup of A_n ($D_n f$ is a square). Neither property is affected by multiplication by a square.

A slight generalization is customary, is more symmetric, and is useful later. As defined, the discriminant is a polynomial in the $\{a_i/a_0\}$, $i = 1, 2, \ldots, n$ which, when expressed as a reduced rational function, has a power of a_0 for denominator. It is convenient to multiply by the denominator (a square) which produces a polynomial function homogeneous in the coefficients. Note that

$$(x_1 x_2 x \cdots x_n)^{2n-2} \prod_{i<j} \left(x_i^{-1} - x_j^{-1}\right)^2 = \prod_{i<j} (x_i - x_j)^2, \quad \text{with } x_1 x_2 \cdots x_n = a_n/a_0.$$

We see that the function $a_0^{2n-2} D_n = \Delta_n(a_0, a_1, \ldots, a_n) = \Delta_n(a_n, a_{n-1}, \ldots, a_0)$ is invariant under the involution taking a_i to a_{n-i}, $i = 0, 1, \ldots, n$ and specializes to $D_n f$ when f is monic ($a_0 = 1$), and, as can be deduced by induction on n, is a homogeneous polynomial in the $\{a_i\}$, $i = 0, 1, \ldots, n$ of degree $2n - 2$.

The construction of the discriminant of a polynomial f seems to be realized most popularly in the literature from the resultant of f and its derivative by using a polynomial remainder sequence, see Childs [**Chi**, Chap. 15]; this can be done so as to produce a Sturm sequence from which the number of real zeros can be read and also the sign of the discriminant — it is a (real) square if and only if the permutation of the zeros representing complex conjugation is even (which is the same as saying that the number of complex zeros is divisible by 4). Berlekamp [**Ber**, §6.6] uses the resultant to find Swan's form of the discriminant for a trinomial

* Supported in part by NSERC and FCAC grants.

which we record as:

$$D_n(x^n + ax^k + b) = (-1)^{n_2} b^{k-l}\left(n^N b^{N-K} - (-1)^N (n-k)^{N-K} k^K a^N\right)^d,$$

$$d = \gcd(n,k), \qquad n = Nd, \qquad k = Kd, \qquad n_2 = \frac{1}{2}n(n-1).$$

Two easily verifiable expressions for $D_n f$ for monic polynomials are:

$$\text{(i)}\ D_n f = (-1)^{n_2}\prod_{i=1}^{n} f'(\alpha_i), \quad \text{and} \quad \text{(ii)}\ D_n f = (-1)^{n_2} n^n \prod_{j=1}^{n-1} f(\beta_j),$$

where f has zeros $\{\alpha_i\}$, $i = 1,2,\dots,n$ and f' has zeros $\{\beta_j\}$ $j = 1,2,\dots,n-1$. Two further alternative forms are described here; the first is as a sum of products of the coefficients of f, and the second is in terms of an $n \times n$ matrix whose entries are symmetric functions of the zeros of f which may be found from Newton's relations.

There is a well-established theory of symmetric functions, see Macdonald **[Mac]**, in which it is shown that any symmetric polynomial in $\mathbb{Z}[x_1, x_2, \dots, x_n]$ can be expressed as a polynomial in $\mathbb{Z}[e_1, e_2, \dots, e_n]$ where the e_r are the elementary symmetric functions defined as

$$e_r = \sum x_{i_1} x_{i_2} \cdots x_{i_r}$$

with the sum over all ${}^nC_r$ choices of subscripts satisfying $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$.

We deduce that $D_n(1, a_1, a_2, \dots, a_n)$ may be expressed as a polynomial in $\mathbb{Z}[a_1, a_2, \dots, a_n]$ since $a_t = (-1)^t e_t$, $t = 1,2,\dots,n$. We shall do this for cubics, quartics and quintics. Each coefficient a_r of $f(x)$ is a homogeneous function of the $\{x_i\}$ of total degree r (this is also called *isobaric of weight* r in the older literature) —by this we mean that

$$a_r(kx_1, kx_2, \dots, kx_n) = k^r a_r(x_1, x_2, \dots, x_n).$$

The discriminant D_n is homogeneous of degree $n^2 - n$ in the $\{x_i\}$ and so may be expressed as an integral sum of products of the coefficients with each product term of weight $n^2 - n$.

For simplicity we work with the reduced monic polynomial for which $a_1 = 0$, as can be assumed by making the transformation $x \leftarrow nx + a_1$, then multiplying the coefficients by n^n.

The cubic. Since $f(x) = x^3 + a_2 x + a_3$ has discriminant of weight 6, we have $D_3 = sa_3^2 + ta_2^3$ for some integers s, t. These integers are found by making use of a sequence of judiciously chosen polynomials; here $x^3 - x$ having zeros $1, 0, -1$ gives $4 = 0 + t(-1)^3$ and $t = -4$, then $x^3 - 3x + 2$ having zeros $1, 1, -2$ gives $0 = s.2^2 - 4(-3)^3$ and $s = -27$, and we finally have

$$D_3 = -27a_3^2 - 4a_2^3.$$

(The generic discriminant is $-27a_3^2 - 4a_2^3 + 18a_1a_2a_3 + a_1^2a_2^2 - 4a_1^3a_3$.)

The quartic. Since $f(x) = x^4 + a_2x^2 + a_3x + a_4$ has discriminant of weight 12, the terms in the sum can be represented (as partitions of 12 with parts 2, 3, or 4) by

the weights of the coefficients of $f(x)$ occurring in the products; these are the seven partitions $4^3, 43^22, 4^22^2, 42^4, 3^4, 3^22^3, 2^6$. Using this notation, (for example $a_4a_3^2a_2$ is denoted by 43^22), we have

$$D_4 = a4^3 + b43^22 + c4^22^2 + d42^4 + e3^4 + f3^22^3 + g2^6.$$

All $f(x)$ for which $a_4 = a_3 = 0$ have repeated zeros and so $D_4 = 0$ and all the terms vanish except the last, hence $g = 0$. The remaining coefficients can be found (solving only one pair of linear equations in the process) by specializing f in the following order: $x^4 - 1$, $x^4 - x$, $x^4 - 3x^2 + 2x$, $x^4 - 2x^2 + 1$, $x^4 + x^2 + 1$, $x^4 - 6x^2 + 8x - 3$. We deduce the following results.

Zeros	$f(x)$	D	Equation	
$1, -1, i, -i$	$x^4 - 1$	-2^8	$= a \cdot (-1)^3$,	$a = 2^8$
$0, 1, \omega, \omega^2$	$x^4 - x$	-3^3	$= e \cdot (-1)^4$, with $\omega = \exp(2\pi/3)$	$e = -3^3$
$0, 1, 1, -2$	$x^4 - 3x^2 + 2x$	0	$= e \cdot 2^4 + f \cdot 2^2(-3)^3$,	$f = -4$
$1, 1, -1, -1$	$x^4 - 2x^2 + 1$	0	$= a \cdot 1^3 + c \cdot 1^2(-2)^2 + d \cdot 1 \cdot (-2)^4$, $c + 4d = -64$	$c = -128$
$\omega, -\omega, \omega^2, -\omega^2$	$x^4 + x^2 + 1$	144	$= a \cdot 1^3 + c \cdot 1^2 \cdot 1^2 + d \cdot 1 \cdot 1^4$, $c + d = -112$	$d = 16$
$1, 1, 1, -3$	$x^4 - 6x^2 + 8x - 3$	0	$= a \cdot (-3)^3 + b \cdot -3.8^2 \cdot -6 + c \cdot (-3)^2(-6)^2$ $+ d \cdot -3 \cdot (-6)^4 + e \cdot 8^4 + f \cdot 8^2(-6)^3$	$b = 2^43^2$

and

$$D_4 = 256a_4^3 - 128a_4^2a_2^2 + 144a_4a_3^2a_2 + 16a_4a_2^4 - 27a_3^4 - 4a_3^2a_2^3;$$

and for generic quartics we add the terms $-192a_4^2a_3a_1 + 144a_4^2a_2a_1^2 - 27a_4^2a_1^4 - 6a_1^2a_3^2a_4 - 80a_4a_3a_2^2a_1 + 18a_4a_3a_2a_1^3 - 4a_4a_2^3a_1^2 + 18a_1a_2a_3^3 - 4a_3^3a_1^3 + a_1^2a_2^2a_3^2$.

Similarly we find for reduced quintics

$$\begin{aligned} D_5 = {} & 3125a_5^4 - 3750a_5^3a_3a_2 + 2000a_5^2a_4^2a_2 + 2250a_5^2a_4a_3^2 - 900a_5^2a_4a_2^3 \\ & + 825a_5^2a_3^2a_2^2 + 108a_5^2a_2^5 - 16000a_5a_4^3a_3 + 560a_5a_4^2a_3a_2^2 - 630a_5a_4a_3^3a_2 \\ & - 72a_5a_4a_3a_2^4 + 108a_5a_3^5 + 16a_5a_3^3a_2^3 + a_4^2D_4(1, 0, a_2, a_3, a_4). \end{aligned}$$

The form of the last term is typical for $n > 2$: by replacing the constant term a_n by zero, we see that D_n contains the term $a_{n-1}^2D_{n-1}$ which, incidentally, shows that $a_1^2a_2^2 \cdots a_{n-1}^2$ occurs with coefficient 1 in D_n. The form of D_{n-1} can be obtained from D_n by putting $a_n = 0$ and dividing by a_{n-1}^2.

According to [**SKW**] the number of non-zero coefficients in the expression for the discriminant increases rapidly with the degree; there are 2, 5, 16, 59, 246, 1103, 5247, 26095 terms for non-reduced (that is, $a_1 \neq 0$) polynomials of degree from 2 up to 9. A general closed form expression for the terms of this infinite sequence seems to be unknown.

From homogeneity of degree $n^2 - n$ in the zeros and of degree $2n - 2$ in the coefficients, we find an upper bound on the terms in the sequence as the number of partitions of $n^2 - n$ into at most $2n - 2$ parts each of which is not greater than n. The corresponding values (which have been computed by Eric Regener) are 2, 5, 18, 73, 338, 1656, 8512, and 45207.

and only if $2k + 1$ is a square, that is for $n = 8m(m + 1)$ or $n = 8m(m + 1) + 1$. The result follows from our earlier remark on the discriminating properties of the discriminant.

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The Geometric, Logarithmic, and Arithmetic Mean Inequality

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Let $0 < a < b$. Because of convexity, the midpoint and trapezoidal approximations to $\int_{\ln a}^{\ln b} e^x dx$ (see Fig. 1) give

$$\left(e^{\frac{\ln a + \ln b}{2}}\right) \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln b} + e^{\ln a}}{2} \cdot (\ln b - \ln a),$$

i.e.,

$$\sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2},$$

the geometric, logarithmic, and arithmetic mean inequality.

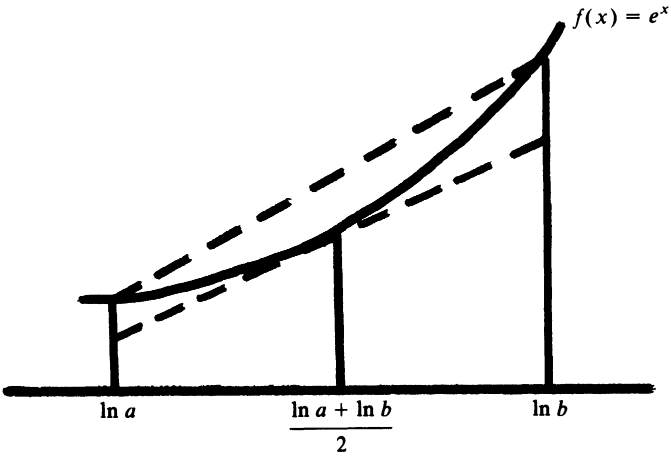

FIG. 1

Incidentally, Tung-Po Lin showed [2]

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \left(\frac{a^{1/3}+b^{1/3}}{2}\right)^3,$$

and these estimates are best possible in the sense of power means,

$$M_p = \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, \; p \neq 0 \quad \text{and} \quad M_0 = \sqrt{ab}.$$

In the spirit of this note, we observe that the second inequality can be obtained by applying Simpson's $\frac{3}{8}$ rule [1],

$$\int_c^d f(x)\,dx = \left[\frac{f(c) + 3f\left(\frac{2c+d}{3}\right) + 3f\left(\frac{c+2d}{3}\right) + f(d)}{8}\right](d-c)$$

$$-\frac{(d-c)^5}{6480} f^{(4)}(\eta) \text{ for some } \eta \text{ between } c \text{ and } d,$$

to the function e^x, replacing c and d with $\ln a$ and $\ln b$, respectively. We have

$$b - a = \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a} + 3e^{\left(\frac{2\ln a + \ln b}{3}\right)} + 3e^{\left(\frac{\ln a + 2\ln b}{3}\right)} + e^{\ln b}}{8} \cdot (\ln b - \ln a)$$

$$= \left(\frac{a^{1/3}+b^{1/3}}{2}\right)^3 (\ln b - \ln a).$$

The inequality arises from the error term associated with this rule.

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Best Approximation by Convex Functions*

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existence and characterization of best uniform approximations from K_c to functions in $C[a, b]$, the class of continuous functions on $[a, b]$. Our theorem is a special case of the more general result announced in **[1]**, for which no complete proof has been published. It turns out that, in order for a function g to be a best convex approximation to a continuous function f, it is necessary and sufficient that the error function $f - g$ achieve its norm thrice with alternating signs, the last being negative, in an interval on which g reduces to a linear polynomial. This is similar to the well-known characterization of best approximation by polynomials **[2]**. The techniques involved are based on fundamental properties of convex functions; in particular, that a convex function remains convex if, on a subinterval, it is replaced by its secant line. A readable text on convex functions is the book **[3]** of Roberts and Varberg.

A convex function g will be called a *best convex approximation* (b.c.a) to $f \in C[a, b]$ if

$$\begin{aligned}\|f - g\|_\infty &:= \sup\{|f(x) - g(x)| : x \in [a, b]\}\\ &= \inf\{\|f - \tilde{g}\|_\infty : \tilde{g} \text{ is convex}\}.\end{aligned}$$

Our main result is the following theorem:

THEOREM 1. *Every $f \in C[a, b]$ has a best convex approximation in K_c. A convex function g is a best convex approximation to $f \in C[a, b] \setminus K_c$ if and only if there exists an interval $[\alpha, \beta] \subseteq [a, b]$ on which g is a linear polynomial, and a point $\tau \in (\alpha, \beta)$ such that*

$$f(\alpha) - g(\alpha) = f(\beta) - g(\beta) = -\|f - g\|_\infty$$

and

$$f(\tau) - g(\tau) = \|f - g\|_\infty.$$

Furthermore, all best convex approximations to f agree on $[\alpha, \beta]$.

Proof. Existence: A simple proof of the existence of best convex approximations is given in **[4]**. For completeness we repeat it here, at the same time filling in some of the details.

Let K denote the class of all convex functions defined on $[a, b]$ (these may fail to be continuous at the endpoints a and b). For $f \in C[a, b]$, choose a 'minimizing sequence' $\{g_k\} \subset K$, one which satisfies

$$\|f - g_k\|_\infty \downarrow \rho := \inf\{\|f - \tilde{g}\|_\infty : \tilde{g} \in K\}.$$

The sequence $\{g_k\}$ is uniformly bounded since

$$\|g_k\|_\infty - \|f\|_\infty \leqslant \|f - g_k\|_\infty \leqslant \|f - g_1\|_\infty \Rightarrow \|g_k\|_\infty \leqslant \|f\|_\infty + \|f - g_1\|_\infty.$$

$\{g_k\}$ also has uniformly bounded variation. This results from the fact that any convex (or concave) function h changes its monotonicity at most once in a given interval, and, therefore,

$$\operatorname{Var}(h; [a, b]) \leqslant 2\Big(\sup_{x \in [a, b]} h(x) - \inf_{x \in [a, b]} h(x)\Big) \leqslant 4\|h\|_\infty.$$

In our case we have

$$\mathrm{Var}(g_k; [a, b]) \leqslant 4(\|f\|_\infty + \|f - g_1\|_\infty).$$

By Helly's Selection Theorem [5] there is a subsequence of $\{g_k\}$, which we will now relabel as $\{g_k\}$, converging pointwise on $[a, b]$ to a function g. Clearly g is bounded and, as the pointwise limit of convex functions, it is also convex. Moreover, for every $x \in [a, b]$

$$|f(x) - g(x)| = \lim_{k\to\infty} |f(x) - g_k(x)| \leqslant \lim_{k\to\infty} \|f - g_k\|_\infty = \rho;$$

hence,

$$\|f - g\|_\infty \leqslant \rho \Rightarrow \|f - g\|_\infty = \rho,$$

from the definition of ρ. Thus g is a b.c.a. to f. If g is not continuous at both of the endpoints, then we may replace g by the convex function g^* that extends g continuously from (a, b) to $[a, b]$. Since f is continuous on $[a, b]$, it follows that $\|f - g^*\|_\infty \leqslant \|f - g\|_\infty$ and, therefore, g^* is a continuous b.c.a. to f.

Sufficiency. Suppose that $g, [\alpha, \beta]$ and τ as described in the theorem exist. If $\tilde{g}$ is any convex function such that

$$\|f - \tilde{g}\|_\infty \leqslant \|f - g\|_\infty,$$

then $h := \tilde{g} - g = (f - g) - (f - \tilde{g})$ satisfies

$$h(\alpha) \leqslant 0, h(\beta) \leqslant 0 \text{ and } h(\tau) \geqslant 0.$$

Since g is assumed to be a linear polynomial on $[\alpha, \beta]$, h is convex and, therefore, these inequalities can only hold if, in fact, there is equality in each case (otherwise the point $(\tau, h(\tau))$ would lie above the line connecting $(\alpha, h(\alpha))$ and $(\beta, h(\beta))$). Thus $h(\alpha) = h(\beta) = h(\tau) = 0$, which in turn implies that $h \equiv 0$ on $[\alpha, \beta]$. This shows that g is a b.c.a. to f and that all such b.c.a.'s agree with g on $[\alpha, \beta]$.

Necessity. For convenience we introduce the following definitions:

DEFINITION. *Let h be defined and bounded on $[a, b]$. We say that h has a negative alternant of length k if there are points $a \leqslant x_1 < \cdots < x_k \leqslant b$ such that $(-1)^{k-i}h(x_i) = -\|h\|_\infty$ $(i = 1, \ldots, k)$.*

DEFINITION. *Let h be defined and bounded on $[a, b]$. We define*

$$\mathrm{crit}(h) := \{x \in [a, b] : |h(x)| = \|h\|_\infty\}.$$

In order to prove the necessity of the conditions in the theorem, we need the following two lemmas.

LEMMA 1. *If g is a b.c.a. to f, then $f - g$ has a negative alternant of length at least* 3.

Proof. This is a result of the type proved for polynomial approximation (see [2, Theorem 1.7] for more detail): If $f - g$ has no negative alternant of length 3, then there is a convex, quadratic polynomial p having the same signs as $f - g$ on

crit$(f-g)$. It follows that, for $\gamma > 0$ small enough, $f - g - \gamma p$ has smaller norm than $f - g$, and, hence, the convex function $g + \gamma p$ is a better convex approximation to f than g, a contradiction to our assumption about g.

LEMMA 2. *There is a negative alternant of length 3 common to all $f - g$ such that g is a b.c.a. to f.*

Proof. We first prove that the assertion of Lemma 2 is valid for a countable collection of b.c.a.'s. Let $\{g_i\}_{i=1}^{\infty}$ be b.c.a.'s to f, and for $\beta_i > 0$ ($i = 1, 2, \ldots$) such that $\sum_{i=1}^{\infty}\beta_i = 1$, define $g := \sum_{i=1}^{\infty}\beta_i g_i$. Since

$$\|g_i\|_\infty - \|f\|_\infty \leqslant \|f - g_i\|_\infty = \rho,$$

we have

$$\|g_i\|_\infty \leqslant \rho + \|f\|_\infty (i = 1, 2, \ldots)$$

and the sequence $\{\|g_i\|_\infty\}$ is uniformly bounded. By the Weierstrass M-test, g is the uniform limit of continuous, convex functions and thus is continuous and convex. Moreover,

$$\rho \leqslant \|f - g\|_\infty = \Big\|f - \sum_i \beta_i g_i\Big\|_\infty = \Big\|\sum_i \beta_i (f - g_i)\Big\|_\infty$$
$$\leqslant \sum_i \beta_i \|f - g_i\|_\infty = \rho,$$

so that equality holds and g is a b.c.a. to f. We now show that

$$\operatorname{crit}(f - g) \subset \bigcap_{i=1}^{\infty} \operatorname{crit}(f - g_i), \tag{1}$$

from which the assertion for countable collections will follow by applying Lemma 1 to $f - g$. Suppose that $x \in \operatorname{crit}(f - g)$ with $f(x) - g(x) = +\rho$. Then

$$\rho = f(x) - g(x) = f(x) - \sum_i \beta_i g_i(x) = \sum_i \beta_i (f(x) - g_i(x))$$
$$\leqslant \sum_i \beta_i \rho = \rho,$$

hence, equality prevails, which is possible only if $f(x) - g_i(x) = \rho$ ($i = 1, 2, \ldots$), i.e.,

$$x \in \bigcap_i \operatorname{crit}(f - g_i). \tag{2}$$

Similarly, if $f(x) - g(x) = -\rho$, then $f(x) - g_i(x) = -\rho$ ($i = 1, 2, \ldots$), so that (2) holds in this case as well. This proves the validity of (1).

Now let $\{g_\alpha\}$ be the collection of all b.c.a.'s to f and define $C_\alpha = \operatorname{crit}(f - g_\alpha)$. The sets C_α are compact and nonempty. Let $C = \bigcap_\alpha C_\alpha$; then C is closed, and, hence,

$$C^c = \Big(\bigcap_\alpha C_\alpha\Big)^c = \bigcup_\alpha C_\alpha^c$$

is an open covering of C^c (here the superscript c denotes the complement). By the Lindelöf property of the real line, this can be reduced to a countable subcovering: $C^c = \bigcup_{i=1}^{\infty} C_{\alpha_i}^c$, and thus $C = \cup_{i=1}^{\infty} C_{\alpha_i}$. As we have shown above, this intersection contains a common negative alternant of length 3, and, therefore, the lemma is proved.

We now have the tools needed to complete the proof of Theorem 1. Let g be a b.c.a. to $f \in C[a, b] \setminus K$, and denote the 'error function' $f - g$ by e. Thus $\|e\|_\infty = \|f - g\|_\infty = \rho$. Let $\alpha \leqslant x_0 < x_1 < x_2 \leqslant b$ be the negative alternant guaranteed by Lemma 3 to be common to all b.c.a.'s to f. If $e(x) = -\rho$ for any $x \in (x_0, x_2)$, we may replace x_0 by $\max\{x \in [x_0, x_2) : e(x) = -\rho\}$ and x_2 by $\min\{x \in (x_0, x_2] : e(x) = -\rho\}$, so that without loss of generality $e(x) > -\rho$ in (x_0, x_2). We first show that either g is a linear polynomial in some neighborhood of x_1, or else there is a b.c.a. $\tilde{g}$ for which x_1 is not in crit$(f - \tilde{g})$, a contradiction to our assumption on x_1. Choose $0 < \varepsilon_1 < \min\{x_1 - x_0, x_2 - x_1\}$. By the continuity of e there is a $0 < \delta < 1$ such that

$$-\delta\rho \leqslant e(x) \leqslant \rho \quad \text{for all } x \in [x_1 - \varepsilon_1, x_1 + \varepsilon_1]. \tag{3}$$

Further, there is a number $0 < \varepsilon_2 \leqslant \varepsilon_1$ such that the secant line l between the points $(y_-, g(y_-))$ and $(y_+, g(y_+))$, where $y_- := x_1 - \varepsilon_2$ and $y_+ := x_1 + \varepsilon_2$, satisfies

$$l(x) \leqslant g(x) + (1 - \delta)\rho \text{ for all } x \in [y_-, y_+]. \tag{4}$$

Since g is convex we also have

$$l(x) \geqslant g(x) \tag{5}$$

in the same interval. Combining (3), (4), and (5) yields

$$\begin{aligned} \rho \geqslant f(x) - g(x) \geqslant f(x) - l(x) &\geqslant f(x) - g(x) - (1 - \delta)\rho \\ = e(x) + \delta\rho - \rho \geqslant -\rho. & \end{aligned} \tag{6}$$

Let $\tilde{g}$ denote the convex function gotten by replacing g by l on $[y_-, y_+]$. By (6) $\tilde{g}$ is a b.c.a. to f. But, unless g was already linear on $[y_-, y_+]$,

$$f(x_1) - l(x_1) < f(x_1) - g(x_1) = \rho,$$

and, therefore, $x_1 \notin$ crit$(f - \tilde{g})$, the desired contradiction.

Thus, g is linear in some neighborhood of the point x_1. Let α and β be the endpoints of the largest interval containing x_1 such that g is linear on $[\alpha, \beta]$ and $e(x) > -\rho$ in (α, β). We wish to show that $e(\alpha) = e(\beta) = -\rho$. To this end we suppose that $e(\beta) > -\rho$ (the proof for $e(\alpha) > -\rho$ is analogous). By reason of continuity there exists a $0 < \delta < 1$ such that

$$\rho \geqslant e(x) \geqslant -\delta\rho \quad \text{for all } x \in [x_1, \beta].$$

Moreover, there is an $\varepsilon_1 > 0$ such that

$$\rho \geqslant e(x) \geqslant -\left(\frac{1 + \delta}{2}\right)\rho \quad \text{for all } x \in [x_1, \beta + \varepsilon_1]. \tag{7}$$

Now choose $0 < \varepsilon_2 < \varepsilon_1$ so that the secant line l between the points $(x_1, g(x_1))$ and

PROPOSITION 3. *If f is an odd function on $[a, -a]$, then p_1 is its best convex (and concave) approximation.*

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A Simple Characterization of the Gamma Function

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The important properties $\Gamma(n+1) = n!$ for natural numbers n and $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ do not characterize the Gamma function. A well-known additional property sufficient for a characterization is the convexity of $\log \Gamma(x)$ [**1**]. We feel that a property originally considered by Euler is less far-fetched and much easier to use.

The Gamma function made its first appearance in a letter of Euler to Goldbach in 1729 when an infinite product was announced. Later Euler produced several representations and, finally, disclosed his original idea [**2**]: For a fixed infinitely large natural number n and all finite natural numbers m, the expression $(n+m)!$ behaves approximately like a geometric sequence. In other words, $L(n+m) = \log(n+m)!$ is infinitesimally close to a linear function of m:

$$\begin{aligned} L(n+m) &= \log n! + \sum_{k=1}^{m} \log(n+k) \\ &= L(n) + m \cdot \log(n+1) + \sum_{k=1}^{m} \log\left(1 + \frac{k-1}{n+1}\right) \\ &\approx L(n) + m \cdot \log(n+1). \end{aligned}$$

Now the assumption of Euler is that this linearity property extends to all finite real numbers x in place of m. A proof in the spirit of Euler should explicitly use infinitely large and small numbers [**3**]. Presently, we shall give the theorem and its proof in terms of limits.

PROPOSITION 3. *If f is an odd function on $[a, -a]$, then p_1 is its best convex (and concave) approximation.*

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THEOREM. *There exists a unique function* Γ, $\Gamma(x) > 0$ *for* $x \geqslant 1$, *such that* $L(x) = \log\Gamma(x+1)$ *has, for* $x \geqslant 0$, *the following properties*:

(i) $L(0) = 0$ (*initial condition*, $1! = 1$),

(ii) $L(x+1) = \log(x+1) + L(x)$ (*functional equation*),

(iii) $L(n+x) = L(n) + x \cdot \log(n+1) + r_n(x)$, *where*

$$\lim_{n\to\infty} r_n(x) = 0. \tag{1}$$

Proof. From (i) and (ii), we obtain

$$L(n) = \log n! = \sum_{k=1}^{n} \log k \tag{2}$$

and, for natural n and real $x \geqslant 0$,

$$L(x+n) = L(x) + \sum_{k=1}^{n} \log(x+k). \tag{3}$$

As a consequence of (3), (iii) and (2),

$$L(x) = L(n) - \sum_{k=1}^{n} \log(x+k) + x \cdot \log(n+1) + r_n(x), \tag{4}$$

$$L(x) = \sum_{k=1}^{n} [\log k - \log(x+k) + x(\log(k+1) - \log k)] + r_n(x), \tag{5}$$

$$L(x) = \sum_{k=1}^{n} [(1-x)\log k + x \cdot \log(k+1) - \log(x+k)] + r_n(x). \tag{6}$$

By $\log(n+1) = -\gamma_n + \sum_{k=1}^{n} \frac{1}{k}, \lim_{n\to\infty}\gamma_n = \gamma = .577\ldots$, we obtain from (4) and (2)

$$L(x) = -\gamma_n x + \sum_{k=1}^{n} \left(\frac{x}{k} - \log(x+k) + \log k\right) + r_n(x), \tag{7}$$

$$L(x) = -\gamma_n x + \sum_{k=1}^{n} \left(\frac{x}{k} - \log\left(1 + \frac{x}{k}\right)\right) + r_n(x). \tag{8}$$

The series of (8) converges, since for $k > x \geqslant 0$

$$0 \leqslant \frac{x}{k} - \log\left(1 + \frac{x}{k}\right) = \frac{x}{k} - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\left(\frac{x}{k}\right)^j \leqslant \frac{x^2}{2k^2}.$$

It follows that the right-hand sides of (8), and consequently of (7), (6), (5) and (4), converge, and that

$$L(x) = -\gamma x + \sum_{k=1}^{\infty} \left(\frac{x}{k} - \log\left(1 + \frac{x}{k}\right)\right)$$

is a uniquely determined real-valued function for $x \geqslant 0$ which can be represented by any of those equations for $n \to \infty$, where $r_n(x) \to 0$.

It remains to show that $L(x)$ has properties (i), (ii), (iii). Here (i) follows, e.g., from (8). From (5) and (1),

$$\begin{aligned} L(x+1) - L(x) &= \sum_{k=1}^{n} [\log(x+k) - \log(x+k+1) + \log(k+1) - \log k] \\ &\quad + r_n(x+1) - r_n(x) \\ &= \log(x+1) - \log(x+n+1) + \log(n+1) \\ &\quad + r_n(x+1) - r_n(x) \\ &= \log(x+1) - \log\left(1 + \frac{x}{n+1}\right) + r_n(x+1) - r_n(x). \end{aligned}$$

For $n \to \infty$ we obtain (ii). Now (3) is a consequence of (ii), and (4) is equivalent to (8). Postulate (iii) follows from (3) and (4). □

Immediate consequences of (6) and (8) are the infinite products for $\Gamma(x+1) = x \cdot \Gamma(x) = \exp L(x)$,

$$x \cdot \Gamma(x) = \prod_{k=1}^{\infty} \frac{k^{1-x}(k+1)^x}{x+k} \quad \text{(Euler 1729)} \tag{6'}$$

and

$$x \cdot \Gamma(x) = e^{-\gamma x} \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + \dfrac{x}{k}}. \tag{8'}$$

It is well known how the products can be used to derive the integral representation and the properties of $\Gamma(z)$ for complex z [**4**].

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Mathematics and Computer Science Journal Subscriptions in Undergraduate Libraries

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Recently, a group of selective liberal arts colleges issued a report, based on a conference at Oberlin College, that emphasized the scholarly contributions these schools make to research and education in science and mathematics. A glance at the data in the Oberlin report shows that, in many respects, mathematics has quite different characteristics than the other sciences. This report and other national studies suggest a need for special indicators of quality in mathematics programs at undergraduate colleges.

Library resources are especially crucial to mathematics, as resources both for students and for faculty. Journal subscriptions pose specially difficult issues because there are so many to choose from and because their costs rapidly exceed the typical budgets of undergraduate institutions. In order to help undergraduate colleges assess their library holdings, I undertook a survey of selective liberal arts colleges to determine patterns in mathematics journal holdings at these "better" undergraduate institutions.

The survey was mailed to all colleges that participated in the Oberlin conference and all colleges in the Associated Colleges of the Midwest and the Great Lakes Colleges Association. This totaled approximately 60 institutions, representing both selective and average liberal arts colleges; 37 institutions, about 60%, returned useable forms. This report is based on an analysis of these 37 responses.

Institutional characteristics. Graduating classes of responding institutions ranged in size from 125 to 950, with a median very close to 400. The mid-range (the middle 50% of institutions) graduated between 250 and 500 students each year.

Half of the institutions offer separate majors in computer science and in mathematics, while half still only offer a mathematics major. Since the overlap in these programs at liberal arts colleges is substantial, for purposes of summary I have combined the mathematics and computer science majors. When this is done, we find a median of 5.8% of the graduating class for the last three years with majors in mathematics or computer science, with a mid-range of 4.2% to 7.3%. The range was from 1.7% to 13.6%.

Mathematics and computer science faculty, combined, range from 4 to 23, with a mid-range of 7 to 12. More interesting, perhaps, is the great unevenness of mathematics and computer science faculty per 100 graduates, which varies by 400% (from 1 to 4) although its mid-range is more limited (from 2 to 3). Typically, these colleges have 2.5 times as many graduating majors as faculty, although some have 10 times as many and others come close to a 1:1 ratio of faculty to majors.

Library resources. Because of the broad range of institutional size and quality, the library budgets for mathematics and computer science varied enormously, from a low of $1,500 to a high of $59,300 per year (for books and journals together). The median figure was $8,000, with a mid-range of $5,000 to $16,800. On a per-graduate basis (not per-major, but per B.A. recipient), the typical institution spent $22.70 per year for mathematics and computer science library resources, with a mid-range of $12.60 to $39.00. The full range was still enormous, from $5.40 to $104.20.

Journal holdings. The major part of our survey was devoted to gathering data on current journal subscriptions with indicators of back holdings. This information has been summarized into a single form, which is printed here by rank order of frequency of journal subscription. Each journal has attached to it five items of information:

— The percentage of responding libraries that currently subscribe.
— The average percentage of back holdings for libraries that subscribe.
— The approximate annual subscription price.
— The year in which the journal began.
— The code for a classification category.

The original survey contained a list of about 120 journals derived from a pre-survey of three large undergraduate libraries. Responding institutions added to this list other journals in the mathematical sciences. Originally, I expected the list to grow to 150 or perhaps 180. In fact, the total number of journals received by the 37 responding institutions is over 320. But only 108 journals are received by 5 or more institutions. The journal list presented here is limited to these 108 more common journals.

Journal subscriptions follow the typical pattern of frequencies fairly well. Only 8 journals are subscribed to by over 80% of the libraries, and only 12 journals by more than 2/3 of the libraries. Exactly 25 journals are received by a majority of the colleges responding to this survey. The remaining 83 journals among the top 108 are taken by only a minority of the institutions. Six of the 25 top journals are general interest journals; ten are mathematics research journals; five are computing or computer science journals; and the remaining four are in applications, statistics, and education.

Mathematics and Computer Science Journal Subscriptions in Undergraduate Libraries
(Listed in Rank Order)

Rank: Percentage of libraries with current subscriptions
Back: Average percentage of back holdings among subscribers
Cost: Approximate annual subscription price
Start: Year in which journal began
Code: A = Applications G = General
C = Computing R = Research
E = Education S = Statistics

Journal Title	Rank	Back	Cost	Start	Code
American Mathematical Monthly	100%	86%	70	1894	G
Mathematics Magazine	95%	59%	28	1926	G
Bulletin of the American Mathematical Society	92%	79%	75	1894	R
Proceedings of the American Mathematical Society	81%	91%	295	1950	R
Transactions of the American Mathematical Society	81%	73%	480	1900	R
Communications of the ACM	81%	68%	50	1958	C
Mathematics Teacher	81%	65%	40	1908	E
Byte	81%	60%	20	1975	C
Computing Reviews	76%	71%	60	1960	C
Mathematical Reviews	73%	94%	2000	1940	R
Mathematical Intelligencer	73%	80%	25	1978	G
Journal of the American Statistical Association	68%	66%	70	1888	S
SIAM Review	65%	78%	85	1959	A
Computing Surveys of the ACM	65%	78%	60	1969	C
Duke Mathematical Journal	62%	87%	120	1935	R
Canadian Journal of Mathematics	62%	83%	50	1949	R
Notices of the American Mathematical Society	62%	70%	50	1953	G
American Journal of Mathematics	62%	68%	115	1878	R
Annals of Mathematics	59%	77%	140	1884	R
Journal of Undergraduate Mathematics	54%	79%	5	1969	G
ACM Transactions on Programming Languages & Systems	54%	68%	20	1979	C
Annals of Statistics	51%	96%	75	1973	S
Pacific Journal of Mathematics	51%	87%	150	1951	R
Abstracts of Papers (American Mathematical Society)	51%	87%	25	1980	R
Journal of Recreational Mathematics	51%	83%	45	1968	G
Journal of Symbolic Logic	49%	89%	65	1936	R
Historia Mathematica	49%	88%	60	1974	G
Current Mathematics Publications	49%	63%	175	1969	R
ACM Transactions on Mathematical Software	49%	58%	20	1975	C
Mathematical Gazette	49%	42%	25	1894	G
Annals of Probability	46%	98%	70	1973	S
Journal of Algebra	46%	95%	570	1964	R
Illinois Journal of Mathematics	46%	92%	50	1957	R
SIAM Journal on Applied Mathematics	46%	60%	120	1953	A
ACM Transactions on Database Systems	46%	55%	20	1976	C
Michigan Mathematical Journal	43%	87%	30	1952	R
UMAP Journal	43%	84%	60	1980	G
Mathematics of Computation	43%	74%	130	1943	C
IBM Systems Journal	43%	71%	15	1962	C

Journal Title	Rank	Back	Cost	Start	Code
Journal of the London Mathematical Society	43%	68%	185	1926	R
Proceedings of the London Mathematical Society	43%	44%	275	1865	R
College Mathematics Journal	43%	35%	33	1970	G
Computers and the Humanities	41%	93%	75	1966	C
Arithmetic Teacher	41%	83%	40	1954	E
Fibonacci Quarterly	41%	82%	25	1963	R
School of Science and Mathematics	41%	74%	25	1901	E
Proceedings of the Royal Society of London	41%	53%	395	1800	R
ACM Transactions on Office Information Systems	38%	90%	20	1983	C
Pi Mu Epsilon Journal	38%	72%	5	1949	G
Journal of Combinatorial Theory, Series B	38%	62%	190	1966	R
Creative Computing	38%	60%	25	1974	C
Advances in Mathematics	35%	82%	375	1961	R
Operations Research	35%	79%	90	1952	A
Journal of Combinatorial Theory, Series A	35%	70%	235	1966	R
Journal of Mathematical Physics	32%	99%	440	1960	A
Topology	32%	99%	150	1962	R
Quarterly of Applied Mathematics	32%	96%	45	1943	A
Fundamenta Mathematicae	32%	90%	105	1920	R
SIAM Journal on Numerical Analysis	32%	74%	120	1964	A
Journal of the Royal Statistical Society A: General	32%	55%	30	1835	S
Soviet Mathematics—Doklady	30%	98%	300	1960	R
Quarterly Journal of Mathematics	30%	86%	85	1930	R
Journal of ACM	30%	65%	50	1954	C
Biometrika	30%	57%	40	1901	S
Cognitive Science	30%	49%	65	1977	A
Journal of Functional Analysis	27%	87%	450	1967	R
Israel Journal of Mathematics	27%	57%	150	1951	R
Journal of the Royal Stat. Society B: Methodology	27%	49%	20	1934	S
Computerworld	27%	18%	45	1967	C
Mathematical Spectrum	24%	87%	10	1968	G
Collegiate Microcomputer	24%	85%	30	1983	C
Acta Mathematica	24%	77%	90	1882	R
SIAM Journal on Control & Optimization	24%	52%	120	1963	A
Software—Practice and Experience	24%	38%	240	1971	C
Infoworld	24%	33%	30	1979	C
Educational Studies in Mathematics	22%	99%	80	1968	E
Journal of Number Theory	22%	89%	200	1969	R
Journal of Mathematical Psychology	22%	84%	120	1964	A
Archive for the History of the Exact Sciences	22%	73%	325	1960	G
American Statistician	22%	60%	20	1947	S
Discrete Mathematics	22%	53%	435	1971	R
Journal of Differential Geometry	19%	93%	180	1967	R
Crux Mathematicorum	19%	92%	20	1975	G
Notre Dame Journal of Formal Logic	19%	85%	35	1960	R
Abacus	19%	81%	20	1983	C
Compositio Mathematica	19%	75%	250	1933	R
Mathematische Zeitschrift	19%	71%	600	1918	R
Indiana University Mathematics Journal	19%	71%	95	1970	R
Journal of Graph Theory	19%	59%	75	1976	R
Artificial Intelligence	19%	41%	175	1972	C

Journal Title	Rank	Back	Cost	Start	Code
Mathematical Systems Theory	16%	98%	100	1966	A
ACM Transactions on Graphics	16%	92%	65	1982	C
Commentarii Mathematici Helvetici	16%	82%	100	1929	R
Journal for Research in Mathematics Education	16%	65%	20	1970	E
Popular Computing	16%	55%	15	1979	C
SIAM Journal of Computing	16%	54%	100	1972	C
Rocky Mountain Journal of Mathematics	16%	54%	95	1971	R
Pentagon	16%	44%	5	1940	G
Computer Magazine	16%	43%	35	1972	C
Communications on Pure and Applied Mathematics	14%	99%	175	1948	R
Canadian Mathematical Bulletin	14%	80%	25	1958	R
Journal of Pure and Applied Algebra	14%	53%	330	1971	R
Indigationes Mathematicae	14%	50%	30	1939	R
SIAM News	14%	48%	15	1968	G
Annals of Pure and Applied Logic	14%	43%	60	1970	R
Personal Computing	14%	38%	25	1977	C
Computing Teacher	14%	34%	25	1979	C
Algebra Universalis	14%	6%	144	1871	R

$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (Proof by exhaustion)

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$$\frac{1}{2}\left(\frac{2}{3} - \frac{2}{4}\right) = \frac{1}{3} - \frac{1}{4};$$

$$\frac{1}{4}\left(\frac{4}{5} - \frac{4}{6}\right) = \frac{1}{5} - \frac{1}{6}, \quad \frac{1}{4}\left(\frac{4}{7} - \frac{4}{8}\right) = \frac{1}{7} - \frac{1}{8};$$

$$\ldots .$$

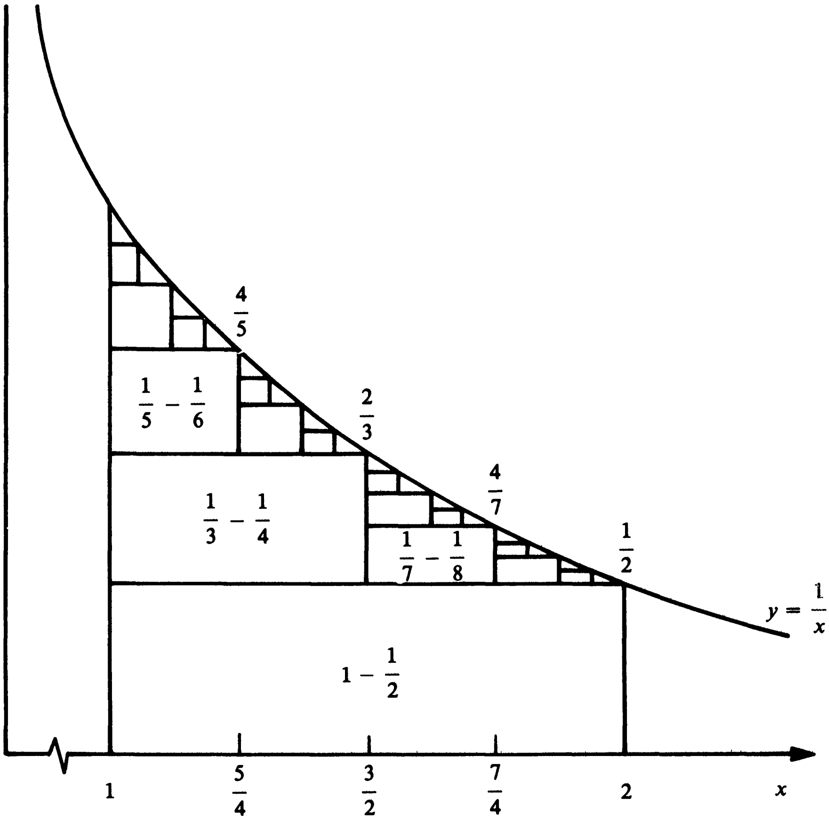

FIG. 1.

In general,

$$\frac{1}{2^n}\left(\frac{2^n}{2^n+2k-1}-\frac{2^n}{2^n+2k}\right)=\frac{1}{2^n+2k-1}-\frac{1}{2^n+2k},$$

$k = 1,2,\dots,2^{n-1}$;
$n = 1,2,\dots$.

$$\begin{aligned}\ln 2 = \int_1^2 \frac{dx}{x} &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\\ &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots .\end{aligned}$$

Thus we may conclude that $\ln(1 + x)$ equals its Maclaurin series when $x = 1$.

Defining the Sign of a Permutation

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We would like to call attention to a way of defining the sign of a permutation that offers several advantages over traditional approaches. We will comment on these advantages after giving the definition.

Consider a permutation σ: $S \to S$. Let n be the number of points in S and let $|\sigma|$ denote the number of orbits of σ. We define the sign of σ by

$$\varepsilon(\sigma) = (-1)^{n-|\sigma|}. \tag{1}$$

For example, if σ is the permutation of 7 points represented by the graph in Figure 1, then $|\sigma| = 4$ and $\varepsilon(\sigma) = -1$. (Note that each orbit of σ corresponds to a component of the graph.)

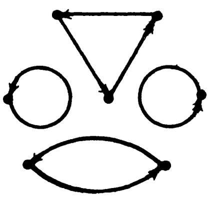

FIG. 1

Starting from (1) the properties of $\varepsilon(\sigma)$ can be derived in a direct and natural way. First we show that

$$\varepsilon((xy) \circ \sigma) = -\varepsilon(\sigma), \tag{2}$$

where $\circ$ denotes the usual composition of functions and (xy) denotes the transposition of x and y. Consider Figures 2 and 3. There, circles represent the orbits of σ that contain x and y while dashed arrows show how $(xy) \circ \sigma$ maps the points x' and y' defined by $\sigma(x') = x$, $\sigma(y') = y$. Let $\tau = (xy) \circ \sigma$. Figure 2 illustrates the case in which x and y belong to different orbits of σ; in that case $|\tau| = |\sigma| - 1$. In

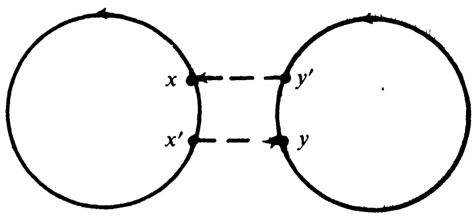

FIG. 2

in [**1**, p. 79]: it would not use an "extraneous" function like the polynomial in (5). For another proof with this feature see [**6**].

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A Refraction Problem in Several Variables

NATHANIEL L. SILVER
3015 Riverdale Ave., Bronx NY 10463

Standard refraction problems in calculus of one variable can be modified to produce instructive boundary minimum problems in calculus of several variables.

1. Statement of the problem. A standard calculus problem is to determine the path of least time when passing from one medium to another: Suppose segments AO and OB are perpendicular with $|AO| = a$ miles and $|OB| = b$ miles. An individual, in a hurry to get from point A to point B, reaches the road OB at some point P between O and B by cutting across swampland along AP at v_1 mph and then proceeding on the road along PB at v_2 mph. Find the path of least time.

One can modify the problem by assuming that the segments AO and OB are both paved stretches of road, the modification being that AO is paved. Does this change the problem in any significant way? It does. Now, the mathematician must consider paths $AQ \cup QP \cup PB$ (see Fig. 1), in which the traveler leaves the road at

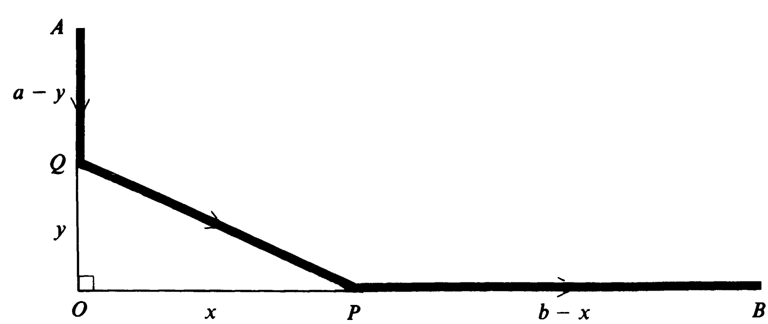

FIG. 1.

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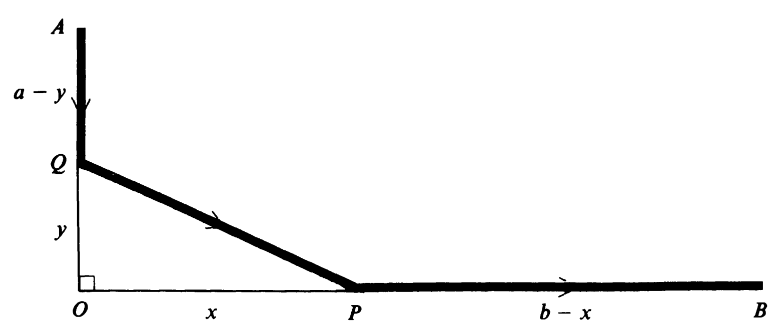

FIG. 1.

Q between A and O, cuts across the swamp along QP, and reenters the road at P between O and B. As before, one seeks the path of least time.

2. An example. For definiteness, set $|AO| = 1$ mi., $|OB| = 10$ mi., $v_1 = 3$ mph, and $v_2 = 5$ mph. Let $x = |OP|$ and $y = |OQ|$. The function T, given by

$$T(x, y) = \frac{1-y}{5} + \frac{\sqrt{x^2+y^2}}{3} + \frac{10-x}{5},$$

maps distance to elapsed time, provided its domain is restricted to the set of points that correspond to *feasible solutions*, i.e., existing physical paths. Note that $x = 0$ iff $y = 0$. Since T is continuous, extending the restricted domain to its closure, $R = \{(x, y): 0 \le x \le 10 \text{ and } 0 \le y \le 1\}$, insures that T has an absolute minimum value on R. It is no coincidence that the minimum value turns out to have a feasible solution, as is explained in Section 3.

Interior Analysis. It may help the analysis to see that T is symmetric in x and y. There are no stationary points, and the only candidate for a nondifferentiable extremum is $T(0,0)$ on the boundary.

Boundary Analysis. In the first two cases T is increasing, and its minimum value is $T(0,0) = 2$ hr. 12 min.

Case 1: $y = 0$.

$$T(x,0) = \frac{2}{15}x + \frac{11}{5}, \qquad 0 \le x \le 10.$$

Case 2: $x = 0$.

$$T(0, y) = \frac{2}{15}y + \frac{11}{5}, \qquad 0 \le y \le 1.$$

The next two cases lead to standard refraction problems.

Case 3: $y = 1$.

$$T(x,1) = \frac{10-x}{5} + \frac{\sqrt{x^2+1}}{3}, \qquad 0 \le x \le 10.$$

The minimum value is $T(3/4, 1) = 2$ hr. 16 min.

Case 4: $x = 10$.

$$T(10, y) = \frac{1-y}{5} + \frac{\sqrt{y^2+100}}{3}, \qquad 0 \le y \le 1.$$

Since T is decreasing, its minimum value is $T(10,1) > 3$ hr., an end-point minimum value. (Ways of posing such problems are discussed at length in [**3**] and [**2**], reprinted in [**1**].) Moreover (if the lower road is washed out), at each point on AO, the path of least time is to head directly toward B.

Comparing all local minima on the boundary shows that the global minimum value is at the origin. The path of least time is to take the road the whole way. The instructor can wax philosophical by drawing a moral: Cutting corners does not always save time.

3. How to construct the problems. Suppose v_1 and v_2 are positive velocities. If $v_1 \geq v_2$ (giving equal time to some reptilian individuals who negotiate swampland at least as fast as they travel the road), then the path of least time is to go straight from A to B.

Next, consider the nontrivial case, $v_1 < v_2$. Let

$$T(x, y) = \frac{(a+b)-(x+y)}{v_2} + \frac{\sqrt{x^2+y^2}}{v_1}, \qquad 0 \leq x \leq b \quad \text{and} \quad 0 \leq y \leq a.$$

Along the coordinate axes, T is linear with positive slope $1/v_1 - 1/v_2$, which, in the final analysis, eliminates all unfeasible solutions.

Solving $T_x = 0 = T_y$ gives $x = y$ and $v_2 = \sqrt{2}\, v_1$. Stationary points are on $x = u = y$, where $0 \leq u \leq \min\{a, b\}$. The value of T along this line is $T(u, u) = (a+b)/v_2$, the same minimum time it takes to traverse the road. The path of minimum time is not unique. Every path, in which $\triangle POQ$ is isosceles, is a path of least time. Each of these paths cuts through swampland, depending on u, for a distance of $\sqrt{2}\, u$ miles.

Uniqueness is restored, if $\sqrt{2}\, v_1 < v_2$. Surprisingly, boundary analysis shows that independent of lengths $|AO|$ and $|OB|$, taking the road the entire way is the path of least time.

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PROBLEMS AND SOLUTIONS

EDITED BY PAUL T. BATEMAN, HAROLD G. DIAMOND, KENNETH B. STOLARSKY (ADVANCED PROBLEMS), AND DOUGLAS B. WEST (ELEMENTARY PROBLEMS)

*For instructions about submitting **proposed** problems for publication in this department see the inside front cover. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the address given on the inside front cover.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed by October 30, 1987, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3213. *Proposed by D. M. Bloom, Brooklyn College, CUNY.*

As in Problem E 2276 [1971, 78; 1972, 90] we consider the "Ehrenfest urn" game in which players A and B have between them n cards labelled $1, 2, \dots, n$. At each move, one of the numbers $1, 2, \dots, n$ is chosen at random and the player who has the card with that number must give it to the other player. The game continues until one player has all the cards. Prove that the expected length of the game is

(a) $2^{n-1} - 1$ if A initially has exactly one card,

(b) $2^{n-1} - \frac{1}{2}(n+1)$ if A is equally likely to start with any number of cards in $\{1, 2, \dots, n\}$.

E 3214. *Proposed by Guo Qiang Zhang, Computer Laboratory, University of Cambridge, England.*

Let f be a real function with $n+1$ derivatives on $[a, b]$. Suppose $f^{(i)}(a) = f^{(i)}(b) = 0$ for $i = 0, 1, \dots, n$. Prove that there is a number ξ in (a, b) such that $f^{(n+1)}(\xi) = f(\xi)$.

E 3215. *Proposed by László Cseh and Imre Merényi, Cluj, Romania.*

Let ϕ denote Euler's arithmetical function and let A be the set of positive integers n for which the equation $\phi(x+n) = 3\phi(x)$ has at least one solution x.

Prove that for every positive integer N at least half of the integers in $[1, N]$ belong to A.

E 3216. *Proposed by Jon Froemke and Jerrold W. Grossman, Oakland University, Rochester, Michigan.*

Fix an integer $a > 1$. For any prime p not dividing a, let $m(p)$ be the multiplicative order of a modulo p, i.e., the smallest positive integer $m(p)$ such that $a^{m(p)} - 1$ is divisible by p. Prove that the ratio $(p-1)/m(p)$ is an unbounded function of p.

E 3217. *Proposed by Richard E. Pfiefer, San Jose State University.*

If R is a plane rectangle of area 1, let $D(R)$ be the average distance between two points of R taken at random (with respect to uniform distribution). Prove that $D(R) \geqslant D(S)$, where S is a square of area 1, and that equality holds only when R is a square.

E 3218. *Proposed by W. O. Egerland, University of Baltimore.*

Suppose a and T are given positive real numbers and suppose n is a given positive integer. Assume that the differential equation $x^{(2n)} = (-1)^n ax$ with boundary conditions

$$\begin{aligned} x^{(m)}(T) &= 0 \qquad (m = 0, 1, \ldots, n-1) \\ x^{(m)}(0) &= 0 \qquad (m = n, n+1, \ldots, 2n-1) \end{aligned}$$

has a nontrivial solution. Prove that this solution is unique up to a constant factor.

SOLUTIONS OF ELEMENTARY PROBLEMS

Another Root Test

E 3017 [1983, 567]. *Proposed by Charles W. Schelin, University of Wisconsin, La Crosse.*

Let $f(z)$ be a real polynomial of degree $n \geqslant 1$ such that $f(-1)f(1) \neq 0$. Put $L = -f'(-1)/f(-1)$ and $R = f'(1)/f(1)$. Show that:

(a) If $L \leqslant n/2$ or $R \leqslant n/2$, then f has a zero in $|z| \geqslant 1$.

(b) If $L \geqslant n/2$ or $R \geqslant n/2$, then f has a zero in $|z| \leqslant 1$.

Solution I by D. Richman, University of South Carolina. Since $L(f_1 f_2) = L(f_1) + L(f_2)$ and since $R(f_1 f_2) = R(f_1) + R(f_2)$, it suffices to consider the cases that f is linear or that f is quadratic with nonreal roots.

If f is linear and r is its root, then $L = 1/(1+r)$. If $L \leq 1/2$, $2/(1+r) \leq 1$, so either $r < -1$ or $r > 1$ (since $r \neq -1, 1$). If $L \geq 1/2$, then $|r| < 1$.

Suppose now that f is quadratic and has a nonreal root $x + iy$. In this case

$$L = 2(1+x)/(1+2x+x^2+y^2).$$

Therefore, $L \leq 1 \Leftrightarrow x^2 + y^2 \geq 1$ and $L \geq 1 \Leftrightarrow x^2 + y^2 \leq 1$. The results for R follow from those for L by replacing $f(X)$ by $f(-X)$.

Solution II by Douglas B. Tyler, Hughes Aircraft Company, Fullerton, California. Let $f(z) = c\prod_{j=1}^{n}(z - a_j)$, c real, f with real coefficients. Now R and L are real numbers, so they are equal to their real parts. Also

$$R = \sum_{j=1}^{n} \frac{1}{1 - a_j} \quad \text{and} \quad L = \sum_{j=1}^{n} \frac{1}{1 + a_j}.$$

Consider the map $z \to 1/(1 - z)$. This map takes the disk $|z| < 1$ to the half-plane $\mathrm{Re}(z) > 1/2$ and takes the set $|z| > 1$ to the half-plane $\mathrm{Re}(z) < 1/2$. Thus if $|a_j| < 1$ for all j, then

$$R = \mathrm{Re}(R) = \sum_{j=1}^{n} \mathrm{Re}\left(\frac{1}{1 - a_j}\right) > n/2,$$

and if $|a_j| > 1$ for all j, then

$$R = \mathrm{Re}(R) = \sum_{j=1}^{n} \mathrm{Re}\left(\frac{1}{1 - a_j}\right) < n/2.$$

The statements about L follow in the same way from considerations of the images of $|z| < 1$ and $|z| > 1$ under the map $z \to 1/(1 + z)$.

Also solved by J. Biasotti, A. Bondesen (Denmark), O. P. Lossers (The Netherlands), D. Moews, J.-M. Monier (France), University of South Alabama Problem Group, R. Stong (student), D. Wells, and the proposer.

Area of a Chain of Circles

E 3098 [1985, 428–429]. *Proposed by Roger Cuculiere, Paris, France.*

Given two circles with diameters $IA = a$ and $IB = b$, and a set of smaller circles between them as in the following figures, find the total area enclosed by the small shaded circles in each of the following cases:

(a) The center of one of the small circles lies on the (common) diameter of the large circles (Fig. 1).

(b) Two of the small circles are tangent to the diameter of the large circles (Fig. 2).

(c) The case with no restrictions (Fig. 3).

Solution by William J. Gilbert, University of Waterloo, Canada. Let a, b and c_n be the diameters of the two large circles and the circle with center C_n. Let $\alpha = 2/a$, $\beta = 2/b$, and $\gamma_n = 2/c_n$ be their curvatures. Soddy's Theorem [H. S. M. Coxeter, *Introduction to Geometry*, pp. 13–15] states that if four circles E_1, E_2, E_3, and E_4, with curvatures ε_1, ε_2, ε_3, and ε_4, are tangent to each other then

$$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2 = 2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2),$$

where $t = \sqrt{ab}/(b-a)$ and $s = t\sqrt{(b-a-c_0)/c_0}$. Using contour integration [M. R. Spiegel, Schaum's *Outline of Complex Variables*, Ch. 7, Solved Problem 25], this doubly infinite sum $\Sigma f(n)$ can be computed as minus the sum of the residues of $\pi\cot\pi z\, f(z)$ at the poles of $f(z)$. Hence, computing the residues at $-s \pm it$ and simplifying, we obtain the area in the general case (c) to be

$$A = \frac{\pi^3 ab}{8}\left[\frac{\sinh y\cosh y}{y(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y)} + \frac{\cos^2 x\sinh^2 y - \sin^2 x\cosh^2 y}{(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y)^2}\right]$$

where $y = \pi t = \pi\sqrt{ab}/(b-a)$ and $x = \pi s = y\sqrt{(b-a-c_0)/c_0}$.

In the case (a), $c_0 = b - a$ so that $x = 0$ and the area is

$$A = \frac{\pi^3 ab}{8}\left[\frac{\coth y}{y} + \frac{1}{\sinh^2 y}\right] = \frac{\pi^3 ab}{8}\left[\frac{\coth y}{y} + \coth^2 y - 1\right].$$

In the case (b), apply Soddy's Theorem to the circles with diameters a, b, c_0 and c_{-1}, where $c_0 = c_{-1}$, to obtain $c_0 = 4ab(b-a)/(a+b)^2$. Hence, $x = \pi/2$ and the area is

$$A = \frac{\pi^3 ab}{8}\left[\frac{\tanh y}{y} - \frac{1}{\cosh^2 y}\right] = \frac{\pi^3 ab}{8}\left[\frac{\tanh y}{y} + \tanh^2 y - 1\right].$$

Also solved by T. Allen, J. R. Gosselin (Canada), H. Guggenheimer, H. Högfors, R. A. Johns, L. Kuipers (Switzerland), I. E. Leonard, O. P. Lossers (The Netherlands), J. Moisan and M. Pagès (France), W. A. Newcomb, M. Pachter (South Africa), E. Salamin, P. Zwier, and the proposer.

Conjugate Algebraic Numbers

E 3100 [1985, 507]. *Proposed by C. J. Smyth, University of Edinburgh, Scotland.*

Given a rational number p/q, show that there are conjugate algebraic numbers, α, α', with $|\alpha| \neq 0, 1$ and $|\alpha|^{p/q} = |\alpha'|$.

Solution by the proposer. Let $\beta = 1.3247\ldots$, β_2, $\bar{\beta}_2$ be the zeros of $x^3 - x - 1$, whose Galois group is the symmetric group on three symbols. Then $\alpha = \beta_2^{q-p}\bar{\beta}_2^{2q+p}$ and $\alpha' = \beta^{q-p}\bar{\beta}_2^{2q+p}$ are conjugate. But $|\beta_2| = \beta^{-1/2}$, so

$$|\alpha| = \beta^{-3q/2}, \qquad |\alpha'| = \beta^{-3p/2} = |\alpha|^{p/q}.$$

When n is the Number of Divisors of nx

E 3101 [1985, 507]. *Proposed by Charles Vanden Eynden, Illinois State University.*

Let $d(n)$ be the number of positive divisors of $n > 0$. Show that the number of solutions x to $d(nx) = n$ is 1 if $n = 4$, $t!$ if n is the product of t distinct primes ($t \geqslant 0$), and infinite otherwise.

Solution by S. F. Barger, Youngstown State University. Let $n = p_1^{s_1}p_2^{s_2}\cdots p_t^{s_t}$, where the p_i's are distinct primes, $s_i \geq 1$. Let

$$x = p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}q_1^{v_1}\cdots q_k^{v_k}, \qquad r_i \geq 0, \qquad v_i \geq 0,$$

where the q_i's are distinct primes and $q_i \neq p_j$. Using a simple counting argument,

$$d(nx) = \prod_{i=1}^{t}(r_i + s_i + 1)\prod_{i=1}^{k}(v_i + 1).$$

Case 1. $s_i = 1, 1 \leq i \leq t$. Then

$$d(nx) = n \quad \text{iff} \quad p_1 p_2 \cdots p_t = \prod_{i=1}^{t}(r_i + 2)\prod_{i=1}^{k}(v_i + 1).$$

Since $r_i + 2 \geq 2$ and there are t such factors, $r_t + 2 = p_j$ for some j and $v_i = 0$ for $1 \leq i \leq k$. Clearly there are $t!$ ways to choose the r_i's.

Case 2. $n = 4$, i.e., $t = 1$, $p_1 = 2$, $s_1 = 2$. Then

$$d(4x) = 4 \quad \text{iff} \quad (3 + r_1)\prod_{i=1}^{k}(v_i + 1) = 4 \quad \text{iff} \quad r_1 = 1, \qquad v_i = 0, \qquad 1 \leq i \leq k.$$

Thus $d(4x) = 4$ has only one solution, $x = 2$.

Case 3. $n > 4$, $p^2|n$ for some prime p. We write $p = p_1$, and exhibit infinitely many solutions with $k = 1$ by choosing q_1 arbitrary, $q_1 \neq p_1, \ldots, p_t$.

Subcase 1. $n = 4p_2 \cdots p_t$, $t \geq 2$. We take

$$r_1 = p_2 - 3, \qquad r_2 = 0, \qquad r_i = p_i - 2, \qquad 3 \leq i \leq t, \qquad v_1 = 1.$$

Subcase 2. Either $p_1 \geq 3$ and $s_1 \geq 2$, or $p_1 = 2$ and $s_1 \geq 3$. We take

$$r_1 = p_1^{s_1 - 1} - s_1 - 1, \qquad r_i = p_i^{s_i} - s_i - 1 \quad \text{for} \quad 2 \leq i \leq t \quad \text{and} \quad v_1 = p_1 - 1.$$

Also solved by 29 other readers and the proposer.

Sets of Distances

E 3104 [1985, 507]. *Proposed by Jay Hook, Florida International University.*

Let A be a closed and bounded subset of $\mathbb{R}$ of Lebesgue measure 1.

(a) Show that A contains two points an integer distance apart.

(b) Show that (a) becomes false if "and bounded" is omitted in the definition of A.

(c)* Does (a) remain true if $\mathbb{R}$ is replaced by $\mathbb{R}^n$?

Solution by Walter Rudin, University of Wisconsin-Madison. For $A \subset \mathbb{R}^n$, let $\Delta(A)$ denote the set of all distances between distinct points of A. Let N denote the set of all positive integers.

Proof of (a). Put $f(t) = e^{2\pi i t}$. Then f maps A onto a compact set $K = f(A) \subseteq T$, where T is the unit circle. Assume, to reach a contradiction, that $N \cap \Delta(A) = \varnothing$. Then f is one-to-one on A, and this, combined with the compactness of A, shows that K is homeomorphic to A; hence $K \neq T$, and, therefore, $m(K) < 2\pi$. However, the definition of f shows that $m(f(A)) = 2\pi$ if f is one-to-one on A, because $m(A) = 1$.

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Case 2. $n = 4$, i.e., $t = 1$, $p_1 = 2$, $s_1 = 2$. Then

$$d(4x) = 4 \quad \text{iff} \quad (3 + r_1) \prod_{i=1}^{k} (v_i + 1) = 4 \quad \text{iff} \quad r_1 = 1, \qquad v_i = 0, \qquad 1 \le i \le k.$$

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Solution by Walter Rudin, University of Wisconsin-Madison. For $A \subset \mathbb{R}^n$, let $\Delta(A)$ denote the set of all distances between distinct points of A. Let N denote the set of all positive integers.

Proof of (a). Put $f(t) = e^{2\pi i t}$. Then f maps A onto a compact set $K = f(A) \subseteq T$, where T is the unit circle. Assume, to reach a contradiction, that $N \cap \Delta(A) = \varnothing$. Then f is one-to-one on A, and this, combined with the compactness of A, shows that K is homeomorphic to A; hence $K \neq T$, and, therefore, $m(K) < 2\pi$. However, the definition of f shows that $m(f(A)) = 2\pi$ if f is one-to-one on A, because $m(A) = 1$.

so that $d(x, y) \in [k + \gamma - 3\varepsilon, k + \gamma + 3\varepsilon]$, for some $k = k_{i,j} \in N$. If $i = j$, then $d(x, y) \leq 2\varepsilon$, so that $d(x, y) \in [0, \delta]$. Hence $\Delta(A) \subset V$.

Also solved by E. Badertscher (Switzerland), P. Erdös (Hungary), F. Galvin, P. Ilacqua, and O. P. Lossers (The Netherlands). Several partial solutions were also received. F. Galvin noted that part (c) relates to a result attributed to R. L. Graham on p. 124 of P. Erdös, Set-Theoretic, Measure-Theoretic, Combinatorial and Number-Theoretic Problems Concerning Point Sets in Euclidean Space, *Real Anal. Exchange*, 4 (1978–79) 113–138.

An appropriate generalization of (a) to $\mathbb{R}^n$ is the following theorem of Blichfeldt: If A is a closed and bounded subset of $\mathbb{R}^n$ of Lebesgue measure 1, then there exist $n + 1$ distinct points $x_1, x_2, \ldots, x_{n+1}$ in A such that each of the differences $x_i - x_j (1 \leq i < j \leq n + 1)$ has all coordinates integral. (Cf. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer, Berlin, 1959, p. 69.)

Closed Mappings on a Compact Space

E 3105 [1985, 590]. *Proposed by Calin P. Popescu, student, Bucharest, Romania, and the editors.*

Let X be a compact space (not necessarily T_1) and $f: X \to X$ a function with the property that $f(A) \subseteq A$ for every closed subset A of X. Show that there is a singleton set P such that $P \subseteq \overline{f(P)}$ and a closed set Q such that $\overline{f(Q)} = Q$.

Solution by Western Maryland College Problems Group. Let $\mathscr{F}$ be the family of closed non-empty subsets of X, and define the ordering $\geqslant$ on $\mathscr{F}$ by $A \geqslant B$ if and only if $A \supseteq B$. Since X is compact, every linearly ordered chain $\mathscr{C}$ of $\mathscr{F}$ has a lower bound, $\bigcap_{A \in \mathscr{C}} A$, in $\mathscr{F}$. Thus, by Zorn's Lemma, $\mathscr{F}$ has a minimal element, which we call Q. Since Q is closed, $f(Q) \subseteq Q$. By the minimality of Q, $\overline{f(Q)} = Q$.

Now let $x \in Q$ and set $P = \{x\}$. Since $\overline{f(Q)} \subseteq Q$, $f(x) \in Q$. Since Q is a minimal closed set, $\overline{\{f(x)\}} = Q$, and thus $P \subseteq \overline{f(P)}$.

Also solved by J. M. Cohen, S. Gopalsamy (India), A. A. Jagers (The Netherlands), D. Neuenschwander (Switzerland), J. Riley, N. Sivakumar (Canada), E. van Douwen, J. W. Walker, D. G. Winslow, and the proposer.

An Inequality

E 3108 [1985, 591]. *Proposed by L. Cseh and I. Merényi (students), Cluj, Romania.*

Let x, y, z be real numbers $k_1, k_2, k_3 \in (0, 1/2)$ and $k_1 + k_2 + k_3 = 1$. Prove that

$$k_1k_2k_3(x + y + z)^2 \geqslant xyk_3(1 - 2k_3) + yzk_1(1 - 2k_1) + zxk_2(1 - 2k_2).$$

When does equality hold?

Solution by C. S. Gardner, University of Texas at Austin. Set $x = k_1u$, $y = k_2v$, $z = k_3w$, and for $i = 1, 2, 3$, $m_i = 1 - 2k_i$. Then $0 < m_i < 1$, $m_1 + m_2 + m_3 = 1$, and the proposed inequality is equivalent to $[m_1(v + w) + m_2(w + u) + m_3(u + v)]^2 \geqslant 4(m_1vw + m_2wu + m_3uv)$. Without loss of generality, we assume $u \leqslant v \leqslant w$.

Then $v(u+w-v) = uw + (w-v)(v-u) \geqslant uw$ with equality if and only if $v = w$ or $v = u$. Also, for arbitrary a, b we have $(a+b)^2 \geqslant 4ab$ with equality if and only if $a = b$. Setting $a = v = (m_1 + m_2 + m_3)v$ and $b = m_1 w + m_2(u + w - v) + m_3 u$, we immediately obtain $[m_1(v+w) + m_2(w+u) + m_3(u+v)]^2 \geqslant 4[m_1 vw + m_2 v(u+w-v) + m_3 uv] \geqslant 4(m_1 vw + m_2 wu + m_3 uv)$. Equality holds if and only if $v = w$ or $v = u$, and $a = b$ which are equivalent to $u = v = w$. That is, equality holds in the proposed inequality if and only if $x/k_1 = y/k_2 = z/k_3$.

Also solved by J. M. Cohen, R. Heller, W. Janous (Austria), L. Kuipers (Switzerland), O. P. Lossers (The Netherlands), S. Marivani, and the proposers.

Fermat's Last Theorem False? What If?

E 3110 [1985, 591]. *Proposed by Dennis Spellman, Sacred Heart University, Bridgeport, CT.*

Let p be an odd prime. Show that if (x_0, y_0) is a pair of positive rational numbers on the curve $x^p + y^p = 1$, then at (x_0, y_0) both dx/ds and dy/ds are quadratic irrational numbers. Here s denotes arc length.

Solution by Z. Franco, University of California, Berkeley. Since $x^p + y^p = 1$, $dy/dx = -(x/y)^{p-1}$ and

$$\frac{dx}{ds} = \left(\frac{ds}{dx}\right)^{-1} = \left(1 + \left(\frac{x_0}{y_0}\right)^{2p-2}\right)^{-1/2}$$

at (x_0, y_0). Clearly if $x_0, y_0 \in Q$, the set of rational numbers, then $(dx/ds)^2 \in Q$. To show that $dx/ds \notin Q$, suppose $x_0/y_0 = m/n$ for relatively prime positive integers m and n and that $dx/ds = b/a$ for relatively prime integers a and b. Then, $1 + (m/n)^{2p-2} = a^2/b^2$. We easily see from $b^2 n^{2p-2} + b^2 m^{2p-2} = a^2 n^{2p-2}$ that $b^2 | n^{2p-2}$ and that $n^{2p-2} | b^2$. Thus, $b = n^{p-1}$. Since p is odd,

$$\left(n^{(p-1)/2}\right)^4 + \left(m^{(p-1)/2}\right)^4 = a^2,$$

which is well known to have no positive integral solutions (see, for example, G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, Clarendon Press, 1965, p. 191). Hence dx/ds is a quadratic irrational and similarly dy/ds is too.

Also solved by R.-Fr. Gloden (Italy), E. Killam, S. Marivani, and the proposer.

Bounds on an Elliptic Integral

E 3111 [1985, 665]. *Proposed by Themistocles M. Rassias, Athens, Greece.*

Prove that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

Then $v(u+w-v) = uw + (w-v)(v-u) \geqslant uw$ with equality if and only if $v = w$ or $v = u$. Also, for arbitrary a, b we have $(a+b)^2 \geqslant 4ab$ with equality if and only if $a = b$. Setting $a = v = (m_1 + m_2 + m_3)v$ and $b = m_1 w + m_2(u + w - v) + m_3 u$, we immediately obtain $[m_1(v+w) + m_2(w+u) + m_3(u+v)]^2 \geqslant 4[m_1 vw + m_2 v(u+w-v) + m_3 uv] \geqslant 4(m_1 vw + m_2 wu + m_3 uv)$. Equality holds if and only if $v = w$ or $v = u$, and $a = b$ which are equivalent to $u = v = w$. That is, equality holds in the proposed inequality if and only if $x/k_1 = y/k_2 = z/k_3$.

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$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

Solution by R. H. Garstang, University of Colorado at Boulder. For $0 < x < 1$ we have

$$4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$$

with the left-hand side becoming an equality for $x = 0$ and the right-hand side at $x = 0$ and $x = 1$. It follows that

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2}} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}},$$

and the result follows upon evaluating the two elementary integrals on the left and right-hand sides of this expression.

The integral is an elliptic integral. It is easily evaluated by numerical integration. A 7-point Gaussian integration gave 0.547962 for the value of the integral. The integral may also be expressed in terms of the incomplete elliptic integral of the first kind, $F(\phi \backslash \alpha)$, where ϕ is the amplitude and α is the modular angle. The necessary formulae are neatly tabulated by M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Washington, D.C., 1964, equations 17.4.70 and 17.4.74, and numerical values can be obtained from Table 17.5 (by interpolation). We find that

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} = 0.598117 \{F(68.883 \backslash 76.083) - F(37.091 \backslash 76.083)\},$$

where ϕ and α are in degrees (since Table 17.5 is given in that form).

Our value of 0.547962 may be compared with the bounds of 0.5236 and 0.5554 given by the problem.

O. P. Lossers (The Netherlands) noted that $1/\sqrt{4 - x^2 - x^3}$ can be expressed in series form as follows:

$$\begin{aligned}\frac{1}{2}\{1 - x^2(1 + x)/4\}^{-1/2} &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{(-1)^n}{2^{2n}}x^{2n}(1 + x)^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2^{2n}n!}x^{2n}(1 + x)^n,\end{aligned}$$

where $(1/2)_n$ denotes Pochhammer's symbol, i.e., the rising factorial.

Consequently,

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2^{2n}n!}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k + 2n + 1)}.$$

Lossers noted that the partial sum $\Sigma_{n=0}^{12}$ gives the approximation 0.547962.

Also solved by 115 other readers and the proposer.

What are the necessary and sufficient conditions on a semigroup S so that S contains a minimal spanning set? For this, I do not have a solution, but I am working on it.

Also solved by 30 other readers.

ADVANCED PROBLEMS

6548. *Proposed by Lee A. Rubel, University of Illinois at Urbana-Champaign.*

Are there uncountably many fields F_α, where α runs over some uncountable index set A, such that no two of the fields F_α are isomorphic but all of the additive groups $(F_\alpha, +)$ are isomorphic and all of the multiplicative groups $(F_\alpha, \cdot)$ are isomorphic? Cf. 6489 [1985, 148; 1986, 744].

6549. *Proposed by L. Matthew Christophe, Jr., Wilmington, Delaware.*

Sum the series

$$\sum_{k=1}^{\infty}\left\{(-1)^{k+1}\left(\frac{(2k)!}{2^{2k}(k!)^2k(2k+1)}\right)\left(\ln 2-\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\right)\right\}.$$

6550. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be integers with $0 \leq k \leq n$. Determine the minimum value of

$$\int_0^1\{P_{n,k}(x)\}^2dx,$$

where $P_{n,k}$ runs through the set of polynomials with real coefficients and degree at most n such that the coefficient of x^k is 1.

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SOLUTIONS OF ADVANCED PROBLEMS

Peculiar Behavior on Cantor Sets

6505. *Proposed by Harry Gonshor, Rutgers University.*

Prove that for any function f: $\mathbb{R} \to \mathbb{R}$ there is a function g which differs from f on a set of measure 0 and has the Darboux (i.e., intermediate value) property but is not continuous.

Composite solution: Construct an infinite sequence $C_1, C_2, C_3, \ldots$ of disjoint Cantor sets in $[0,1]$ such that (i) each is homothetic to the classical "middle thirds removed" Cantor set and (ii) $S = \bigcup_{i=1}^{\infty} C_i$ is dense in $[0,1]$. Construct functions h_i that map C_i onto the set of all real numbers. Define

$$g(x) = f(x) \quad \text{for } x \notin S,$$
$$g(x) = h_i(x) \quad \text{for } x \in C_i.$$

Then $g(x)$ clearly has the Darboux property, since between any two distinct values of x it takes on every real value infinitely often. Since S has measure zero, the problem is solved. (Cf. R. P. Boas, Jr., *A Primer of Real Functions*, §13.)

Most of the solutions were equivalent to the above, and many carried out the construction in considerable detail. By considering different constructions, the proposer easily showed that every function is the sum of two Darboux noncontinuous functions. G. J. Foschini (AT & T Bell Labs., Holmdel, New Jersey) deduced the result from his theorem that every $f: R \to R$ is equal almost everywhere to a function g whose graph is a dense connected subset of R^2. See Theorem 3.3 of G. J. Foschini, The Pathology of Functions Which Are A. E. Constant, *Bulletin Mathématique* (Romania), 11 (59), 3, 1967, 35–54. The San Bernardino Problem Solving Group has located constructions of g that preserve various properties of f (such as Baire classification and measurability) in papers of A. B. Gurevič (1966), S. Marcus (1967), and A. M. Bruckner, J. G. Ceder, and R. Keston (1968). They also remark that in a 1974 doctoral dissertation (Univ. of California at Santa Barbara) S. Agronsky showed how to construct a Darboux Baire 1 function that agrees with a given continuous function on a dense set, and differs on a set of measure 0.

Finally, George Piranian points out that in 1975 a high school student discovered and later published the construction of a "pathological" function similar to the functions g arising here. See David Anderson, An Outrageously Discontinuous Function, *Journal of Undergraduate Mathematics*, 8 (1976) 11–12.

Also solved by Daniel Cass, A. K. Desai (India), Eric van Douwer, Gerd H. Fricke, J. P. Holmes, Robert B. Israel (Canada), O. P. Lossers (The Netherlands), Andy Martin, J. G. Mauldon, Mark D. Meyerson, Daniel Neuenschwander (Switzerland), Victor Pambuccian (Romania), M. Petrakakis and T. Vidalis (Greece; jointly), Imrich Pokorný (Czechoslovakia), George Piranian, D. Ramachandran, and the proposer.

MISCELLANEA

Hear No Evil—Speak No Evil—Zeno Evil

WILLIAM DUNHAM
Department of Mathematics, Hanover College, Hanover, IN 47243

I'd always thought space could be split willy-nilly
Til Zeno's "Achilles" made me feel pretty silly.
"Alas," I concluded, "I must have been wrong."
"All things are just *atoms* lumped into a throng."
Then I read Zeno's "Arrow" and instantly fainted—
The atomic idea seemed equally tainted!

I pondered the problem, unable to face
A motionless arrow just hanging in space,
Nor Achilles, as swift as a hurricane's gust,
Condemned to a tortoise's slow-settling dust.
This MUST be illegal by logic's own laws,
Yet try as I might, I could not find the flaws.

So baffled was I that my brain split a seam
And my sleep was disturbed by this nightmarish dream:
Achilles, I dreamt, interrupted his chase
To snare that darn arrow (STILL hanging in space)
Then used that projectile the tortoise to slay—
And promptly got sued by the SPCA!

O, my sanity's vanished. I'm filled with despair.
I've lost my composure and most of my hair.
For I failed to take heed of that old caveat:
"Beware Greeks bearing paradoxes" (or something like that).
And my downfall I owe to the cunning of Zeno,
Who showed that the truth lies NOWHERE in between-*o*.

REVIEWS

EDITED BY JOSEPH KONHAUSER

A First Course in the Mathematical Foundations of Thermodynamics. By David R. Owen. Springer-Verlag, 1984, xvii + 130.
Rational Thermodynamics. By C. Truesdell. Springer-Verlag, 1984, xvii + 555.

RUTHERFORD ARIS
Department of Chemical Engineering and Materials Science, University of Minnesota, Minneapolis MN 55455

The Thermodynamic Scene has certain elements in common with the Public Aspect of Christendom, or indeed of other major religions. The Scientist is by nature a Thermodynamic Animal and some sort of observance of thermodynamics is essential to the health of science and technology even as some degree of religious adherence is to the human condition. If one is not prepared to accept the framework of a particular Tradition one would have to invent one's own Great Thermodynamician. Then there often arise prophets—here and there a Saul also among them—who, overwhelmed by the importance of their message and born along by their Enthusiasm or Caloric, willy-nilly find themselves Sectaries. There are the Falwells of No-nonsense-just-give-me-the-formulas Majority endorsing the Moons of the Unification Church of Irreversible Thermodynamics. There are the Peales and Shulers of Positive and Possibility Thinking for whom Perpetual Motion is just round the corner. Eschatology abounds, for Thermodynamics has its own Cosmological Argument, while, for the Confirmed Secularizer, there is the Vineyard of Sociological Application. Following in the footsteps of Dr. Bowdler, Thermodynamic Textbooks have been rewritten in S. I. Units with the same zeal that Service Books have been translated into Sexually Indifferent language. There is an Eastern Orthodoxy which, though not quite up to a Great Schism, can at least claim a Principle of Inaccessibility. The Kalendar of its Saints must surely start with St. Willard of Yale whose fund of merit seems to be inexhaustible, but it extends backwards and forwards in time to St. Sadi, Patron of all Cycles, St. Ambrose and St. Bernard, Doctors of the Church, and St. James who has the keys to it all. But I, a very Gallio in these matters, must not pursue this Analogy too far lest I be trapped in the Morass of my own Metaphor.

The principal volume under review is a second edition of a 1969 volume of the same title, itself an outgrowth of that slim, but powerful, "Six Lectures on Modern Natural Philosophy." Something of the original structure remains in the design of this volume though it is much amplified and augmented by representative work of the school of rational thermodynamics. In a historical introduction Truesdell traces the development of rational thermodynamics from its roots in the late eighteenth-century concern with the conduction of heat and with thermometry and calorimetry. The first develops into the Fourier line of workless dissipation and explicit time

merely competent but distinguished. Truesdell is well known for the pungency [sometimes even pugnacity] of his prose and examples enough will be found here, but it is for the carefully balanced sentence and nice choice of word which, paragraph after paragraph, give the reader a rare pleasure for which we should thank him most.

David Owen's first course in the mathematical foundations of the subject is a useful introduction and a valuable tool for anyone who would teach it in this way. The heart of the book (Chapters III and IV) is a modern treatment of the first and second laws. By these are meant the assertions that, in any thermodynamic system, a cycle that absorbs no heat does no work and if the accumulation function is not negative for any hotness level then it is zero. Naturally these terms and concepts are carefully defined, explained, and related to the physical world. Two introductory chapters on classical thermodynamics and systems with perfect accessibility precede the two laws, and they are followed by discussions of actions with Clausius, Conservation and Dissipation Properties, isothermal processes of homogeneous filaments, and homogeneous bodies with viscosity. Altogether it is a satisfying brief introduction which is intended to be accessible to the undergraduate in science and engineering and will serve to orient any reader to the important contemporary developments.

An Introduction to Discrete Mathematics. By Steven Roman. CBS Publishing Co., 1983, vii + 455.

Discrete Mathematics. By Richard Johnsonbaugh. Macmillan Publishing Co., 1986, v + 480.

CHARLES F. KELEMEN,
Department of Computer Science and Mathematics, Swarthmore College, Swarthmore, PA 19081

Both these books are worthy of consideration by someone considering teaching a discrete mathematics course at the freshman or sophomore level. Before I say more I would like to state some of my predjudices about mathematics courses in general and discrete mathematics courses in particular so that the reader can properly weigh my later comments.

All mathematics courses should do some real mathematics, i.e., state and prove theorems. Intuition should be stressed and encouraged throughout; but the final test of whether intuition is correct is a formal statement of a theorem and a proof. Not everything has to be treated formally or proved, but there should be many careful definitions, theorems, and proofs. Students should finish a mathematics course with considerably more mathematical maturity than when they entered. This should include improved intuition, and comfort with notation and manipulation of the mathematical objects studied. The level of a course should not be flat but monotonically increasing almost everywhere (an occasional respite is not unreasonable).

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Every course should have at least some topics that the instructor obviously enjoys and feels are important. These feelings should be conveyed to the students. Some example topics for me are: the fundamental theorem of calculus, application of calculus to physics, group theory (as a beautiful abstract entity), applications of group theory to coding, Galois Theory, completeness of the first-order predicate calculus, Godel's Incompleteness results, undecidability, almost any open question, etc.

For me discrete mathematics courses should begin at a level appropriate for students who have had a good four years of high school mathematics and have some mathematical interest and ability. Students who finish the course should have the mathematical maturity with respect to discrete mathematics that was associated with the completion of linear algebra 20 years ago. (I associate less maturity with many of the "applied" linear algebra courses given today.) That is, students should be comfortable with mathematical abstractions, logical reasoning, appropriate notation, definitions, theorems, and proofs. Recursive definitions and proof by mathematical induction are particularly important. Some students will not make it through this course.

There are numerous topics that could be covered in a discrete mathematics course. Topics I prefer are: propositional logic and first-order predicate calculus, mathematical induction, introductory set theory, relations and functions, introductory graph theory with an emphasis on trees, counting and finite probability, recursion and recurrence equations, application is to proving algorithms and analyzing algorithms. I like to push my students a little (some say a lot).

Frequently students do not like the books I choose. Several years ago I taught discrete mathematics using Stanat and McAllister [5] and some handouts. Students in the course had all had a year of college calculus. I covered the first five chapters and some additional material on finite probability and logic. At the end of the course, most students thought the course was hard. Most students did not like the book. Two years later many of those students told me that the course had been an extremely valuable experience. I would probably use [5] again. Now that you know some of my quirks and prejudices, let's talk about these books.

Despite the lack of the words 'computer science' in the title, both Johnsonbaugh and Roman are strongly oriented toward CS. Johnsonbaugh recommends a prerequisite of one programming course and Roman lists as the first of his two main goals to "acquaint the students with a variety of mathematical concepts that will be needed in the study of computer science and in the further study of mathematics." Both books are well written, cover a reasonable selection of topics, have copious examples and ample exercises. Roman has more examples and is written at a lower level than Johnsonbaugh. [5] requires considerably more sophistication than either of these books. So instructors who consider [5] too hard for their students should give these books a good look. My guess is that most students would like Roman better but that some bright students would prefer Johnsonbaugh. For most students, Roman would be quite good for self-study. Roman would also be quite nice for a senior level high school course.

binary search trees. Backtracking is also discussed. In most cases, the material in Roman is descriptive and not too quantitative. The material in Johnsonbaugh is more formal, more quantitative, with more details. He does a nice job with decision trees and the proof of a lower bound for comparison sorting. Johnsonbaugh also does game trees, matching, and network flow.

Roman has a short section on finite state machines. Johnsonbaugh has a chapter on automata, grammars, and languages. My opinion is that the material suggested above on formal logic and finite probability is more important in a discrete mathematics course than automata theory. So I would not be able to get to this material. For those who disagree, it's nice to have this material.

Both books have answers to plenty of exercises. Johnsonbaugh provides excellent notes at the end of each chapter suggesting further references.

To summarize, both Johnsonbaugh and Roman are reasonable texts for a discrete mathematics course. They are both well written and cover a nice range of topics. I would supplement either with the material on formal logic and finite probability mentioned above. Roman is written at a level that should be appropriate for any college student. Thus, it might be particularly appropriate for courses that are designed to satisfy distribution requirements. Johnsonbaugh is written at a level more appropriate for students with a serious interest in mathematics or computer science. Both cover essentially the same topics but there is more material in Johnsonbaugh than in Roman.

REFERENCES

1. J. N. Crossley et al, What is Mathematical Logic, Oxford University Press, 1972.
2. David Gries, The Science of Programming, Springer, 1981.
3. Richard Johnsonbaugh, Discrete Mathematics, Macmillan, 1984.
4. Steven Roman, An Introduction to Discrete Mathematics, Saunders, 1986.
5. D. F. Stanat and D. F. McAllister, Discrete Mathematics in Computer Science, Prentice-Hall, 1977.
6. Mitchell Wand, Induction, Recursion, and Programming, North Holland, 1980.

Groups: A Path to Geometry. By R. P. Burn. Cambridge University Press, 1985, xi + 242.

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Like the book under review, I will get down to business right away.

Introduction. This textbook implicitly raises some very interesting and important pedagogical problems, including the following:

1. Do groups provide a path to the geometry that we should be teaching?
2. How appropriate is the "Moore method" (or any variant of it) for the teaching of undergraduate mathematics?

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1. J. N. Crossley et al, What is Mathematical Logic, Oxford University Press, 1972.
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1. Do groups provide a path to the geometry that we should be teaching?
2. How appropriate is the "Moore method" (or any variant of it) for the teaching of undergraduate mathematics?

3. What is the place of the history of mathematics in the undergraduate mathematics curriculum?

It is impossible to discuss any of these issues adequately in a brief review, but that does not make it any less important to raise them. I hope that the following questions will at least be sufficiently provocative to generate lively debate. (The format of this essay is that of the book, except that I do not provide any answers.)

I. *Groups and Geometry*. "A geometry, Klein said, is defined by a group of transformations, and investigates everything that is invariant under the transformations of this given group" (Hermann Weyl, *Symmetry*). This was the substance of the famous Erlanger Programm, which Klein announced in 1872. Over the past century, his point of view has become so well established that even in our high schools transformation geometry is competing with Euclid for control of the Euclidean plane. We all know, or think we know, what is gained when the study of a geometry is replaced by the study of its group of automorphisms. *But what is lost*? This question can be broken down into several parts.

1. Is there more to the regular solids than the permutation groups associated with their vertices? Than the identification of these groups with finite subgroups of 0(3)?
2. What is the role of the classical theorems of projective geometry under the transformation group approach? Should we continue to teach them?
3. Why do geometries have nontrivial groups of automorphisms? That is, what is the relation between local geometric properties and global invariants? (Think of tiling a floor!)
4. Can we understand the role of symmetry in art, nature and science without asking these questions?
5. Discuss the following statement: "Geometry is what geometers do." (What do they do?)
6. Does group theory provide the most suitable context for understanding contemporary geometric problems?
7. Shouldn't the title of this book really be *Geometry*: *a Path to Groups*? If not, why is conjugacy first introduced in chapter 17?
8. What should be the content of an introductory undergraduate geometry course?

Concurrent reading:

H. S. M. Coxeter, *Introduction to Geometry*, Wiley, N.Y., 1961.
B. Grünbaum, Regular polyhedra, old and new, Aequationes Mathematicae, 16 (1977) 1–20.
B. Grünbaum and G. S. Shephard, Tiling with congruent tiles, Bulletin of the A.M.S., 3 (1980) 951–973.
M. Senechal, Symmetry revisited, preprint.
T. Thompson, *From Error-correcting Codes through Sphere Packings to Simple Groups*, Mathematical Association of America, Carus Monograph 21, 1983.
H. Weyl, *Symmetry*, Princeton University Press, Princeton, 1952.

II. *The Moore Method*. Legend has it that the late R. L. Moore taught his graduate

topology classes without saying a word. Instead, the students were given definitions and the statements of theorems, which they had to prove without help from the instructor, the literature, or each other. An unusually large percentage of Moore's students became productive research mathematicians.

1. What are the difficulties that teachers have had in adapting the Moore method to the undergraduate classroom? what techniques have proved successful in overcoming some of them? Which problems are *inherent* in the method?
2. Discuss the following statement (anon.): "The mathematics that can be taught by the Moore method is dead mathematics."
3. Which of the mathematics courses that you took as a student was the most stimulating, and why?
4. Discuss the following statement (Burn, preface): "Mathematics is something we do rather than something we learn, and, all too often, lectures give the opposite impression." (Warning: brief answers are not acceptable.)
5. Is there such a thing as an *apprentice method* of classroom teaching? What would be its strengths and weaknesses?

Concurrent reading:

D. Cohen, A modified Moore method for teaching undergraduate mathematics, American Mathematical Monthly, 89 (1982) 473–474, 487–490.
R. P. Burn, Geometry: A Path to Groups, Cambridge University Press, 1985.
G. Polya, Mathematical Discovery: on understanding, learning and teaching problem solving, Wiley, N.Y., 1962.
P. Davis and R. Hersh, The Mathematical Experience, Birkhauser Boston, 1980.

III. *History in the mathematics curriculum*. It has become fairly standard to include brief historical notes (if only some combination of names, dates, photographs, and anecdotes) in undergraduate mathematics texts. Do such notes convey an understanding of the contexts in which theorems were discovered and proved? (What should historical notes convey?) Should mathematics majors be made aware of the role of mathematics in history and in our culture? If so, how? The following subquestions may or may not be relevant.
1. Why is the principal theorem about the "wallpaper groups" known as the "crystallographic restriction" (Burn, page 213)?
2. How and why did anyone ever come up with the ideas of projective geometry?
3. Why did Klein propose his Erlanger Programm? What did it achieve? Why are some people beginning to question its continued usefulness?
4. What motivates professional mathematicians to work on particular problems? How can we help students develop a context for their work?
5. Discuss the following statement (Burn, p. 34): "Strangely enough, the analysis of isometries of three-dimensional space into translations, rotations, screws and products of these three types with a reflection, by L. Euler (1776), long precedes the corresponding analysis for two dimensions which appears in the work of F. M. Chasles (1831)." (Is it really strange? Why didn't Burn give complete citations?)

6. Why do students often find abstract algebra difficult?
7. Who was Rip van Winkle?

Concurrent reading:

W. Irving, "Rip van Winkle", in The Sketch Book, 1819. Currently available in paperback by Signet classics, NAL.
F. Klein, Vergleichende Betrachtugen über neuere geometrische Forschungen, Werke, I, 460–498.
M. Senechal, Introduction to Mathematical Crystallography, IHES/P/85/47, Bures-sur-Yvette, 1985.

Summary. Groups: A Path to Geometry consists of 827 well-planned and challenging problems, through which transformation geometry is developed and the abstract concepts of elementary group theory are made concrete. The only narrative is the brief introduction to each chapter. Concurrent readings are suggested, and answers to the problems are given at the end of each chapter. Short historical notes provide names, dates, and occasionally some idea of the development of the subject. As the author explains in the preface, the book is intended to be the basis for a seminar, rather than a lecture course.

As far as I can judge without testing the book in class, and modulo the order of the topics, it seems to be excellent. But:
1. Is transformation geometry the best way to introduce our students to the geometric problems of today?
2. Are there middle grounds between the seminar and the lecture methods of teaching which retain the best features of each?
3. Wouldn't a serious discussion of the development of the subject matter be a valuable supplement to any course?

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatabilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, S, L*.** *International Mathematical Olympiads, 1978-1985 and Forty Supplementary Problems.* Murray S. Klamkin. New Math. Lib., V. 31. MAA, 1986, xii + 141 pp, (P). [ISBN: 0-88386-631-X] A sequel to NML Volume 27, giving 42 problems from seven recent Olympiads 1978-85 supplemented by 40 other problems that were submitted to but not chosen for the Olympiads. Solutions were rewritten and expanded by Klamkin, a former coach of the U.S. team. Excellent references; glossary of uncommon terms; scoring records of all IMO's. LAS

General, S(16-17), P. *Einführung in die klassische Mathmatik I.* Helmut Koch. Springer-Verlag, 1986, 326 pp, DM68. [ISBN: 0-387-16665-3] First part of an unusual and interesting two-volume work on the mathematics of the nineteenth and early twentieth centuries. Thirty largely unrelated chapters, presenting many different results in modern language but in the spirit of their origins. JD-B

General, P, L?. *Discrete Thoughts: Essays on Mathematics, Science, and Philosophy.* Mark Kac, Gian-Carlo Rota, Jacob T. Schwartz. Birkhauser Boston, 1986, xii + 264 pp, $65. [ISBN: 0-8176-3285-9] A *very* miscellaneous collection of old writings (all reprints) on the nature of mathematics and its relation with philosophy, science, and computing. Some are essays; others book reviews. Extraordinary price of $.25 per page for straight text. LAS

General, S*, L. *Symmetry Through the Eyes of a Chemist.* István and Magdolna Hargittai. VCH (Suite 909, 220 E. 23 St., NY 10010), 1986, xii + 458 pp, $95. [ISBN: 0-89573-520-2] This beautiful book, along the lines of Weyl's classic *Symmetry,* looks at symmetry as a unifying theme in the nature of things (especially in chemistry). Richly illustrated with photographs, drawings, and figures. Accessible descriptions accompany the examples of symmetry in molecular geometry, molecular vibrations, electronic structure, groups, and crystals. LCL

General, S(13-15), L. *Measures in Science and Engineering: Their Expression, Relation and Interpretation.* B.S. Massey. Math. & Its Applic. Halsted Pr, 1986, 216 pp, $41.95. [ISBN: 0-470-20331-5] An examination of the various units used to measure physical quantities, with related discussions concerning precision of results, dimensional analysis, and tables of conversion factors and dimensionless parameters. LCL

Precalculus, T(13: 1). *Precalculus: Functions and Graphs, Fifth Edition.* Earl W. Swokowski. Prindle, Weber & Schmidt, 1987, ix + 581 pp. [ISBN: 0-87150-060-4] This *Fifth Edition* puts more emphasis on graphs and geometric interpretations and on the use of calculators instead of tables. The order is now more flexible, with a larger variety of applied problems in more disciplines. The section on trigonometry is expanded for more clarity. GF

Precalculus, T(13). *College Algebra.* Jerome E. Kaufmann. Prindle, Weber & Schmidt, 1987, xiv + 526 pp. [ISBN: 0-87150-109-0] Covers standard material required for calculus, finite math, and business courses. Major themes are solving equations and inequalities; solving problems (emphasized throughout); developing graphing techniques; and the concept of a function. Familiarizes the student with graphing before encountering the definition of a func-

tion. Plenty of examples and problems. Chapter summaries appear especially useful. Very student-oriented text. MR

Precalculus, T(13: 1, 2). *College Algebra and Trigonometry.* Jerome E. Kaufmann. Prindle, Weber & Schmidt, 1987, xiv + 688 pp. [ISBN: 0-87150-014-0] Text for college students written to serve as a prerequisite for the standard calculus sequence. Covers basic algebra, solution of linear and quadratic equations, inequalities and absolute values, graphing, functions, exponentials and logarithms, trigonometry of right triangles and the unit circle, systems of equations, matrices, and conic sections. There is an extensive set of problems with answers to odd exercises. Also includes sets of special problems which may be used as supplementary material. AM

Precalculus, T(13). *College Algebra, Second Edition.* Max A. Sobel, Norbert Lerner. Prentice-Hall, 1987, xvi + 621 pp. [ISBN: 0-13-141839-4] Text with exercises and examples emphasizing skills needed in calculus (inequalities, simplifying expressions common in differentiating, etc.). Also includes chapters on matrices and determinants, sequences and series, and elementary combinatorics. Lots of problems and chapter tests. Cautionary notes in main body and margin of text give *incorrect* ways to perform certain calculations. (*First Edition,* TR, June-July 1983.) LC

Finite Mathematics, T(13-14: 1). *Applied Finite Mathematics, Second Edition.* S.T. Tan. Prindle, Weber & Schmidt, 1987, xiv + 668 pp. [ISBN: 0-87150-074-4] Quantitative techniques for students in managerial, social, and life sciences. Covers systems of equations, linear programming, mathematics of finance, probability and some statistics, Markov chains, games, and logic. Filled with worked-out examples, nice exercises, a wealth of applications. (*First Edition,* TR, June-July 1983.) RM

Finite Mathematics, T(13: 2). *Mathematics with Applications in the Management, Natural, and Social Sciences, Fourth Edition.* Margaret L. Lial, Charles D. Miller. Scott Foresman, 1987, 740 pp, $23.16. [ISBN: 0-673-18464-1] Aimed at undergraduate non-science students who want to see the usual collection of finite mathematics topics and a short calculus course collected in one book. (*First Edition,* TR, November 1974; Extended Review, June-July, 1975; *Second Edition,* TR, April 1979; *Third Edition,* TR, August-September 1983.) AWR

Education, L. *Teaching Thinking Skills: Theory and Practice.* Ed: Joan Boykoff Baron, Robert J. Sternberg. WH Freeman, 1986, xi + 275 pp, $13.95 (P). [ISBN: 0-7167-1791-3] A collection of 12 essays on the theory of teaching and evaluation of thinking skills. Although these essays do not focus on mathematics alone, they are of value to any teacher of college mathematics. SG

History, S, P, L. *Native American Mathematics.* Ed: Michael P. Closs. U of Texas Pr, 1986, 431 pp, $35. [ISBN: 0-292-75537-1] An innovative multidisciplinary exploration of the number systems (primarily), notation, calendars, and geometry of New World natives: Innit, Ojibway, Incas, Mayans, Aztecs, and Amazon Indians. A blend of archeology, linguistics, anthropology, and mathematics. LAS

History, S(14-16), P, L.** *Episodes in the Mathematics of Medieval Islam.* J.L. Berggren. Springer-Verlag, 1986, xiv + 197 pp, $23. [ISBN: 0-387-96318-9] A sample of Islamic source documents dealing primarily with high school mathematics—algebra, geometry, trigonometry—tied together with rich commentary on the historical and Islamic context of these ideas. Contains algorithms for finding fifth roots of 15 digit numbers; Islamic inheritance word problems involving fractional distributions; proofs for trigonometric identities; and enough spherical trigonometry to find the direction of Mecca. A valuable resource for historians and teachers of school mathematics. LAS

History, P, L*. *Hermann Weyl 1885-1985.* Ed: K. Chandrasekharan. Springer-Verlag, 1986, 119 pp, $30. [ISBN: 0-387-16843-5] Three centenary lectures on Weyl's contributions to physics (by C.N. Yang), to geometry (by Roger Penrose), and to Lie groups (by Armand Borel), together with Weyl family photographs and other details of a celebration at the Swiss Federal Institute of Technology (ETH) on the anniversary of Weyl's birth. LAS

History, P. *Le Calcul Simplifié: Graphical and Mechanical Methods for Simplifying Calculation.* Maurice d'Ocagne. Transl: J. Howlett, M.R. Williams. MIT Pr, 1986, x + 167 pp, $35. [ISBN: 0-262-15032-8] First English translation of an important although obscurely written treatise on nomography—graphical versions of slide rules constructed for special calculations. Originally written in the 1890's, this volume is a translation of the 1928 third edition. Eleventh volume in the Charles Babbage Institute Reprint Series for the History of Computing. LAS

Logic, P. *Predicative Arithmetic.* Edward Nelson. Math. Notes, V. 32. Princeton U Pr, 1986, viii + 189 pp, $21 (P). [ISBN: 0-691-08455-6] From page 80: "The infant counts on its fingers, the mathematician counts on ω—but the infant at least knows its fingers to exist." Predicative arithmetic removes the induction principle and looks at how much theory of numbers remains. (An "impassable barrier" is reached midway through!) BC

Logic, T(18), S, P. *A Manual of Intensional Logic.* Johan van Benthem. CSLI Lect. Notes, No. 1. CSLI (Ventura Hall, Stanford U., Stanford, CA 94305),

1985, 74 pp, \$8.95 (P). A survey of intensional logic as a general research program, whose core idea is that intensional notions may be modelled by families of ordinary extensional models from standard logic, spread out over suitable points in time, possible worlds, etc. ("multiple reference" strategy). LCL

Foundations, P, L. *Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science.* Michael Friedman. Princeton U Pr, 1983, xvi + 385 pp, \$14.50 (P). [ISBN: 0-691-02039-6] A philosophical treatise (with an appendix on differential geometry) aimed at resolving incompatible conceptions of the role of geometry in physics: the physicalization of geometry in which its features become as empirical as, say, an election, or the idealization of geometry in which it is merely a pattern imposed on nature by human intelligence. (The author is a professor of philosophy at the University of Illinois at Chicago—not the homonymous Fields medalist mathematician who works in the same area.) LAS

Foundations, P. *Filters and Ultrafilters Over Definable Subsets of Admissible Ordinals.* J.C.M. Baeten. CWI Tract, No. 24. Math Centrum, 1986, 77 pp, Dfl. 12.50 (P). [ISBN: 90-6196-301-X] A study of recursive analogues of measurable cardinals. LCL

Foundations, S(15-17), P, L. *New Directions in the Philosophy of Mathematics: An Anthology.* Ed: Thomas Tymoczko. Birkhauser Boston, 1986, xvii + 323 pp, \$57.50. [ISBN: 0-8176-3163-1] A compelling collection of reprints, each with a careful introduction, on recent challenges to the traditional philosophies of mathematics and on current practice that may illuminate a new direction. Authors Reuben Hersh, Imre Lakatos, Philip Kitcher, Philip Davis, *et al.*, emphasize the empirical, exploratory, fallible roots of mathematical practice. An excellent resource for a stimulating undergraduate seminar. LAS

Combinatorics, T(17: 1), P. *Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability.* Béla Bollobás. Cambridge U Pr, 1986, xii + 177 pp, \$39.50; \$13.95 (P). [ISBN: 0-521-33059-9; 0-521-33703-8] Thorough grounding in set systems and hypergraphs. Chapter headings include representing sets, Sperner systems, random sets, Helly families, and partitioning sets of vectors. Over a hundred exercises. References included. LC

Number Theory, T*(17-18: 1, 2), S, P*, L*. *Elliptic Curves.* Dale Husemöller. Grad. Texts in Math., V. 111. Springer-Verlag, 1986, xv + 350 pp, \$48. [ISBN: 0-387-96371-5] Another fine introduction to the hottest topic in number theory. (If you haven't heard, Fermat's Last Theorem is now known to follow from the Weil-Taniyama conjecture on elliptic curves.) Combines algebraic and analytic approaches. The first few chapters are accessible to undergraduates. Lots of exercises—includes solutions. BC

Number Theory, P. *Lecture Notes in Mathematics-1231: Drinfeld Modular Curves.* Ernst-Ulrich Gekeler. Springer-Verlag, 1986, xiv + 107 pp, \$12.80 (P). [ISBN: 0-387-17201-7] An introduction to the theory of modular forms in characteristic p. The author includes a summary of needed background material based on the work of Drinfeld, Hayes, and Gass as well as a presentation of some of his own recent results. This work underscores the deep and beautiful analogy between the field of rational numbers and the field of algebraic functions in one variable over a finite field. SG

Number Theory, P?. *1012 Problems.* A.S. Moiseenko (10-12 Kimball St., Belleville, NJ), 1986, 585 pp, (P). A collection of 1012 Diophantine equations, almost all of which are solved. CEC

Number Theory, S(18), P. *Diophantine Analysis.* Ed: J.H. Loxton, A.J. van der Poorten. London Math. Soc. Lect. Note Ser., V. 109. Cambridge U Pr, 1986, 170 pp, \$19.95 (P). [ISBN: 0-521-33923-5] A collection of nine number theory papers which were presented at the Third Australasian Mathematics Convention which was held at the University of New South Wales from May 13-17, 1985. A particularly distinguished group of number theorists give an overview of the field of Diophantine analysis and a guide to problems of current interest. CEC

Linear Algebra, T(17-18: 2), S, P. *Matrizentheorie.* Felix R. Gantmacher. Springer-Verlag, 1986, 654 pp, DM138. [ISBN: 0-387-16582-7] German translation of *Second Edition* of a comprehensive Russian text on matrices and their applications. JD-B

Linear Algebra, T*(17: 1), S, P, L. *Invariant Subspaces of Matrices with Applications.* I. Gohberg, P. Lancaster, L. Rodman. Canadian Math. Soc. Ser. of Mono. & Adv. Texts. Wiley, 1986, xv + 692 pp, \$59.95. [ISBN: 0-471-84260-5] An advanced linear algebra book in which invariant subspaces of matrices are the central notion and the main tool. Contains a comprehensive treatment of geometrical, algebraic, topological, and analytic properties of invariant subspaces. Begins with work suitable for advanced undergraduates and moves into recent achievements which have not appeared before in books. Includes exercises and a list of references. CEC

Linear Algebra, T(14: 1). *Elementary Linear Algebra, Alternate Second Edition.* Stewart Venit, Wayne Bishop. Prindle, Weber & Schmidt, 1987, xiii + 415 pp. [ISBN: 0-87150-094-9] Contains the same material as *Second Edition* but the first chapter, geometry of R^m, has become chapter 3 in this version (following chapters on linear equations and matrices,

and determinants). (*First Edition,* TR, April 1981.) JNC

Topological Groups, P. *Harmonic Analysis on the Heisenberg Nilpotent Lie Group, With Applications to Signal Theory.* W. Schempp. Res. Notes in Math. Ser., V. 147. Longman Scientific & Technical (US Distr: Wiley), 1986, 199 pp, $39.95 (P). [ISBN: 0-470-20374-9] The Heisenberg group is a non-commutative, non-compact Lie group whose Lie algebra is defined by the canonical commutation relations from quantum mechanics. The goal of this book is to study the harmonic analysis of the Heisenberg group, i.e., its unitary linear representations. Includes applications in the theory of analog and digital signals. AM

Topological Groups, P. *Group Representations, Ergodic Theory, Operator Algebras, and Mathematical Physics.* Ed: C.C. Moore. Math. Sci. Res. Inst. Pub., V. 6. Springer-Verlag, 1987, ix + 278 pp, $27. [ISBN: 0-387-96471-1] Ten papers from a conference in honor of George Mackey (who has made profound contributions to each subject listed in the title) held at MSRI, Berkeley, in 1984. Includes excellent expository lectures on dual vector spaces by Irving Kaplansky, and on mathematical physics by Irving Segal. BC

Algebra, T(15-16: 1, 2), L. *Modern Abstract Algebra.* David C. Buchthal, Douglas E. Cameron. Prindle, Weber & Schmidt, 1987, xii + 548 pp. [ISBN: 0-87150-057-4] After preliminaries on mathematical proof and abstract mappings, the presentation is a fairly straightforward look at the standard topics of groups, rings, and fields. More unusual topics included are chapters on group codes, polynomial codes, and lattices. Exercises (some answers and solutions), references, index. JS

Algebra, P. *Representation Theory of Infinite Groups and Finite Quasigroups.* Jonathan D.H. Smith. Pr U Montreal, 1986, 132 pp, $18 (P). [ISBN: 2-7606-0776-3] The purpose of these notes is to initiate a study of the representation theory of finite quasigroups. One of the main theses is that the representation theory of a finite quasigroup is equivalent to the representation theory of certain infinite groups, the universal multiplication groups of the quasigroup. A consequence of this is that the study of a finite Latin square (classical finite combinatorics) may be related to the representation theory of free groups (infinite group theory). LCL

Algebra, T(14-16: 1, 2), S, L. *Essential Student Algebra.* T.S. Blyth, E.F. Robertson. Chapman & Hall, 1986, (P). *V. 1: Sets and Mappings,* 120 pp, [ISBN: 0-412-27880-4]; *V. 2: Matrices and Vector Spaces,* 120 pp, [ISBN: 0-412-27870-7]; *V. 3: Abstract Algebra,* 120 pp, [ISBN: 0-412-27860-X]; *V. 4: Linear Algebra,* 120 pp, [ISBN: 0-412-27850-2]; *V. 5: Groups,* 120 pp. [ISBN: 0-412-27840-5] Compact, concise, modular units, moving through the basics to the fundamentals of the following topics: (1) equivalence relations, permutations, cardinal numbers, (2) eigenvalues and eigenvectors, (3) groups, rings, fields, quotient structures, (4) Jordan forms, bilinear and quadratic forms, (5) Sylow theorems, composition series. LCL

Algebra, T(15-16). *Undergraduate Algebra.* Serge Lang. Undergrad. Texts in Math. Springer-Verlag, 1987, ix + 256 pp, $36. [ISBN: 0-387-96404-5] The first five chapters cover the integers, groups, rings, polynomials, vector spaces and modules. The remaining five chapters focus on linear groups, Galois theory, finite fields, the real and complex number systems, and set theory. The emphasis is on breadth rather than depth although the treatment is somewhat more sophisticated than that of many other algebra texts. SG

Calculus, T(13-14: 2). *Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences, Fifth Edition.* Ernest F. Haeussler, Jr., Richard S. Paul. Prentice-Hall, 1987, xiii + 815 pp. [ISBN: 0-13-501941-9] Expanded to include applications in biology, sociology, psychology, ecology, statistics and archeology. No prior knowledge of applications areas assumed. Algebra, probability, matrix algebra, linear programming, functions and both single- and multi-variable calculus. Interesting examples and exercises. (*First Edition,* TR, May 1974; *Second Edition,* TR, December 1976; *Third Edition,* TR, December 1980; *Fourth Edition,* TR, August-September 1983.) JK

Calculus, T(13: 2). *Calculus and Its Applications, Fourth Edition.* Larry J. Goldstein, David C. Lay, David I. Schneider. Prentice-Hall, 1987, xix + 682 pp. [ISBN: 0-13-111030-6] An intuitive presentation illustrated with numerous applications to biological, social, and management sciences. Changes in this edition include additional examples and exercises, an alteration in the definition of the definite integral, some revisions in the chapter on differentiation, new material on Taylor polynomials and infinite series. (*First Edition,* TR, February 1977; *Second Edition,* TR, December 1980; *Third Edition,* TR, April 1984.) JNC

Calculus, T(13: 1, 2). *Applied Calculus.* Soo Tang Tan. Prindle, Weber & Schmidt, 1986, ix + 662 pp. [ISBN: 0-87150-954-7] An intuitive introduction to the basics of elementary differential and integral calculus, mainly of algebraic, logarithmic, and exponential functions. Trigonometric functions are handled only briefly. Not rigorous—e.g., no mean value theorem. Economics and life-science examples predominate over physical examples. Elementary numerical methods (integration, equation-solving, differen-

tial equations) are covered. PZ

Real Analysis, P. *Lecture Notes in Mathematics-1189: Fine Topology Methods in Real Analysis and Potential Theory.* Jaroslav Lukeš, Jan Malý, Luděk Zajíček. Springer-Verlag, 1986, x + 472 pp, $40.20 (P). [ISBN: 0-387-16474-X] In 1940, H. Cartan defined the *fine topology* as the coarsest topology on R^n making all superharmonic functions continuous. Text develops general properties of this topology and details its uses in potential theory. Numerous historical notes and references, as well as an extensive bibliography. BH

Real Analysis, P. *Recent Progress in Fourier Analysis.* Ed: I. Peral, J.-L. Rubio de Francia. Math. Stud., V. 111. Elsevier Science, 1985, v + 268 pp, $37 (P). [ISBN: 0-444-87745-2] Proceedings of the seminar on Fourier analysis held in El Escorial, Spain, June 30-July 5, 1983. BH

Complex Analysis, P. *Contributions to Several Complex Variables: In Honor of Wilhelm Stoll.* Ed: Alan Howard, Pit-Mann Wong. Aspects of Math., V. E9. Friedr Viewig & Sohn, 1986, xi + 353 pp, (P). [ISBN: 3-528-08964-4] 18 papers on many topics in several complex variables constitute the proceedings of a conference in honor of Wilhelm Stoll, held in 1984 at the University of Notre Dame. PZ

Complex Analysis, T(18: 1, 2), S, P. *Univalent Functions and Teichmüller Spaces.* Olli Lehto. Grad. Texts in Math., V. 109. Springer-Verlag, 1986, xii + 257 pp, $46. [ISBN: 0-387-96310-3] From the introduction: "The interplay between the theory of univalent functions and the theory of Teichmüller spaces is the main theme... ." Assuming graduate-level complex analysis, this monograph begins with basics of quasiconformal mappings and univalent function theory. Later chapters treat Teichmüller spaces and Riemann surfaces, culminating with Teichmüller spaces *of* Riemann surfaces. With chapter summaries; no exercises. PZ

Complex Analysis, S(18), P. *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie.* Edmund Landau, Dieter Gaier. Springer-Verlag, 1986, xi + 201 pp, DM96. [ISBN: 0-387-16886-9] Text of *Second Edition* of Landau's classic book, with two appendices by Gaier. The first contains brief comments on most sections of Landau and on more recent related results. The second covers in some detail several further "pearls of function theory." Extensive bibliography, no index. JD-B

Differential Equations, T(16-17: 2), S. *Gewöhnliche Differentialgleichungen: Eine Einführung in Theorie und Praxis.* Helmut Werner, Herbert Arndt. Hochschultext. Springer-Verlag, 1986, x + 335 pp, DM38 (P). [ISBN: 0-387-15288-1] An introductory but sophisticated text. Treats existence and uniqueness theorems as well as numerical methods. Chapter on stiff equations. No exercises. JD-B

Differential Equations, T(16-17: 1, 2), S*, P, L*. *Nonlinear Stochastic Operator Equations.* George Adomian. Academic Pr, 1986, xv + 287 pp, $64.50. [ISBN: 0-12-044375-9] From the Preface: "This book addresses the need for realistic solutions of the nonlinear stochastic equations arising in the modeling of frontier problems in every area of science... . Thus the solution sought is that of the problem at hand, rather than one tailored to machine computation or the use of existing theorems of mathematics." A worthy endeavor, well embarked on! BC

Differential Equations, T(16-17: 2, 3). *Applied Analysis.* Allan M. Krall. Math. & Its Applic. D Reidel, 1986, xi + 561 pp, $89.50. [ISBN: 90-277-2328-1] Textbook on applied mathematics from the mathematician's viewpoint—i.e., applications secondary. Covers linear spaces and linear operators, linear ordinary differential equations, and the Sturm-Liouville problem. Also includes an introduction to partial differential equations, using a distributional setting to discuss the Laplace equation, the heat equation, and the wave equation. Examples and exercises included. LC

Differential Equations, T(14). *The Solution of Ordinary Differential Equations.* E.L. Ince, I.N. Sneddon. Math. Texts. Longman Scientific & Technical (US Distr: Wiley), 1987, x + 234 pp, $24.95 (P). [ISBN: 0-470-20680-2] Just a little addition here and there by Sneddon to update the immensely popular (*Seventh Edition*) compact differential equations book by Ince. Still attractive and effective. AWR

Differential Equations, P*. *Wave Interactions and Fluid Flows.* Alex D.D. Craik. Mono. on Mech. & Appl. Math. Cambridge U Pr, 1985, xii + 322 pp, $59.50. [ISBN: 0-521-26740-4] Comprehensive account of current theory and experiment. Focusses on nonlinear wave interactions in incompressible fluids. Includes wave-driven mean flows, multi-wave resonance, conservative and dissapative cases, and a discussion of numerical methods. MR

Differential Equations, P. *Lecture Notes in Mathematics-1223: Differential Equations in Banach Spaces.* Ed: A. Favini, E. Obrecht. Springer-Verlag, 1986, viii + 299 pp, $27.80 (P). [ISBN: 0-387-17191-6] Proceedings of a July 1985 conference held at the University of Bologna. LAS

Differential Equations, T(14-15: 1, 2), L.** *Introduction to Differential Equations.* Richard K. Miller. Prentice-Hall, 1987, x + 628 pp. [ISBN: 0-13-481003-1] Very readable and carefully written. A multitude of problems, carefully chosen, of varying degrees of difficulty. Traditional and nonstandard

applications. Reviews complex numbers, infinite series, linear algebra. Concludes with chapters on numerical analysis (computer-oriented, and suitable for self-study), qualitative analysis, Fourier series, separation of variables in partial differential equations. A very nice book. DFA

Differential Equations, P. *Asymptotics of High Order Differential Equations.* R.B. Paris, A.D. Wood. Res. Notes in Math. Ser., V. 129. Longman Scientific & Technical (US Distr: Wiley), 1986, 344 pp, $49.95 (P). [ISBN: 0-470-20375-7] Develops asymptotic solutions about an irregular singular point at infinity for differential equations with coefficients that are powers of or polynomials in z. Methods employed are classical and require familiarity with complex function theory and hypergeometric functions. Many examples and applications to physics and spectral theory. MR

Differential Equations, T(17-18), P, L. *Solving Ordinary Differential Equations I: Nonstiff Problems.* E. Hairer, S.P. Nørsett, G. Wanner. Ser. in Comput. Math., V. 8. Springer-Verlag, 1987, xiii + 480 pp, $69. [ISBN: 0-387-17145-2] Three chapters on classical mathematical theory; Runge-Kutta and extrapolation methods; multistep and general linear methods. Applications from physics, chemistry, biology, astronomy. Appendix contains Fortran codes. Engagingly written, with historical development and remarks throughout. Exercises. Over 350 bibliographic citations. DFA

Differential Equations, P. *Lecture Notes in Mathematics-1192: Equadiff 6.* Ed: J. Vosmanský, M. Zlámal. Springer-Verlag, 1986, xx + 424 pp, $36.10 (P). [ISBN: 0-387-16469-3] Proceedings of an international conference on differential equations and their applications held in Brno, Czechoslovakia in August, 1985. Nine of the ten plenary lectures, plus sectional papers on ordinary differential equations, partial differential equations, numerical methods and applications. LAS

Differential Equations, P. *Lecture Notes in Mathematics-1191: The Isomonodromic Deformation Method in the Theory of Painlevé Equations.* Alexander R. Its, Victor Yu. Novokshenov. Springer-Verlag, 1986, iv + 313 pp, $25 (P). [ISBN: 0-387-16483-9]

Differential Equations, T(16-17), S. *Stability of Functional Differential Equations.* V.B. Kolmanovskii, V.R. Nosov. Math. in Sci. & Eng., V. 180. Academic Pr, 1986, xiv + 217 pp, $29.95 (P); $50. [ISBN: 0-12-417941-X; 0-12-417940-1] A functional differential equation involves a function $g(t)$ of one scalar argument t and its derivatives for several values (translations) of t. They occur in problems involving time log, delay, etc. Authors discuss retarded equations, neutral, and stochastic functional differential equations. Calculus is the only prerequisite. Attractively printed. AWR

Partial Differential Equations, T(17-18). *Partial Differential Equations, Second Edition.* P.R. Garabedian. Chelsea, 1986, xii + 672 pp, $27.50. [ISBN: 0-8284-00325-2] A reprint of the *First Edition* originally published in 1964. Assumes a knowledge of separation of variables and Fourier series. Written for scientists and engineers as well as for mathematicians, this text includes the method of characteristics, Hamilton-Jacobi theory, the Cauchy problem, Dirichlet's principle, integral equations, and fluid dynamics. AM

Partial Differential Equations, P. *Advances in Microlocal Analysis.* Ed: H.G. Garnir. NATO ASI Ser. C, V. 168. D Reidel, 1985, xvii + 390 pp, $69. [ISBN: 90-277-2195-5] Collection of 15 lectures given at the 1985 Castelvecchio-Pascoli NATO Advanced Study Institute, September 2-12, 1985. These lectures cover applications of microlocal analysis to solutions of linear partial differential equations, and also cover recent work using microlocal analysis in non-linear partial differential equations. AM

Partial Differential Equations, S(16-17). *Analysis of a Finite Element Method: PDE/PROTRAN.* Granville Sewell. Springer-Verlag, 1985, x + 154 pp, $24. [ISBN: 0-387-96226-3] Use of a finite element package to solve partial differential equations in 2-dimensions. Methods for elliptic, hyperbolic, parabolic, and eigenvalue problems. RWN

Partial Differential Equations, P. *Solitons in Mathematics and Physics.* Alan C. Newell. SIAM, 1985, xvi + 244 pp, $29.50. [ISBN: 0-89871-196-7] Solitons were first observed as coherent solitary waves propagating through canals in England. They arise as local, traveling pulse solutions to non-linear wave equations. This book describes solitons as they appear in mathematics and physics. Includes the KDV equation, non-linear Schroedinger equation, and the completely integrable structure of these equations. AM

Partial Differential Equations, P. *Wave Propagation: An Invariant Imbedding Approach.* Richard Bellman, Ramabhadra Vasudevan. Math. & Its Applic. D Reidel, 1986, xiv + 367 pp, $54.50. [ISBN: 90-277-1766-4] Invariant imbedding is the idea that it is often useful to study a problem, not in isolation, but as a member of a family of similar problems depending on an additional parameter. Explains basic principles of imbedding concepts and their application to obtaining solutions for wave equations propagating in homogeneous media. AM

Partial Differential Equations, P. *Lecture Notes in Mathematics-1054: Galerkin Finite Element Methods for Parabolic Problems.* Vidar Thomée. Springer-Verlag, 1984, vii + 237 pp, $11 (P). [ISBN:

0-387-12911-1] Summary of recent results and theoretical background for applying Galerkin-like finite element methods to a variety of problems including inhomogeneous, discontinuous, nonlinear, mixed and singular problems. RWN

Partial Differential Equations, P. *Lecture Notes in Mathematics-985: Asymptotic Analysis II—Surveys and New Trends.* Ed: F. Verhulst. Springer-Verlag, 1983, 497 pp, $26.50 (P). [ISBN: 0-387-12286-9] A collection of 19 survey and research papers on asymptotic and perturbation methods and their applications. RWN

Partial Differential Equations, P. *On the Cauchy Problem.* Sigeru Mizohata. Notes & Rep. in Math. in Sci. & Eng., V. 3. Academic Pr, 1985, 177 pp, $36 (P). [ISBN: 0-12-501660-3] Rather stilted English translation of series of lectures given by the author at Wuhan University, China. Topics include Lax-Mizohata theorem, Cauchy problems in the Gevrey class, and micro-local analysis in the Gevrey class. Numerous references at end of each chapter. Assumes basic knowledge of pseudo-differential operators. BH

Partial Differential Equations, P. *Foundations of Algebraic Analysis.* Masaki Kashiwara, Takahiro Kawai, Tatsuo Kimura. Transl: Goro Kato. Math. Ser., V. 37. Princeton U Pr, 1986, xii + 254 pp, $38. [ISBN: 0-691-08413-0] Microlocal analysis refers to the local study of differential equations on cotangent bundles of manifolds. This book provides a thorough description of microlocal analysis. It includes microfunction theory and its application to differential equations and theoretical physics. It also describes microdifferential equations and the microlocalization of linear differential equations. AM

Partial Differential Equations, T(17-18), P, L. *An Introduction to Fast Fourier Transform Methods for Partial Differential Equations, With Applications.* Morgan Pickering. Appl. & Eng. Math. Ser., V. 4. Research Studies Pr (US Dist: Wiley), 1986, xi + 178 pp, $49.95. [ISBN: 0-471-91261-1] Mathematical theory; problems susceptable to direct solution; the cyclic reduction method; problems on irregular regions; applications to two more general problems. Traces modern developments. Cites appropriate software packages. DFA

Partial Differential Equations, P. *Partial Differential Relations.* Mikhael Gromov. Ser. of Mod. Surv. in Math., B. 9. Springer-Verlag, 1986, ix + 363 pp, $60. [ISBN: 0-387-12177-3] Given a fibration of X over a space V, a differential relation imposed on sections of this fibration is a subset R of the bundle of n-jets of smooth sections over V. A solution of the differential relation corresponds to a section of X over V whose n-jet lies in R. For example, consider the fibration of the plane over the line taking coordinates (x, y) to x. The bundle of one jets may be identified as 3-space with coordinates (x, y, y'). The differential equation $y' = xy$ then defines a subset of 3-space which is the differential relation. This book examines examples of differential relations arising in differential geometry and discusses the existence and classification of their solutions. AM

Partial Differential Equations, T*(15-16: 1). *Boundary Value Problems, Third Edition.* David L. Powers. Harcourt Brace Jovanovich, 1987, x + 419 pp, $35.95. [ISBN: 0-15-505535-6] Many sections have been rewritten, two chapters have been completely reorganized. New material includes two new sections (on Green's functions in the opening chapter reviewing ordinary differential equations; a convergence proof in the chapter on Fourier series and integrals), some Basic program segments, 200 new exercises, an appendix collecting mathematical formulas. (*First Edition,* TR, January 1973; Extended Review, March 1974; *Second Edition,* TR, November 1979.) DFA

Partial Differential Equations, P. *Nonstrictly Hyperbolic Conservation Laws.* Ed: Barbara Lee Keyfitz, Herbert C. Kranzer. Contemp. Math., V. 60. AMS, 1987, ix + 133 pp, $19 (P). [ISBN: 0-8218-5069-5] Proceedings of a special session held at the January 1985 meeting of AMS in Anaheim. LAS

Partial Differential Equations, P. *Lecture Notes in Mathematics-1232: Asymptotic Analysis of Soliton Problems: An Inverse Scattering Approach.* Peter Cornelis Schuur. Springer-Verlag, 1986, vii + 180 pp, $15.80 (P). [ISBN: 0-387-17203-3] Concerns nonlinear evolution equations solvable by the inverse scattering method. Studies emergence of the solitons and uses an almost uniform method to obtain the asymptotic behavior for large time of solutions of soliton problems. Rigorous. DFA

Partial Differential Equations, P. *Semigroups, Theory and Applications.* Ed: H. Brezis, M.G. Crandall, F. Kappel. Longman Scientific & Technical (US Distr: Wiley), 1986, $34.95 (P) each. *Volume I,* Res. Notes in Math. Ser., V. 141, 252 pp, [ISBN: 0-470-20372-2]; *Volume II,* Res. Notes in Math. Ser., V. 152, 252 pp. [ISBN: 0-470-20383-8] Papers from a course in 1984 at the International Center for Theoretical Physics, in Trieste, Italy. Applications of semigroups to differential equations and physics, mostly via operator algebras. *Volume I:* twenty-eight short papers, mostly research results. *Volume II:* seven longer papers, partly expository. BC

Numerical Analysis, T(18), P. *Nonlinear Approximation Theory.* Dietrich Braess. Ser. in Comput. Math., V. 7. Springer-Verlag, 1986, xiv + 290 pp, $69.50. [ISBN: 0-387-13625-8] Monograph on advanced aspects of approximation theory directed towards researchers, but can be used as a text by stu-

dents with knowledge of analysis and functional analysis. Covers methods in local and global best approximation, best rational approximation, approximation by exponential sums, and Chebyshev approximation by γ-polynomials. Includes examples, exercises, references. LC

Numerical Analysis, P. *Lecture Notes in Mathematics-1228: Multigrid Methods II.* Ed: W. Hackbusch, U. Trottenberg. Springer-Verlag, 1986, vi + 335 pp, $27.80 (P). [ISBN: 0-387-17198-3] A selection of papers from an October 1985 conference at the University of Cologne emphasizing the application of multigrid techniques to problems of fluid and aerodynamics, and touching on new opportunities provided by parallel processing. LAS

Numerical Analysis, T(15-16), S, L. *Computational Numerical Methods.* Chris Phillips, Barry Cornelius. Ser. in Comput. & Their Applic. Halsted Pr, 1986, 375 pp, $51.95. [ISBN: 0-470-20336-6] Attempts to describe root finding, linear systems of equations, approximation, quadrature, and ordinary differential equations in a way not mathematically demanding. Written with computer scientists and engineers in mind, the book discusses relative performance of various numerical methods and the limitations of various methods. Code for some programs is included in both Pascal and Algol 68. AWR

Numerical Analysis, P. *Numerical Analysis.* Ed: D.F. Griffiths, G.A. Watson. Res. Notes in Math. Ser., V. 140. Longman Scientific & Technical (US Distr: Wiley), 1986, 262 pp, $42.95 (P). [ISBN: 0-470-20669-1] Papers of 16 invited talks at the 11th Dundee biennial conference on numerical analysis, held at the University of Dundee in June 1985. DFA

Numerical Analysis, P. *Numerical Methods for Fluid Dynamics II.* Ed: K.W. Morton, M.J. Baines. Inst. of Math. & Its Applic. Conf. Ser., V. 7. Clarendon Pr, 1986, xv + 679 pp, $95. [ISBN: 0-19-853610-0] Proceedings of an April 1985 conference on computational methods in aerodynamics and fluid dynamics at the Institute of Mathematics and Its Applications at Reading, England. LAS

Numerical Analysis. *Supplement to Table of Sines and Cosines to Ten Decimal Places at Thousandths of a Degree.* Herbert E. Salzer, Norman Levine. Applied Science Pub (POB 5399, Grand Central Station, NY 10163), 1986, 68 pp, $3.50 (P). [ISBN: 0-915061-02-3] An addendum to a 1962 Pergamon Press *Table of Sines and Cosines* containing a proof of error bounds in the original volume and a supplementary table for small angles giving ten significant figures instead of the original volume's ten decimal places. LAS

Functional Analysis, S(17-18), P. *Lecture Notes in Mathematics-1227: The Spectral Theorem.* Henry Helson. Springer-Verlag, 1986, vi + 104 pp, $12.80 (P). [ISBN: 0-387-17197-5] Based on lectures given at the summer school for graduate students, Mathematical Institute, Nankai University, Tianjin, China, June-July 1985. Chapters on multiplicity of spectral measures, the spectral theorem, Bochner's theorem, distribution of cocycles, cocycles on the line. References. RJA

Functional Analysis, T(17), S, P. *Fundamental Principles of the Theory of Extremal Problems.* Vladimir M. Tikhomirov. Transl: Bernd Luderer. Wiley, 1986, 136 pp, $27. [ISBN: 0-471-90563-1] Beautiful little book, set in the context of normed linear spaces, that cites three basic facts (contraction mapping principle, Banach's theorem on inverse mapping, the Hahn-Banach theorem) from which one develops a unified point of view for extremal problems in the calculus of variations, optimal control theory, and convex programming. Compact! AWR

Functional Analysis, P. *Transform Analysis of Generalized Functions.* O.P. Misra, J.L. Lavoine. Math. Stud., V. 119. Elsevier Science, 1986, xiv + 332 pp, $48.25 (P). [ISBN: 0-444-87885-8] Concentrating on finite parts of integrals, generalized functions and distributions, this work gives a unified treatment of the distributional setting with transform analysis, i.e., Fourier, Laplace, Stieltjes, Mellin, Hankel, and Bessel series. It includes solutions to sample problems in mathematical physics as well as information on distributional solutions of differential, partial differential, and integral equations. MU

Functional Analysis, P. *Two Applications of Functional Analysis.* Sudarsan Nanda. Papers in Pure & Appl. Math., No. 74. Queen's U, 1986, 150 pp, (P). Notes used by the author in two series of lectures delivered in several places. The first series, matrix transformations and sequence spaces, is a review article giving "almost all known results" on the topic; the second series, complementarity and generalized convexity in mathematical programming, includes a table summarizing "most of the important types of generalized convex sets and functions," together with references. Set from typewritten copy. AWR

Functional Analysis, P. *Functional Analysis and Two-point Differential Operators.* John Locker. Res. Notes in Math. Ser., V. 144. Longman Scientific & Technical (US Distr: Wiley), 1986, 257 pp, $44.95 (P). [ISBN: 0-470-20382-X] Studies two-point differential operators and linear boundary value problems in the Hilbert space $L^2[a,b]$, emphasizing modern operator theory viewpoint. LC

Functional Analysis, P. *Geometry of Normed Linear Spaces.* Ed: R.G. Bartle, *et al.* Contemp. Math., V. 52. AMS, 1986, xi + 171 pp, $18 (P). [ISBN: 0-8218-5057-1] Proceedings of a June 1983 conference on geometry of normed linear spaces, held in honor of M.M. Day, one of the founders of the subject. In-

cludes sixteen papers on geometric properties of and geometric methods applied to normed linear spaces. With a valedictory poem, in Poe's style. PZ

Analysis, P. *Thirteen Papers in Analysis.* R.R. Suncheleev, *et al.* Transl: Ben Silver. AMS Transl. Ser. 2, V. 133. AMS, 1986, v + 122 pp, $46. [ISBN: 0-8218-3109-7] Selection of papers from various USSR journals originally published between 1977 and 1983. LAS

Algebraic Geometry, P. *The Curves Seminar at Queen's, Volume IV.* Ed: Anthony V. Geramita. Papers in Pure & Appl. Math., No. 76. Queen's U, 1986, 273 pp, (P). A collection of 13 expository papers covering a variety of topics including zeta functions, locally complete intersections, and standard bases for ideals. SG

Algebraic Geometry, S(18), P. *Residues and Traces of Differential Forms via Hochschild Homology.* Joseph Lipman. Contemp. Math., V. 61. AMS, 1987, vii + 95 pp, $16 (P). [ISBN: 0-8218-5070-9] Provides an elementary development of the theory of residues as a formal algebraic construct, bypassing duality theorems as motivation and thus simplifying many proofs. "Hard" results include a formula for residues with respect to powers of quasi-regular sequences, and a trace formula expressing an adjoint-type relationship between trace and cotrace maps in the Hochschild formulism. MR

Algebraic Geometry, P. *Topics on Families of Projective Schemes.* Edoardo Sernesi. Papers in Pure & Appl. Math., No. 73. Queen's U, 1986, 203 pp, (P). An introduction to the study of families of projective schemes with an emphasis on Hilbert schemes. SG

Differential Geometry, P. *Separation of Variables for Riemannian Spaces of Constant Curvature.* E.G. Kalnins. Mono. & Surv. in Pure & Appl. Math., V. 28. Longman Scientific & Technical (US Distr: Wiley), 1986, 172 pp, $56.95. [ISBN: 0-470-20366-8] Aims to show how all the actual inequivalent separable coordinate systems can be computed for the Hamilton-Jacobi and Helmholtz equations on real positive definite Riemannian spaces of constant curvature. Includes results on the classification of separable coordinate systems on the n-sphere, on Euclidean n-sphere, and on the upper sheet of the n-hyperboloid. AM

Differential Geometry, P. *Einstein Manifolds.* Arthur L. Besse. Ser. of Mod. Surv. in Math., B. 10. Springer-Verlag, 1987, xii + 510 pp, $89. [ISBN: 0-387-15279-2] An Einstein manifold is a manifold that admits a Riemannian structure with constant Ricci curvature. For two-dimensional manifolds, this notion coincides with the notion of constant Gaussian curvature. In this case it is known that every compact surface admits at least one Riemannian structure of constant curvature and these structures form a finite dimensional submanifold in the moduli space of Riemannian structures. The book explores the conjecture that constant Ricci curvature is the appropriate generalization of the concept of constant curvature. AM

Differential Geometry, P*.** *Nonlinear Analysis in Geometry.* Shing Tung Yau. L'Enseignement Math, 1986, 54 pp, (P). An account of three lectures given by the author at the ETH-Zurich on November 20, 27, and December 1, 1981 under the sponsorship of the International Mathematical Union. Surveys recent work and trends in Kahler geometry, minimal surfaces, semi-linear equations, and questions about the spectrum of the Laplacian. AM

Differential Geometry, P. *Variational Methods for Free Surface Interfaces.* Ed: Paul Concus, Robert Finn. Springer-Verlag, 1987, x + 204 pp, $36. [ISBN: 0-387-96396-0] In the Plateau problem, one is looking for a minimal surface spanning a fixed boundary. If one prescribes the boundary of the minimal surface to lie in a given submanifold, one has an example of a free boundary problem. This book is a collection of 23 papers recording the proceedings of a conference held at Vallombrosa Center, Menlo Park, California, September 7-12, 1985. AM

Differential Geometry, P. *Lecture Notes in Mathematics-1201: Curvature and Topology of Riemannian Manifolds.* Ed: K. Shiohama, T. Sakai, T. Sunada. Springer-Verlag, 1986, vii + 336 pp, $26.40 (P). [ISBN: 0-387-16770-6] Proceedings of the seventeenth Taniguchi International Symposium at Katato, Japan in August 1985, and of a conference at the Research Institute for Mathematical Science, Kyoto University, September 1985. BH

Differential Geometry, S(17-18), P. *Lecture Notes in Mathematics-1207: Spectral Geometry: Direct and Inverse Problems.* Pierre H. Bérard. Springer-Verlag, 1986, xiii + 272 pp, $23.40 (P). [ISBN: 0-387-16788-9] Spectral geometry refers to the relationship between the geometry of a Riemannian manifold and the set of eigenvalues of its Laplace-Beltrami operator. The book discusses the background of this problem, surveys recent work, and includes several new results related to isoperimetric inequalities and the topology of Riemannian manifolds. AM

Differential Geometry, S(16-18), P. *The Classical Differential Geometry of Curves and Surfaces.* Georges Valiron. Transl: James Glazebrook. Lie Groups: History, Frontiers and Applications, V. XV. Math Science Pr, 1986, viii + 268 pp, $50. [ISBN: 0-915692-39-2] Translation from French of volume two of Valiron's *Cours d'analyse mathematique* dealing with curves and surfaces in three-dimensional space

and first- and second-order partial differential equations in two variables. Part of Robert Hermann's series of classics in differential geometry. BC

Geometry, S(16-17), P, L*. *An Adventure in Multidimensional Space: The Art and Geometry of Polygons, Polyhedra, and Polytopes.* Koji Miyazaki. Wiley, 1986, vii + 112 pp, $49.95. [ISBN: 0-471-81648-5] Remarkable figures, photographs and drawings illustrating a variety of views of the role of polygons (2-polytopes), polyhedra (3-polytopes), and 4-polytopes in two-, three-, and four-dimensional worlds, covering various ages and countries, with emphasis on Japan's past. In the author's words, "The stars of the show are Plato and polygons, Kepler and polyhedra, Fuller and polytopes." This slim, expensive volume was translated from the original 1983 Japanese language edition with (English) title *Forms of Space.* JK

Geometry. *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry.* Ed: J. Carrell, A.V. Geramita, P. Russell. Conf. Proc., V. 6. AMS, 1986, viii + 503 pp, $56 (P). [ISBN: 0-8218-6010-0] Proceedings of the conference held July 2-12, 1984 at the University of British Columbia, Vancouver, British Columbia, Canada. JAS

Differential Topology, P. *Lecture Notes in Mathematics-1214: Global Analysis—Studies and Applications II.* Ed: Yu. G. Borisovich, Yu. E. Gliklikh. Springer-Verlag, 1986, 275 pp, $23.60 (P). [ISBN: 0-387-16821-4] A follow-up to *LNM-1108*: fourteen papers translated from Russian, on new developments (1985-86) in global analysis. Varying amounts of detail. BC

Topology, P. *Topological Dynamix.* J.C.S.P. van der Woude. CWI Tract, V. 22. Math Centrum, 1986, 298 pp, Dfl. 44.90 (P). [ISBN: 90-6196-298-6] Develops structure theory of minimal topological transformation groups (ttgs) with primary focus on quasi-factors of minimal ttgs, (weak) disjointness of homomorphisms of ttgs, and the equicontinuous structure relation. BH

Topology, P. *On the Existence of Natural Non-topological, Fuzzy Topological Spaces.* R. Lowen. Res. & Expos. in Math., V. 11. Heldermann Verlag, 1985, xvi + 183 pp, $34 (P). [ISBN: 3-88538-211-3] A *fuzzy topology* on a set X is a collection of functions $\mu : X \to [0,1]$ which are closed for finite infima and arbitrary suprema and contain all constant functions. Text develops basic theory of fuzzy topologies and applies this theory to spaces of probability measures and spaces of upper semicontinuous functions on a uniform space. Extensive bibliography. BH

Topology, P. *Convergence Structures 1984.* Ed: Josef Novák, *et al.* Akademie-Verlag, 1985, 254 pp, (P). Proceedings of a conference held in Bechyně, Czechoslovakia from September 24-28, 1984. JAS

Game Theory, P. *Non-Antagonistic Games.* Yu. B. Germeier. Transl: Anatol Rapoport. Theory & Decision. Lib., V. 46. D Reidel, 1986, xiii + 331 pp, $69. [ISBN: 90-277-2023-1] A study of strategies for competitive games in which the players interests' partially coalesce, emphasizing iterated games, voluntarily revealed information, bluffing, and other realistic options. Translation of a 1976 Russian monograph *Igry s Nieprotivopolozhnymi Interesami.* LAS

Game Theory, S*(13-15), L*. *The Compleat Strategyst: Being a Primer on the Theory of Games of Strategy.* J.D. Williams. Dover, 1986, xvi + 268 pp, $5.95 (P). [ISBN: 0-486-25101-2] An unabridged republication of the 1966 revised edition (TR, January 1967) of a classic first published in 1954. Written as the very first lay introduction to the new theory of games that emerged from war-related research, it remains a lucid, entertaining account of enduring value. LAS

Optimization, S(15-16), P. *Quasidifferential Calculus.* Vladimir F. Dem'yanov, Alexander M. Rubinov. Transl. Ser. in Math. & Eng. Optimization Software, 1986, xi + 289 pp, $72. [ISBN: 0-911575-35-9] A preliminary chapter explains the relationship of quasidifferentials to optimization. The remainder of the book develops properties of quasidifferentiability (a concept introduced in 1979) and applies them to standard topics in optimization. Reads nicely. AWR

Optimization, P. *Tree Network and Planar Rectilinear Location Theory.* A.J.W. Kolen. CWI Tract, V. 25. Math Centrum, 1986, iii + 85 pp, Dfl. 12.50 (P). [ISBN: 90-6196-300-1] Location theory deals with the problem of finding points whose sums of distances from a fixed set of points is a minimum. This monograph considers discrete location theory, in which the underlying structure is a tree, and planar location theory in which the underlying structure is a plane. Assumes some mathematical programming. LC

Dynamical Systems, P. *Lecture Notes in Mathematics-1211: Reversible Systems.* M.B. Sevryuk. Springer-Verlag, 1986, v + 319 pp, $27.80 (P). [ISBN: 0-387-16819-2] Kolmogorov-Arnold-Moser (KAM) theory describes the preservation of quasi-periodic motion under a small perturbation of a non-degenerate integrable Hamiltonian system. A system of differential equations associated with a vector field V is called reversible if there exists an involution transforming V into $-V$. This book examines the generalization of KAM theory to the context of integrable reversible systems. AM

Dynamical Systems, P*. *Inverse Problems in Vibration.* G.M.L. Gladwell. Martinus Nijhoff (US Distr: Kluwer Academic), 1987, x + 263 pp, $79.50. [ISBN: 90-247-3408-8] Non-rigorous introduction to

the problem of constructing a model given spectral data for which there is a unique vibrating system of a specified type. Covers discrete systems and one-dimensional continuous sytems governed by differential equations of order 2 or 4. JK

Dynamical Systems, P. *Multiparameter Bifurcation Theory.* Ed: Martin Goklubitsky, John M. Guckenheimer. Contemp. Math., V. 56. AMS, 1986, xvii + 387 pp, $34 (P). [ISBN: 0-8218-5060-1] Proceedings of an AMS summer research conference held in Arcata, California in July 1985. Participants included "scientists working on fluid instabilities and chemical reactor dynamics as well as mathematicians interested in multiparameter bifurcation." BH

Dynamical Systems, P. *Lecture Notes in Mathematics-1222: Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities.* Anatole Katok, Jean-Marie Strelcyn. Springer-Verlag, 1986, viii + 283 pp, $23.60 (P). [ISBN: 0-387-17190-8] Existence of invariant manifolds for smooth maps with singularities; absolute continuity; estimation of entropy from below and from above through Lyapunov characteristic exponents; plane billiards as smooth dynamical systems with singularities. DFA

Control Theory, P. *Geometric Measure Theory and the Calculus of Variations.* Ed: William K. Allard, Frederick J. Almgren, Jr. Proc. of Symp. in Pure Math., V. 44. AMS, 1985, xiv + 464 pp, $59. [ISBN: 0-8218-1470-2] Proceedings of the Thirty-Second Summer Research Institute of the AMS held at Humboldt State University in Arcata, California, July 16-August 3, 1984. JAS

Systems Theory, P. *Dynamics of Hierarchical Systems: An Evolutionary Approach.* J.S. Nicolis. Ser. in Synergetics, V. 25. Springer-Verlag, 1986, xv + 397 pp, $66. [ISBN: 0-387-13323-2] A mathematical investigation of the interaction of complex systems in a changing environment with emphasis on the physical bases for the exchange of information. LAS

Probability, P. *Probability, Statistical Mechanics, and Number Theory: A Volume Dedicated to Mark Kac.* Ed: Gian-Carlo Rota. Adv. in Math., Suppl. Stud., V. 9. Academic Pr, 1986, xi + 194 pp, $59.50. [ISBN: 0-12-598543-6] A collection of twelve papers dedicated to the memory of Marc Kac. CEC

Probability, P. *Random Polynomials.* A.T. Bharucha-Reid, M. Sambandham. Prob. & Math. Stat. Academic Pr, 1986, xv + 206 pp, $29.95 (P); $49.50. [ISBN: 0-12-095711-6; 0-12-095710-8] The first book to present a fairly comprehensive treatment of random algebraic, orthogonal, and trigonometric polynomials, including in-depth coverage of expectation, variance, maxima, and distribution of the number of real zeros of random polynomials. Computer generated results are included to illustrate the theory. LCL

Probability, P. *Random Mappings.* Valentin F. Kolchin. Transl. Ser. in Math. & Eng. Optimization Software, 1986, xiv + 207 pp, $80. [ISBN: 0-911575-16-2] Considers probabilistic methods used to study combinatorial questions, in particular, the behavior of one-to-one mappings of a finite set onto itself as the size of the set tends to infinity. LC

Probability, P. *Lecture Notes in Mathematics-1210: Probability Measures on Groups VIII.* Ed: H. Heyer. Springer-Verlag, 1986, x + 386 pp, $31.70 (P). [ISBN: 0-387-16806-0] Proceedings of a November 1985 Oberwolfach conference, the eighth in a series begun in 1970: six survey lectures and 20 research papers. LAS

Probability, P*. *Monte Carlo Methods, Volume I: Basics.* Malvin H. Kalos, Paula A. Whitlock. Wiley, 1986, ix + 186 pp, $29.95. [ISBN: 0-471-89839-2] Unified treatment of Monte Carlo methods (methods that involve "deliberate use of random numbers in a calculation that has the structure of a stochastic process"). Two main themes are random walks and variance reduction techniques (particularly importance sampling). Includes applications to statistical physics, radiation transport, and to the solution of some general integral equations. RSK

Probability, T(15-16: 1), S, L. *Probability, An Introduction.* Geoffrey Grimmett, Dominic Welsh. Clarendon Pr, 1986, ix + 211 pp, $19.95 (P). [ISBN: 0-19-853264-4] Basic ideas of probability up to the central limit theorem by way of moment generating functions, plus branching processes, random walks, and random processes in continuous time. FLW

Probability, P, L. *Kiyosi Itô: Selected Papers.* Ed: Daniel W. Stroock, S.R.S. Varadhan. Springer-Verlag, 1987, xxi + 647 pp, $44. [ISBN: 0-387-96326-X] "K. Itô is the Lebesgue of [stochastic] integration theory (Paley and Wiener were its Riemann)." Beginning with Itô's 1942 doctoral thesis in Tokyo that defines the tangent to an integral curve of probability measures, this *Selecta* offers the major part of Itô's life's work on stochastic integrals, and stochastic differential equations. Includes an introduction by the editors and a professional autobiography by Itô. LAS

Probability, T(17-18: 1). *Probability and Statistics, Volume II.* Didier Dacunha-Castelle, Marie Duflo. Transl: David McHale. Springer-Verlag, 1986, xiv + 410 pp, $32.50. [ISBN: 0-387-96213-1] Translation of the 1983 French edition (see TR, November 1986 of *Volume I*). *Volume II* deals primarily with stochastic processes, statistics of processes, and asymptotic theories. RSK

Probability, P, L. *Detection of Changes in Random Processes.* Ed: Laimutis Telksnys. Trans. Ser. in Math. & Eng. Optimization Software, 1986, xiii + 226 pp, $78. [ISBN: 0-911575-20-0] A collection of

papers on recent advances reported by research centers across the Soviet Union on minimal time detection of change in the properties of a random process, including diverse applications (e.g., seismic data, image segmenting). Includes brief biographies on each of the 32 authors. LAS

Statistics, T(18: 2), P. *Empirical Processes with Applications to Statistics.* Galen R. Shorack, Jon A. Wellner. Wiley, 1986, xxxvii + 938 pp, $59.95. [ISBN: 0-471-86725-X] In the Wiley Series in Probability and Mathematical Statistics. Thorough presentation of the theory of one-dimensional empirical processes, particularly for independent identically distributed random variables. Includes theoretical applications to such topics as tests of fit, bootstrapping, linear combinations of order statistics, rank tests, spacings, and censored data. RSK

Statistics, T(13: 1). *Elementary Statistics, Third Edition.* Mario F. Triola. Benjamin/Cummings, 1986, xviii + 663 pp, $26.95. [ISBN: 0-8053-9327-7] Revision of the author's 1983 *Second Edition* (TR, June-July 1983). New material includes stem-and-leaf plots, box-and-whisker diagrams, one-way analysis of variance with unequal sample sizes, and sections on the nature of data, counting, and p-values. Also includes several new features, such as case study activities at the end of each chapter, and 40% more exercises. RSK

Statistics, T*(18: 1-2), P*. *Testing Statistical Hypotheses, Second Edition.* E.L. Lehmann. Wiley, 1986, xx + 600 pp, $45.95. [ISBN: 0-471-84083-1] In the Wiley Series in Probability and Mathematical Statistics. Updated and expanded version of the author's classic 1959 text. More emphasis is placed on robustness; coverage of confidence intervals, simultaneous inference procedures, admissibility, and multivariate tests has been expanded; a chapter on conditional inference has been added; material on sequential analysis has been removed. Companion volume to the author's 1983 text *Theory of Point Estimation* (TR, April 1984). RSK

Statistics, P*. *COMPSTAT: Proceedings in Computational Statistics.* Ed: F. De Antoni, N. Lauro, A. Rizzi. Physica-Verlag (US Distr: Springer-Verlag), 1986, xv + 512 pp, $54.50 (P). [ISBN: 0-387-91286-X] Selection of 72 papers presented at the 7th COMPSTAT Symposium held in Rome in 1986. Includes papers in the following general areas: information science and statistics, probabilistic models in exploratory data analysis, computational approach of inference, numerical aspects, three-mode data matrices, cluster analysis, robustness in multivariate analysis, computer graphics, expert systems, statistical software, clinical trials, econometric computing, statistical data base management systems, and teaching of computational statistics. RSK

Statistics, P.** *The Collected Works of John W. Tukey.* Ed: Lyle V. Jones. Stat. & Prob. Ser. Wadsworth, 1986. *Volume III: Philosophy and Principles of Data Analysis: 1949-1964*, lxviii + 569 pp, [ISBN: 0-534-03305-9]; *Volume IV: Philosophy and Principles of Data Analysis: 1965-1986*, lxviii, + 553 pp. [ISBN: 0-534-05101-4] Further volumes of a projected series covering Tukey's many contributions to statistics (see TR's, May 1985 of Series and *Volume I,* February 1986 of *Volume II*). These volumes contain 30 papers, including some previously unpublished, in which the dominant issue is his philosophy of research and data analysis. Emphasis is on the need for exploratory as well as confirmatory data analysis. Includes comments by the editor on each of the papers. RSK

Statistics, S(16-17), P. *A Guide to Statistical Methods and to the Pertinent Literature.* Lothar Sachs. Springer-Verlag, 1986, xi + 212 pp, $25 (P). [ISBN: 0-387-16835-4] Contains approximately 5500 alphabetically arranged keywords and subject headings related to statistical methodology, with references to a bibliography of 1449 articles and books where further details can be found. Includes many German terms and references. RSK

Statistics, T(18), S, P. *Recursive Estimation and Control for Stochastic Systems.* Han-Fu Chen. Wiley, 1985, x + 378 pp, $39.95. [ISBN: 0-471-81566-7] Text and reference on convergence of recursive estimates in discrete- and continuous-time systems. Main tools are probability theory and ordinary differential equations. Topics include stochastic approximation algorithms, linear unbiased minimum variance estimates for continuous-time systems, singularity problems, and Gauss-Markov estimation for continuous-time systems. KK

Statistics, S(13), L. *Misused Statistics: Straight Talk for Twisted Numbers.* A.J. Jaffe, Herbert F. Spirer. Popular Stat., V. 5. Dekker, 1987, xi + 237 pp, $29.75. [ISBN: 0-8247-7631-3] A popular polemic against abuses of numbers, with extensive examples: bad data, faulty interpretation, "horror pictures," surveys and polls. Neither teaches nor uses statistical methods. A good resource for examples. LAS

Statistics, T(13-14: 1), L. *Think and Explain with Statistics.* Lincoln E. Moses. Addison-Wesley, 1986, xii + 483 pp, $28.95. [ISBN: 0-201-15619-9] Presupposes no college mathematics. The usual topics with much use of graphs and intuitive explanations for mathematical results. FLW

Statistics, P. *Selected Tables in Mathematical Statistics, Volume 10.* Ed: R.E. Odeh, J.M. Davenport. AMS, 1986, xi + 347 pp, $39. [ISBN: 0-8218-1910-0] Contains two tables: percentile points of the distribution of positive definite quadratic forms for samples up through size 10, and confidence limits on

the correlation coefficient for various sample sizes up through 1000. RSK

Statistics, T*(13-15: 1), S*, P, L*. *Counting for Something: Statistical Principles and Personalities.* William S. Peters. Texts in Stat. Springer-Verlag, 1986, xviii + 275 pp, $33. [ISBN: 0-387-96364-2] An elementary introduction to descriptive statistics "counting and measuring"—embedded in a thoughtful, thorough discussion of the history of probability (Laplace) and statistics (Pearson, Fischer, Neyman). Exquisite examples from the social sciences, ancient and modern, enrich this literate exposition of the basic ideas of modern statistics. LAS

Statistics, P. *Lecture Notes in Statistics-38: Survey Research Designs: Towards a Better Understanding of Their Costs and Benefits.* Ed: R.W. Pearson, R.F. Boruch. Springer-Verlag, 1986, v + 129 pp, $15.80 (P). [ISBN: 0-387-96428-2] General approaches to the question of allocating resources or choosing among various research designs followed by presentations and comparisons of the costs and benefits associated with specific large-scale research designs—cross sectional, longitudinal, and social experiments. LCL

Statistics, P. *Small Area Statistics: An International Symposium.* Ed: R. Platek, *et al.* Ser. in Prob. & Math. Stat. Wiley, 1987, xiv + 278 pp, $39.95. [ISBN: 0-471-84456-X] A report on the symposium held in Ottawa in May 1985 to consider recent results related to the collection and use of data from geographically-integrated small areas. FLW

Statistics, P. *Analysis of Categorical Data.* Gary G. Koch, Peter B. Imrey, *et al.* Pr U Montreal, 1985, 288 pp, $22 (P). [ISBN: 2-7606-0733-X] Based on lectures given by Koch at the 21st Session of the Séminaire de mathématiques supérieurs—NATO Advanced Study Institute on Data Analysis at the Université de Montréal in the summer of 1982. Emphasizes weighted least squares methods, with comparisons to maximum likelihood methods for fitting log-linear models and randomization model methods. RSK

Statistics, S(15-18), P, L*. *New Developments in Statistics for Psychology and the Social Sciences.* Ed: A.D. Lovie. British Psychological Society (US Distr: Methuen), 1986, ix + 177 pp, $55. [ISBN: 0-901715-46-8] Brief discussions of new methods concerning graphics, robust procedures, outliers, cross-classified data, longitudinal studies, mixtures of distributions, sample size and power, and the ranking and selection of populations. Extensive bibliographies. FLW

Statistics, P. *Ordination and Classification.* P.G.N. Digby, J.C. Gower. Pr U Montreal, 1986, 87 pp, $16 (P). [ISBN: 2-7606-0742-9] Based on ten lectures given by Gower at the 21st Session of the Séminaire de mathématiques supérieurs—NATO Advanced Study Institute on Data Analysis at the Université de Montréal in the summer of 1982. Primarily an exploratory data approach to multidimensional scaling, with some general comments on classification. RSK

Statistics, T(18: 1), P. *Foundations of Optimum Experimental Design.* Andrej Pázman. Math. & Its Applic. D Reidel, 1986, xv + 228 pp, $39. [ISBN: 90-277-1865-2] Translation with some revisions of the 1980 Czechoslovakian *First Edition.* Presents the mathematical background for developing and computing optimal designs, emphasizing the linear theory of estimation and the convex theory of experimental design. RSK

Statistics, S(16-18). *Multivariate Statistical Methods: A Primer.* Byran F.J. Manly. Chapman & Hall, 1986, x + 159 pp, $15.95 (P); $35. [ISBN: 0-412-28620-3; 0-412-28610-6] Attempts to give an "idea of what can and what cannot be achieved" by some multivariate methods. Brief discussions of matrix algebra, principal component analysis, factor analysis, discriminant analysis, cluster analysis, canonical correlation, and multidimensional scaling. FLW

Statistics, P*, L. *Statistics and the Law.* Ed: Morris H. DeGroot, Stephen E. Fienberg, Joseph B. Kadane. Wiley, 1986, xviii + 484 pp, $39.95. [ISBN: 0-471-09435-8] In the Wiley Series in Probability and Mathematical Statistics. Interesting collection of articles describing a wide variety of applications of statistical ideas in legal settings, many of which are controversial, together with cases in which statistical analyses were important elements. RSK

Statistics, S(15-17). *A Topical Dictionary of Statistics.* Gary L. Tietjen. Chapman & Hall, 1986, ix + 171 pp, $22.50. [ISBN: 0-412-01201-4] A dictionary in which the terms are not defined in isolation but rather within the context of one of fifteen topical chapters. An alphabetic index indicates the appropriate page. RSK

Statistics, T(18: 1, 2), P*. *Asymptotic Methods in Statistical Decision Theory.* Lucien Le Cam. Ser. in Stat. Springer-Verlag, 1986, xxvi + 742 pp, $49.95. [ISBN: 0-387-96307-3] Well-organized treatment of asymptotic methods within the framework of Wald's decision theory. Main theme is the approximation of complex statistical experiments by experiments that are known and mathematically tractable. RSK

Statistics, P. *Jack Carl Kiefer, Collected Papers: Supplementary Volume.* Ed: Lawrence D. Brown, *et al.* Springer-Verlag, 1986, vi + 56 pp, $20. [ISBN: 0-387-96383-9] Contains additional commentaries on papers from *Volume I* and *Volume II* of Kiefer's *Collected Papers* (TR, October 1985), which apparently were inadvertently omitted. RKS

Statistics, T(14-17: 1, 2), P. *Quality Control and Industrial Statistics, Fifth Edition.* Acheson J. Duncan. Richard D Irwin, 1986, xxii + 1123 pp, $39.95. [ISBN: 0-256-03535-0] Revision of the author's 1974 *Fourth Edition,* designed to bring the book up-to-date with respect to sampling standards, to make it more cost-oriented, and to bring it more in harmony with the use of computers. Contains major sections on acceptance sampling plans and control charts, in addition to standard statistical topics. RSK

Statistics, S(14-16), P*, L. *Graphical Exploratory Data Analysis.* S.H.C. du Toit, A.G.W. Steyn, R.H. Stumpf. Texts in Stat. Springer-Verlag, 1986, ix + 314 pp, $28. [ISBN: 0-387-96313-8] Provides a survey of the best known and most widely used methods of analyzing and portraying data graphically, amply illustrated with real data. Includes working computer programs for most cases, particularly using the SAS and BMDP packages. RSK

Statistics, T(16-18: 1, 2), S, P, L. *Density Estimation for Statistics and Data Analysis.* B.W. Silverman. Mono. on Stat. & Appl. Prob. Chapman & Hall, 1986, ix + 175 pp, $29.95. [ISBN: 0-412-24620-1] Readable treatment of density estimation (estimation of unknown probability density function from observed data). Emphasis on nonparametric estimation. Includes a survey of existing methods with concentrated treatments of some of these. KK

Statistics, T(13-14: 1), L. *Introduction to Statistics.* J.S. Milton, J.J. Corbet, P.M. McTeer. DC Heath, 1986, xiii + 593 pp, $29.95. [ISBN: 0-669-06209-X] Presupposes no college mathematics. Takes up the usual topics with more emphasis on probability than usual at this level. Optional sections give an introduction to SAS. FLW

Computer Literacy, S, P, L. *The Control Revolution: Technological and Economic Origins of the Information Society.* James R. Beniger. Harvard U Pr, 1986, xi + 493 pp, $25. [ISBN: 0-674-16985-9] A scholarly historical treatise liberally laced with trite aphorisms of the information age ("organization man," "micro millenium," "third wave"). Traces the origin of feedback and control in social systems in an effort to determine why it is information rather than, say, production that plays an increasingly crucial role in society. LAS

Computer Programming, S, L. *Advanced Applications for Introduction to Pascal with Applications in Science and Engineering.* Susan Finger, Ellen Finger. DC Heath, 1986, vi + 138 pp, $1.95 (P). [ISBN: 0-669-12059-6] Intended as a companion to the book named in the title, but useable with other texts. Nine applications and case studies: linear interpolation, discrete-time systems, numerical differentiation, numerical integration, roots of equations, quick sort, determinants, linear systems, matrix operations. Not all students will have the necessary mathematical prerequisites. DFA

Computer Programming, T(14-15: 1), S, L. *The World of Programming Languages.* Michael Marcotty, Henry Ledgard. Books on Prof. Comput. Springer-Verlag, 1987, xiv + 360 pp, $29.95. [ISBN: 0-387-96440-1] A systematic introduction to elements of programming languages (e.g., data types, nesting, dynamically varying structures), one per chapter, with apt illustrations to appropriate programming languages (e.g., PL/I, Pascal, Ada). LAS

Computer Programming, T?(1). *Programming in Micro-Prolog.* Hugh de Saram. Ser. in Comp. & Their Applic. Halsted Pr, 1985, 166 pp, (P). [ISBN: 0-470-20218-1] Aimed at home and high school this book presents a substantial introduction to Prolog via a variety of applications including databases and natural language translation. Unfortunately the variant of the language, called micro-PROLOG, is not close to other versions of Prolog, and there are no comparisons with or references to the more standard implementations. Micro-PROLOG does include its own turtle graphics. JAS

Computer Programming, T*(13: 1). *Introduction to Pascal and Structured Design, Second Edition.* Nell Dale, Chip Weems. DC Heath, 1987, xxi + 825 pp, $27.95 (P). [ISBN: 0-669-09570-2] Over 200 more pages than the *First Edition* (TR, January 1985). More emphasis on problem solving, testing, control and data abstraction, interactive programming. Earlier coverage of procedures. More material on recursion and linked structures; more exercises and nearly twice as many examples. One chapter is devoted entirely to common algorithms applied to arrays. Supplements available for dialects. DFA

Computer Programming, S*(13-18). *6502: Assembly Language Programming, Second Edition.* Lance A. Leventhal. Osborne McGraw-Hill, 1986, xix + 737 pp, $19.95 (P). [ISBN: 0-07-881216-X] This edition conforms to the fine standards of this series and includes full information about the enhanced 65C02 and associated processors. JAS

Computer Programming, T(13-18: 1), S. *A Prolog Primer.* Jean B. Rogers. Addison-Wesley, 1986, xii + 223 pp, $19.95 (P). [ISBN: 0-201-06467-7] Presents many non-numerical examples. Takes a spiral approach. Lots of explanation and relevant discussion connected to topics and examples. First part is a tutorial and begins with a look at language. The second part—advanced topics—covers efficiency, built-in predicates, I/O, arithmetic, building large programs. Exercises; chapter summaries; appendices; indexes. RJA

Computer Programming, T*(13: 1). *Pascal: Problem Solving and Structured Program Design.* Henry M. Walker. Little Brown, 1986, xxiv + 542

pp, $29.75 (P). [ISBN: 0-316-91848-2] Very readable text for the CS-I course. True problem-solving orientation. Good choice of examples and programming exercises. Spiral approach to functions, procedures, loops. A separate chapter on program correctness and accuracy, and on trees and recursion. Worth a good look. DFA

Computer Programming, T*(13: 1). *VAX-11 BASIC By Design: Structured Programming in BASIC.* Andrew Kitchen. Prentice-Hall, 1987, xvii + 492 pp, $21.95 (P). [ISBN: 0-13-940974-2] For beginners. Assumes only high school algebra. Pleasant, even entertaining style. Attractive format. Many examples and exercises. Stresses good programming style but avoids much formalism of structured programming. Suitable for shorter courses and self-study. DFA

Computer Programming, S(13-15). *Advanced Modula-2.* Herbert Schildt. Osborne McGraw-Hill, 1987, x + 379 pp, $18.95 (P). [ISBN: 0-07-881245-3] Advanced programming topics (e.g., sorting, searching, lists, queues, stacks, trees, data compression and codes, expression parsing, some discussion of interfacing and concurrency) usually treated in a data structures or second programming course, all in Modula-2. Readable, but no exercises. RM

Computer Programming, T(14-15: 1). *Programming in Assembly Language: MACRO-11.* Edward F. Sowell. Addison-Wesley, 1984, xix + 492 pp, $28.95. [ISBN: 0-201-07788-4] Number systems, computer arithmetic, assembly language programming, computer organization, addressing modes, I/O, subroutines, macros. RWN

Software Systems, P. *Lecture Notes in Computer Science-236: TEX for Scientific Documentation.* Ed: Jacques Désarménien. Springer-Verlag, 1986, vi + 204 pp, $19.80 (P). [ISBN: 0-387-16807-9] Contributions from the second European TEX conference held in Strasbourg in June 1986. Topics range from interactive environments to multilingual TEX and "theological typesetting"—setting biblical scholarship using mixtures of Latin, Greek, Hebrew, Fraktur, and Gothic fonts. LAS

Software Systems, P. *Product Data Interfaces in CAD/CAM Applications: Design, Implementation and Experiences.* Ed: J. Encarnacão, R. Schuster, E. Vöge. Symbolic Computation. Springer-Verlag, 1986, xiv + 254 pp, $68. [ISBN: 0-387-15118-4] Rapidly increasing applications of graphics in engineering and science has led to a great variety of hardware and software products. The incompatibility of these products poses a major obstacle to the goal of systems integration and has brought about efforts to develop standard product data interfaces. This book describes on-going work in this area and contains papers presented in a seminar of the Zentrum für Graphische Datenverarbeitung held at the Technical University Darmstadt from December 1984 to February 1985. AM

Software Systems, S(13-15), P, L. *UNIX Survival Guide.* Elizabeth A. Nichols, Sidney C. Bailin, Joseph C. Nichols. Holt, Rinehart & Winston, 1987, vii + 311 pp, $15.25 (P). [ISBN: 0-03-000773-9] A well-written introduction to UNIX especially designed for the increasing numbers of new users who are also serving as system managers—because they are using UNIX on small office systems. After files, directories, and editors are introduced, major later chapters deal with processes, utilities, and file system maintenance. A good book from which to learn how UNIX works, not just what the commands do. LAS

Computer Science, P. *Pyramidal Systems for Computer Vision.* Ed: Virginio Cantoni, Stefano Levialdi. NATO ASI Ser. F, V. 25. Springer-Verlag, 1986, viii + 392 pp, $82.50. [ISBN: 0-387-17165-7] Workshop proceedings on pyramid computers (stacks of smaller arrays linked vertically by trees; generalizing arrays and hypercubes). Papers discuss pyramidal languages, algorithm paradigms, digital transforms, VLSI fabrication, and applications to vision, image segmentation, tactile systems. RM

Computer Science, S, L. *Taming the Tiger: Software Engineering and Software Economics.* Leon S. Levy. Books on Prof. Comput. Springer-Verlag, 1987, viii + 248 pp, $25 (P). [ISBN: 0-387-96468-1] An idiosyncratic personal essay on a philosophy of programming (illustrated by AWK) and on models for estimating software development costs, together with a program (with sample runs) to produce these estimates. LAS

Computer Science, P, L. *Software Engineering Education: The Educational Needs of the Software Community.* Ed: Norman E. Gibbs, Richard E. Fairley. Springer-Verlag, 1987, xvi + 439 pp, $32. [ISBN: 0-387-96469-X] Proceedings of a February 1986 workshop held at Carnegie-Mellon University at which leaders in computer science and software engineering from academia, industry, and government assessed the current state of undergraduate and master's education in software engineering. LAS

Computer Science, P, L. *Annual Review of Computer Science, Volume 1, 1986.* Ed: Joseph F. Traub, *et al.* Annual Reviews, 1986, xiv + 459 pp, $39. [ISBN: 0-8243-3201-6] First volume in a new series of *Annual Reviews.* Fifteen survey papers on a wide variety of topics, from natural language interfaces to dataflow architectures. Integrated subject index. LAS

Computer Science, P. *Lecture Notes in Computer Science-213: ESOP 86.* Ed: B. Robinet, R. Wilhelm. Springer-Verlag, 1986, 374 pp, $22.70 (P). [ISBN: 0-387-16442-1] The proceedings of the European sym-

posium on programming held in Saarbrücken, West Germany, March 17-19, 1986. JAS

Computer Science. *Lecture Notes in Computer Science-212: Interval Mathematics 1985.* Ed: K. Nickel. Springer-Verlag, 1986, vi + 227 pp, $16.40 (P). [ISBN: 0-387-16437-5] Proceedings of a symposium held in Freiburg, West Germany, from September 23-26, 1985. JAS

Computer Science, P. *Design and Analysis of Coalesced Hashing.* Jeffrey Scott Vitter, Wen-Chin Chen. Intern. Ser. of Mono. on Comput. Sci. Oxford U Pr, 1987, xii + 160 pp, $29.95. [ISBN: 0-19-504182-8] Algorithms and analysis of hashing methods for internal search where colliders with different hash addresses are coalesced into single chains. Memory divided into fixed size address portion for hash function, and "cellar" for colliders (which may overflow back into address portion). Analysis of tradeoffs between relative sizes of partition, and of algorithms for insertion to produce optimal performance. RM

Computer Science, P. *New Computing Environments: Parallel, Vector and Systolic.* Ed: Arthur Wouk. SIAM, 1986, 270 pp, $29.50. [ISBN: 0-89871-201-7] Survey articles on recent accomplishments (interaction between hardware, software, algorithms) in parallelism in large scale scientific computations. Discussion of communication costs as well as operation count measures, tools for detecting and utilizing implicit parallelism, granularity at which parallelism is implementable, systolic architecture, local/global memory organization. RM

Computer Science, S(16-18), P, L. *Lecture Notes in Computer Science-138: Negation and Control in Prolog.* Lee Naish. Springer-Verlag, 1986, ix + 119 pp, $14.90 (P). [ISBN: 0-387-16815-X] Work based on author's Ph.D. thesis and done in conjunction with the development of the MU-Prolog system. Text is divided into two parts. First, discussion of how negation as failure can be implemented soundly, including enhancements to curent systems and suggestions for future systems. Second, the control of logic programs is treated. Introduces control primitives for database and recursive predicates and uses them to generate control information automatically. This is followed by a re-examination of the theoretical foundations of Prolog systems with flexible computation rules. Two appendices, one being a MU-Prolog reference manual. References. RJA

Computer Science, T(13-18: 1, 2), S, P, L. *Programming with Sets: An Introduction to SETL.* J.T. Schwartz, *et al.* Texts & Mono. in Comput. Sci. Springer-Verlag, 1986, xv + 493 pp, $45. [ISBN: 0-387-96399-5] SETL, a language that manipulates general finite sets and maps, is a tool for experimenting with algorithms and program design and for prototyping large systems. Presents major data objects, control structures, program development, testing, backtracking. Latter sections contain advanced material including substantial applications. Exercises; appendices; index. RJA

Computer Science, T(16-18: 1, 2), S, L. *Denotational Semantics: A Methodology for Language Development.* David A. Schmidt. Allyn & Bacon, 1986, xiii + 331 pp. [ISBN: 0-205-08974-7] Begins with a survey of semantics specification methods. Covers syntax, semantic domains, denotational semantics, semantics of computer storage, least fixed point semantics, block structure, and data structures. Advanced topics are treated in chapters toward the end of the book. Exercises; suggested readings; bibliography; index. RJA

Computer Science, P, L. *Lecture Notes in Computer Science-235: Accurate Scientific Computations.* Ed: Willard L. Miranker, Richard A. Toupin. Springer-Verlag, 1986, xiii + 205 pp, $19.80 (P). [ISBN: 0-387-16798-6] Concerns applications in mathematics. Different concepts and definitions of "accuracy," ways to achieve it efficiently, algorithms to prove or validate it. Nine papers and three abstracts on evaluating elementary functions, axioms for computer arithmetic, computing inclusions, computer architectures, probabilistic algorithms, computer algebra. From a March 1985 symposium at Bad Neuenahr, West Germany. DFA

Computer Science, P. *Lecture Notes in Computer Science-242: Combinators and Functional Programming Languages.* Ed: Guy Cousineau, Pierre-Louis Curien, Bernard Robinet. Springer-Verlag, 1986, v + 208 pp, $20 (P). [ISBN: 0-387-17184-3] Proceedings of the Thirteenth Spring School of the Laboratoire Informatique Théorique et Programmation, Universités Paris VI-VII and CNRS, held May 6-10, 1985. Papers are mostly tutorial and present the different available formalisms of functions, new and efficient implementation techniques, and two functional programming languages (Amber and Graal). RJA

Computer Science, P. *Lecture Notes in Computer Science-241: Foundations of Software Technology and Theoretical Computer Science.* Ed: Kesav V. Nori. Springer-Verlag, 1986, xii + 519 pp, $36.60 (P). [ISBN: 0-387-17179-7] Proceedings of the Sixth Conference, New Delhi, December 1986. Contains invited talks and sessions on software technology, logic programming and functional programming, algorithms, theory, distributed computing, scheduling, complexity, parallel algorithms. RJA

Applications, T(16), P, L. *Decision Theory: An Introduction to the Mathematics of Rationality.* Simon French. Math. & Its Applic. Halsted Pr, 1986, 448 pp, $54.95. [ISBN: 0-470-20308-0] Aimed at undergraduates, cognizant that not many undergradu-

ates take a course entirely on decision theory, so written as background reading for students in management science, psychology, political science as well as operations research and applied statistics. Good bibliographies on selected topics. Preface claim of prerequisites (a little calculus, a little probability, mathematical maturity for following an argument and understanding notation) seems realistic. An attractive, albeit unusual book, worth looking at. AWR

Applications. *Deterministic Aspects of Mathematical Demography.* John Impagliazzo. Biomath., V. 13. Springer-Verlag, 1985, xi + 186 pp, $34. [ISBN: 0-387-13616-9] A concise exposition of the traditional stable theory of population, from mortality tables to age-specific matrix models, illustrated with extensive tables and graphs of data from Denmark for several centuries. An excellent primer on notation and basic theory. LAS

Applications, P. *Lecture Notes in Biomathematics-66: Nonlinear Oscillations in Biology and Chemistry.* Ed: H.G. Othmer. Springer-Verlag, 1986, vi + 289 pp, $24 (P). [ISBN: 0-387-16481-2] Papers on biological systems, chemical systems, and mathematical methods involving nonlinear dynamics, chaotic behavior, and oscillating systems. From a May 1985 conference sponsored by the Mathematics Department of the University of Utah. LAS

Applications, P. *Mathematics Applied to Fluid Mechanics and Stability: Proceedings of a Conference Dedicated to Richard C. DiPrima.* Ed: Donald A. Drew, Joseph E. Flaherty. SIAM, 1986, xii + 295 pp, $38.50. [ISBN: 0-89871-208-4] The state-of-the-art and future directions in singular perturbations, bifurcation and stability as applied to fluid mechanics and lubrication. Based on 22 lectures given at Rensselaer Polytechnic Institute, September 9-11, 1985. DFA

Applications (Artificial Intelligence), T*(16-18: 1, 2), S, L. *The Elements of Artificial Intelligence: An Introduction Using LISP.* Steven L. Tanimoto. Princip. of Comput. Sci., V. 11. Computer Science Pr, 1987, xxii + 529 pp, $35.95. [ISBN: 0-88175-113-8] Presents principles and main programming techniques of artificial intelligence. Begins with introduction to field, its relationships to other disciplines, and a word about the literature on artificial intelligence. Chapter two presents LISP which is used in the rest of the text to illustrate examples and present algorithms. Chapters on methodology, knowledge representation, search, logical and probabilistic reasoning, learning, natural language understanding, vision, expert systems. Concluding chapter focuses on the future. Chapter references. Exercises. Appendix of LISP functions. Author and subject indexes. RJA

Applications (Astronomy). *Building Blocks of the Universe.* Lorenzo Eric Sepulveda. Water Chemistry Eastex (POB 6432, Longview, TX 75608), 1986, 119 pp, $10 (P). It's a happy thought that mathematics can still be advanced by "amateurs." This treatise, by a self-proclaimed amateur mathematician (a chemist by profession), is primarily "a mathematical treatise about the geometry of four spatial dimensions, concerning those aspects of hypergeometry that are directly relevant to cosmology." In review of the manuscript, H.M.S. Coxeter writes "Your monograph...might well be regarded as an up-to-date version of my joint paper with G.J. Whitrow: 'World Structure and Non-Euclidean Honeycombs,' *Proc. Royal Soc.* A201 (1950) 417-437." LCL

Applications (Biology), P, L. *Lecture Notes in Biomathematics-69: The Mathematical Structure of the Human Sleep-Wake Cycle.* Steven H. Strogatz. Springer-Verlag, 1986, viii + 239 pp, $22.80 (P). [ISBN: 0-387-17176-2] A well-written comprehensive analysis of the "Rosetta Stone" of data from 22 human subjects who have undertaken lengthy free-sleep experiments in environments totally isolated from clues to the 24 hour day. This analysis leads to surprisingly strong laws of correlation and independence between natural length of sleep, gaps between sleep, and the circadian rhythm of body temperature, as well to striking evidence for alternating periods of cyclic order and nearly random chaos. Concluding with various mathematical models and computer simulation. LAS

Applications (Biology), P. *Lecture Notes in Biomathematics-67: Intrinsic Geometry of Biological Surface Growth.* Philip H. Todd. Springer-Verlag, 1986, iv + 128 pp, $14.20 (P). [ISBN: 0-387-16482-0] An exploration of how the intrinsic geometry of living organisms changes as they grow, based on a detailed application of differential geometry (e.g., Dirichlet integral, quadric surface, numerical approximations) applied to the post-natal folding of ferret brains. LAS

Applications (Economics), P, L*. *Lecture Notes in Economics and Mathematical Systems-271: The Cowles Commission in Chicago, 1939-1955.* Clifford Hildreth. Springer-Verlag, 1986, v + 176 pp, $19.30 (P). [ISBN: 0-387-16774-9] An intellectual history of econometric ideas developed during the Chicago phase of the Cowles Commission for Research in Economics under the directorships of Jacob Marshak and Tjalling Koopmans: identification in linear probability models, activity analysis, equilibrium, social choice. LAS

Applications (Economics), T(18), S, P. *Lecture Notes in Economics and Mathematical Systems-254: Arbitrage Pricing of Contingent Claims.* Sigrid Müller. Springer-Verlag, 1985, viii + 151 pp, $14.80 (P). [ISBN: 0-387-15973-8] "This book is intended as a contribution to the theory of contingent claim valuation based on arbitrage considerations. It is con-

cerned with preference-free valuations of contingent claims (such as options written on a stock) in frictionless multiperiod securities markets that do not permit arbitrage profits. Besides the question of pricing it considers the possibility of hedging in securities markets." KK

Applications (Engineering), T(15-16: 1, 2), S, L. *Mathematical Foundations for Communication Engineering.* Kenneth W. Cattermole. Halsted Pr, 1985, $33.95 each. *Volume 1: Determinate Theory of Signals and Waves,* x + 287 pp [ISBN: 0-470-20176-2]; *Volume 2: Statistical Analysis and Finite Structures,* ix + 357 pp. [ISBN: 0-470-20177-0] Mathematics for communication engineering, based on the unity of various linear transforms. *Volume 1*: Fourier transforms in one and several variables. *Volume 2*: Probability and abstract algebra, including coding. Applications throughout to signal processing and filtering. BC

Applications (Engineering), T(15-16). *Mathematical Methods, Second Edition.* Merle C. Potter, Jack Goldberg. Prentice-Hall, 1987, xvi + 639 pp, $45.95. [ISBN: 0-13-561184-9] Topics for engineers: differential equations, series, Laplace transforms, matrices, Fourier series, numerical methods, complex variables. *First Edition* by Potter alone tried to make topics from advanced mathematics accessible to undergraduate students (TR, February 1979); *Second Edition* includes Goldberg to add some mathematical precision while retaining the undergraduate focus. AWR

Applications (Engineering), T(17: 1), S, P, L. *Theory of Matrix Structural Analysis.* J.S. Przemieniecki. Dover, 1985, xi + 468 pp, $10 (P). [ISBN: 0-486-64948-2] A corrected and unabridged republication of a book which originally appeared in 1968. Includes an overview of matrix methods applied to the design of aircraft, basic equations of elasticity, equilibrium and compatibility equations, energy theorems, structural idealization, Castigliano's theorem, structured synthesis and nonlinear structural analysis. Also exercises and many illustrations. CEC

Applications (Engineering), S(14-16), L. *Worked Examples in Engineering Mathematics.* L.R. Mustoe. Wiley, 1986, x + 111 pp, $14.95 (P). [ISBN: 0-471-91171-2] Fifty-three mostly two-part sample problems, with solutions, taken from college engineering exams. The problems aren't bad, but the book is too slim for the price. BC

Applications (Engineering), P. *Design Sensitivity Analysis of Structural Systems.* Edward J. Haug, Kyung K. Choi, Vadim Komkov. Math. in Sci. & Eng., V. 177. Academic Pr, 1986, xvi + 381 pp, $34.95 (P); $60. [ISBN: 0-12-332921-3; 0-12-332920-5] For both engineer and mathematician. Considers linear structural mechanics, where the equations (matrix, ordinary differential, partial differential) are linear in the state variables once the design variable is fixed. Finite-dimensional problems, distributed parameter structural components, structural components with shape as the design, built-up structures. Two of the authors are with the Center for Computer Aided Design at the University of Iowa. DFA

Applications (Engineering), P. *Accuracy Estimates and Adaptive Refinements in Finite Element Computations.* Ed: I. Babuska, *et al.* Ser. in Num. Methods in Engin. Wiley, 1986, xiv + 393 pp, $84.95. [ISBN: 0-471-90862-2] 21 papers by invited participants at an international conference in Lisbon in June 1984. Thirteen papers address stationary, elliptic problems; the remainder address transient and flow problems. DFA

Applications (Engineering), T(18: 2), S. *Application of Structural Systems Reliability Theory.* Palle Thoft-Christensen, Yoshisada Murotsu. Springer-Verlag, 1986, viii + 343 pp, $58. [ISBN: 0-387-16362-X] A graduate text covering relatively new theory of reliability of systems (e.g., offshore oil platforms). Brief introduction to the theory of single structures (e.g., single beams), and level two methods. Extensive examples, including numerical, and exercises. MR

Applications (Engineering), T(18: 2), S. *Variational Principles of Continuum Mechanics with Engineering Applications, Volume 1: Critical Points Theory.* Vadim Komkov. Math. & Its Applic. D Reidel, 1986, viii + 387 pp, $59. [ISBN: 90-277-2157-2] New approach to "old" theory of variational methods normally taught in mathematical methods course. Primarily focuses on critical point theory and its applications to continuum mechanics. Attempts to incorporate modern functional analysis into standard engineering problems such as elastic stability. Many excellent examples and problems interspersed in text to illustrate techniques. MR

Applications (Linguistics), P. *Foundations and Applications of Montague Grammar, Part 1: Philosophy, Framework, Computer Science.* T.M.V. Janssen. CWI Tract V. 19. Math Centrum, 1986, iv + 205 pp, Dfl. 31.30 (P). [ISBN: 90-6196-292-7] Based on author's dissertation. Each chapter starts with an abstract. Chapters on the principle of compositionality of meaning, the algebraic framework for the study, intensional logic. Montague grammar and programming languages. Appendix; index of names; references. RJA

Applications (Physics), P. *Proceedings Seminar 1983-1985: Mathematical Structures in Field Theories, V. 1, Geometric Quantization.* G.M. Tuynman. CWI Syllabus, V. 8. Math Centrum, 1985, iii + 158 pp, Dfl. 22.70 (P). [ISBN: 90-6196-293-5] The notes for the author's lectures which were part of the

1983-85 seminar held at the Centrum voor Wiskunde en Informatica in Amsterdam. Emphasizes thorough coverage, with proofs, of parts of geometric quantization theory. JAS

Applications (Physics), T(16-18: 1, 2), S, P. *A Course in Mathematical Physics, V. 2: Classical Field Theory, Second Edition.* Walter Thirring. Transl: Evans M. Harrell. Springer-Verlag, 1986, x + 261 pp, $35. [ISBN: 0-387-96266-2] In addition to correcting minor mistakes in the original edition, this version contains a section on gauge theories. The mathematical language, differential geometry, remains unchanged. MU

Applications (Physics), T(16-18: 1), S, L. *An Introduction to Twistor Theory.* S.A. Huggett, K.P. Tod. London Math. Soc. Stud. Texts, V. 4. Cambridge U Pr, 1985, 145 pp, $13.95 (P). [ISBN: 0-521-31361-9] This introduction to twistor theory and modern geometrical approaches to space-time structure evolved from graduate lectures given in London and Oxford. Topics include: spinor algebra, compactified Minkowski space, the geometries of null congruences and twistor space, sheaf cohomology, the active twistor constructions which solve the self-dual Yang-Mills and Einstein equations, and Penrose's quasi-local-mass construction. MU

Applications (Physics), S(18), P. *Ill-posed Problems of Mathematical Physics and Analysis.* M.M. Lavrent'ev, V.G.Romanov, S.P. Shishatskii. Transl. of Math. Mono., V. 64. AMS, 1986, vi + 290 pp, $98. [ISBN: 0-8218-4517-9] Problems, ill-posed in the Hadamard sense, arising from the interpretation of geophysical data, are considered in the context of advanced applied mathematics. The authors attempt to fill the following gaps in the current literature: ill-posed problems for concrete types of differential equations, problems of analytic continuation, inverse problems for differential equations, and problems of integral geometry. MU

Applications (Physics), P. *Current Algebra and Anomalies.* Sam B. Treiman, *et al.* Ser. in Physics. Princeton U Pr, 1985, xi + 537 pp, $54; $26 (P). [ISBN: 0-691-08397-5; 0-691-08398-3] Current algebras are used in the analysis of fundamental particle interactions. Contains surveys on current algebra and anomalies derived in part from lectures given at the Brookhaven Summer School in Theoretical Physics in 1970, and lectures given at the 1983 Les Houches Summer School. AM

Applications (Physics), P. *Dynamical Problems in Continuum Physics.* Ed: J.L. Bona, *et al.* IMA, V. 4. Springer-Verlag, 1987, xii + 321 pp, $28. [ISBN: 0-387-96463-0] Sixteen contributions to the 1984-85 IMA program in Minnesota on nonlinear behavior of matter and waves, such as fluid jet instability, piezoelectricity, and acoustic waves in (stressed) elastic materials. BC

Applications (Physics), P. *Wave Propagation and Scattering.* Ed: B.J. Uscinski. Inst. of Math. & Its Applic., Conf. Ser., V. 5. Clarendon Pr, 1986, x + 381 pp, $59. [ISBN: 0-19-853607-0] A collection of papers presented at the University of Cambridge in April 1984. MU

Applications (Physics), S(18), P. *Essays on Supersymmetry.* Ed: C. Fronsdal, *et al.* Math. Phy. Stud., V. 8. D Reidel, 1986, x + 270 pp, $54.95. [ISBN: 90-277-2207-2] Organized into four sections: "Unitary Representations of Supergroups," a study of unitarizable representations of noncompact superalgebras; "3 + 2 de Sitter Superfields," examining the reduction of de Sitter superfields into irreducible representations of $osp(4/1)$; "Spontaneously Generated Field Theories, Zero-center Modules, Colored Singletons and the Virtues of $N = 6$ Supergravity," which is devoted to electrodynamics; and "Massless Particles, Orthosymplectic Symmetry and Another Type of Kaluza-Klein Theory," which was inspired by twistor theory. MU

Applications (Physics), S(18), P. *Homogenization and Effective Moduli of Materials and Media.* Ed: J.L. Ericksen, *et al.* IMA Vol. in Math. & Its Applic., V. 1. Springer-Verlag, 1986, x + 263 pp, $22.50. [ISBN: 0-387-96306-5] A collection of papers presented at a workshop on homogenization of differential equations and the determination of effective moduli of materials and media, primarily in the context of continuum theory. Applications include elastic and dielectric responses of composites, and the effective properties of shales and soils. MU

Applications (Physics), P. *Supersymmetry and Its Applications: Superstrings, Anomalies, and Supergravity.* Ed: G.W. Gibbons, S.W. Hawking, P.K. Townsend. Cambridge U Pr, 1986, xiv + 481 pp, $49.50. [ISBN: 0-521-30721-X] In physical terms, supersymmetry is a system which describes the symmetry of bosons (particles with integer spin), and fermions (particles with half integer spin) within a unified framework. Contains the proceedings of the workshop on supersymmetry and its applications which took place in the department of applied mathematics and theoretical physics of Cambridge University, June 23-July 14, 1985. AM

Applications (Physics), P. *Computational Methods for Kinetic Models of Magnetically Confined Plasmas.* J. Killeen, *et al.* Ser. in Computat. Physics. Springer-Verlag, 1986, viii + 199 pp, $38. [ISBN: 0-387-13401-8] Applied kinetic theory can be used to develop computer simulations of magnetically confined plasmas. In particular, kinetic models based on the Boltzmann equation with Fokker-Planck collision terms have been successful in rep-

resenting plasma behavior. Describes a number of Fokker-Planck computational models developed over the last decade. AM

Applications (Physics), P. *Generalized Coherent States and Their Applications.* A. Perelomov. Texts & Mono. in Physics. Springer-Verlag, 1986, xi + 320 pp, $65. [ISBN: 0-387-15912-6] Coherent states arise in quantum mechanics. The standard coherent states describe nonspreading wave packets for the harmonic oscillator. This book explores the relation between the standard coherent states system and the Heisenberg-Weyl group. This relationship generalizes the notion of coherent states systems to other Lie groups. These systems are then applied to a variety of physical problems. AM

Applications (Physics), P*.** *Quantum Mechanics from General Relativity: An Approximation for a Theory of Inertia.* Mendel Sachs. Fund. Theo. of Physics. D Reidel, 1986, xvi + 227 pp, $69. [ISBN: 90-277-2247-1] Proposes a new theory of matter in opposition to the Copenhagen school. Assuming Einstein's general relativity, a new quantum mechanics is shown to follow as a linear approximation of inertia. Makes new relativistic and non-relativistic predictions. Demonstrates logical incompatibility of special relativity and old quantum mechanics. Proves an equivalent of Pauli Exclusion Principle. Philosophical tenets and ramifications thoroughly discussed. Very interesting. MR

Applications (Physics), S(18). *Relativistic Electrodynamics and Differential Geometry.* Stephen Parrott. Springer-Verlag, 1987, xi + 308 pp, $48. [ISBN: 0-387-96435-5] Mathematical formulation of classical relativistic electrodynamics (i.e., no general relativity). Addresses problem of choice of Lorentz-Dirac equation to govern motion of charged particle. Assumes differential geometry, basic electrodynamics and special relativity. Good book for mathematicians. MR

Applications (Physics), P. *Unification and Supersymmetry: The Frontiers of Quark-Lepton Physics.* Rabindra N. Mohapatra. Contemp. Physics. Springer-Verlag, 1986, xii + 309 pp, $34. [ISBN: 0-387-96285-9] This book traces the development of theoretical models of elementary particle interactions. Based on a graduate course offered at CCNY; assumes as prerequisites knowledge of quantum field theory, the theory of Lie groups, basic particle theory, and familiarity with spontaneously broken gauge theories. AM

Applications (Physics), P. *Computer Simulation in Physical Metallurgy.* Ed: Gianni Jacucci. D Reidel, 1986, viii + 311 pp, $58. [ISBN: 90-277-2192-0] Lectures from a May 1984 conference held in Ispra, Italy on mathematical and computer models of solids--fractures, surface effects, flows and plasticity, etc. LAS

Applications (Physics), T(16-17: 1). *A Concept of Mathematical Physics: Models in Mechanics.* Tamás Matolcsi. Akademiai Kiado, 1986, 335 pp, $34. [ISBN: 963-05-3627-7] Introduces the topics of classical mechanics, quantum mechanics, special relativity, and field theory together with the appropriate mathematical background from probability, functional analysis, symplectic geometry, and group representations. Emphasizes the notion of relativity—the idea that a physical theory must establish a rule connecting measurements made by observers with absolute objects. AM

Applications (Physics), P. *Rotations, Quaternions, and Double Groups.* Simon L. Altmann. Clarendon Pr, 1986, 317 pp, $49. [ISBN: 0-19-855372-2] Written for an audience such as molecular and solid-state theorists, and crystallographers. Considers rotations from geometric and quaternionic viewpoint and avoids Lie group theory. LC

Applications (Physics), P. *Lecture Notes in Mathematics-1174: Categories in Continuum Physics.* Ed: F.W. Lawvere, S.H. Schanuel. Springer-Verlag, 1986, 126 pp, $11.60 (P). [ISBN: 0-387-16096-5] Proceedings of a 1982 conference held at SUNY at Buffalo in May 1982 which presents some efforts to find better categories of geometrical objects for physics. JAS

Applications (Physics), P. *Hamiltonian Structure and Lyapunov Stability for Ideal Continuum Dynamics.* D.D. Holm, J.E. Marsden, T.S. Ratiu. Pr U Montreal, 1986, 208 pp, $20 (P). [ISBN: 2-7606-0771-2] Two long papers on mathematical physics; lecture notes from an August 1985 NATO seminar on nonlinear dynamical systems, held at the University of Montreal. PZ

Reviewers

RJA: Richard J. Allen, St. Olaf; DFA: David F. Appleyard, Carleton; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; LC: Laura Chihara, St. Olaf; BC: Barry Cipra, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; JD-B: John Dyer-Bennet, Carleton; GF: Giovanna Fjelstad, St. Olaf; SG: Steven Galovich, Carleton; BH: Bruce Hanson, St. Olaf; PH: Paul Humke, St. Olaf; KK: Kenneth Kaminsky, St. Olaf; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; AM: Alan Magnuson, St. Olaf; SM: Steve McKelvey, St. Olaf; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; MR: Matthew Richey, St. Olaf; AWR: A. Wayne Roberts, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MU: Milton Ulmer, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; MZ: Michael Zielinski, St. Olaf; PZ: Paul Zorn, St. Olaf.

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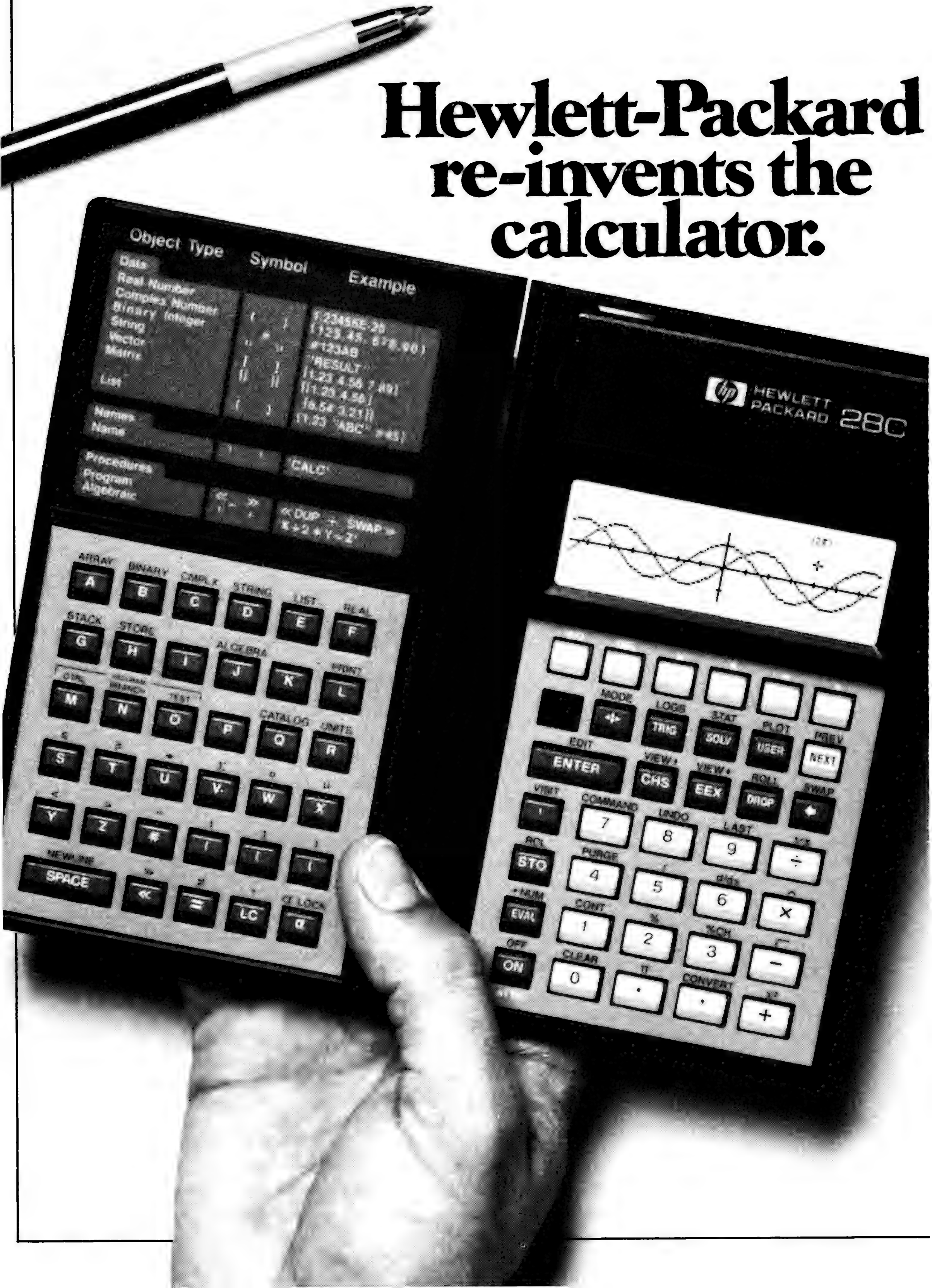

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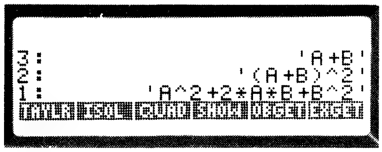

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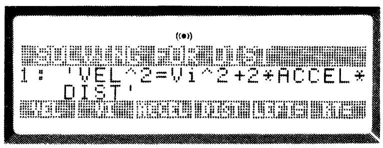

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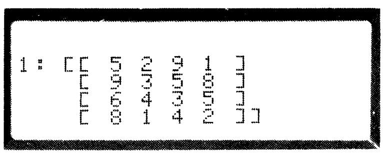

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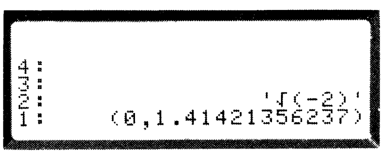

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THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 7 August-September 1987

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(ISSN 0002-9890)

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Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

Fourteen Proofs of a Result About Tiling a Rectangle

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STAN WAGON: I received my undergraduate degree at McGill and my doctorate at Dartmouth, in set theory under James Baumgartner. Recently my work has centered around expository writing: *The Banach-Tarski Paradox* was published in 1985 (Cambridge), and a series of eight articles on numerical evidence for various conjectures appeared in *The Mathematical Intelligencer* in 1985–6.

1. Introduction. In [2] (see also [5]) N. G. de Bruijn proved a result about packing n-dimensional bricks into an n-dimensional box that, when $n = 2$, implies that if an $a \times b$ rectangle is tiled with copies of a $c \times d$ rectangle, then each of c, d divides one of a, b. By a *tiling* we mean a covering with interior pairwise-disjoint sets. De Bruijn's proof has been generalized to yield the following more general theorem (illustrated in Figure 1), which implies his result on bricks (in the case $n = 2$, divide each side of the box by c (resp., d)).

THEOREM 1. *Whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.*

At the 1985 Summer Meeting of the MAA in Laramie, Wyoming, Hugh Montgomery mentioned this theorem and the proof using double integrals, in the hope of stimulating a search for more elementary proofs. That he did, as proofs have been forthcoming from various countries. Indeed, the variety of techniques that have been brought to bear is striking. Paul Erdös [1, p. 87] has suggested that "[God] has a transfinite book of theorems in which the best proofs are written." It is by no means clear which of the many proofs that follow is the best (the criteria for inclusion in the book are not readily available!). Perhaps none of these proofs is in the book, and the "best" proof has yet to be discovered. Even if simplicity is taken as the criterion, it is not completely clear which proof wins—the checkerboard and bipartite graph proofs seem to be the top candidates. And if strength is taken into account, that is, the ability to yield, perhaps with modification, more general results, then the situation is complicated. Variations of the theorem are true on the cylinder and torus, in higher dimensions, and for multiple tilings, but no one of the proofs is best in terms of its ability to generalize. Before reading Section 3 the reader might enjoy trying to predict which of the proofs are most likely to generalize.

Max Zorn has pointed out that Dehn considered similar questions in 1903. Dehn [3, p. 327] proved, as a corollary to a rather different sort of investigation, that if a rectangle is tiled as in Theorem 1, then one of the sides is rational.

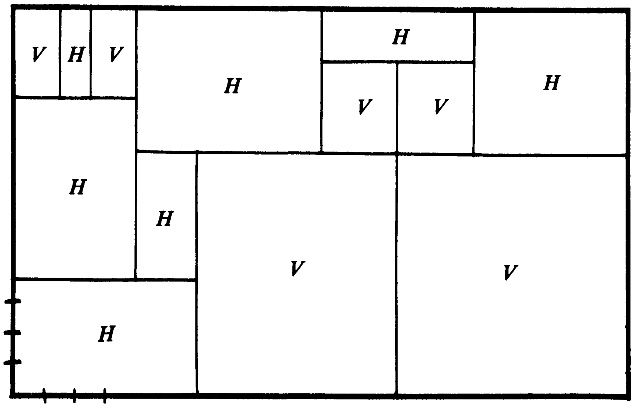


FIG. 1. An example of a tiling in which each tile has at least one integer side. The tiles labelled "H" have integer width; those labelled "V" have integer height.

2. The proofs. The *width* (resp., *height*) of a rectangle denotes its horizontal (resp., vertical) dimension. Given a tiling as in Theorem 1, let R denote the ambient rectangle. Let a tile with integer width be called an *H-tile* ("horizontal tile"); the other tiles, necessarily having integer height, are called *V-tiles* ("vertical tiles"). It is often assumed that R is in *standard position*, that is, its lower left corner is at the origin and its sides are parallel to the coordinate axes in the x-y plane.

(1) *Complex double integral* (extends original method of de Bruijn) First observe that $\int_a^b \sin 2\pi x\,dx = 0$ if and only if one of $a \pm b$ is an integer and $\int_a^b \cos 2\pi x\,dx = 0$ if and only if one of $a - b$, $a + b - 1/2$ is an integer. It follows that for any rectangle T in the x–y plane with sides parallel to the axes,

$$\iint_T e^{2\pi i(x+y)}dA = 0$$

if and only if at least one side of T has integer length. Now, the hypothesis implies that the double integral over each tile vanishes and therefore, by additivity of integrals, the double integral over R is zero. This implies that either the width or height of R is an integer. ■

(2) *Real double integral* (variation of complex double integral proof) Assume R is an $a \times b$ rectangle in standard position. As in the preceding proof, $\iint_T \sin 2\pi x \sin 2\pi y\,dA = 0$ for each tile T. Therefore, the double integral over R vanishes, which, because R has a corner at $(0,0)$, implies that at least one of a, b is an integer. (One could use other integrands as well, for example, $(x - [x] - 1/2)$, $(y - [y] - 1/2)$.) ■

(3) *Checkerboard* (Richard Rochberg, Washington Univ.; Sherman K. Stein) Place R in standard position. Color the square lattice generated by a $(1/2) \times (1/2)$ square with lower left corner at $(0,0)$ in black/white checkerboard fashion. Since

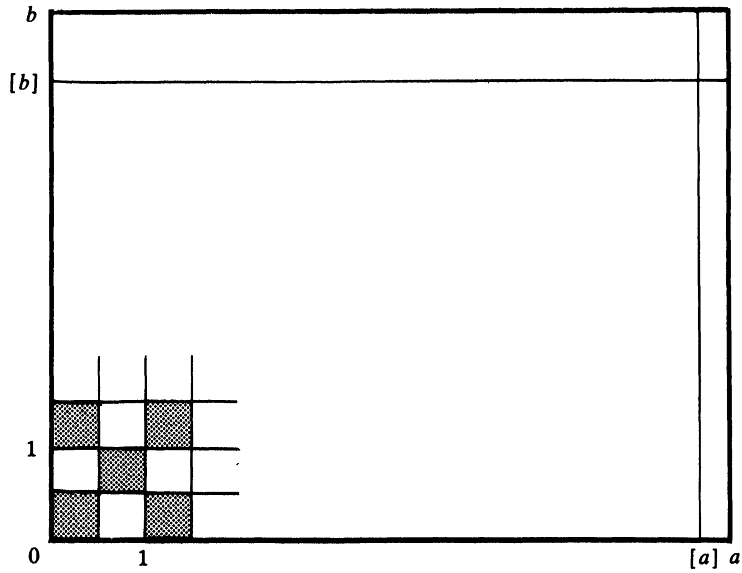


FIG. 2. If neither a nor b is an integer then the upper right corner has more black than white.

each tile has an integer side, each tile contains an equal amount of black and white. Therefore, the same is true of R. But then R must have an integer side for otherwise it can be split into four pieces (see Figure 2), three of which have equal amounts of black and white while the fourth does not. (This proof is derived from the preceding proof by using the integrand $(-1)^{[2x]}(-1)^{[2y]}$.) ■

(4) *Counting squares* (Imre Z. Ruzsa, Mathematical Institute of the Hungarian Academy of Sciences, Budapest; Peter Gilbert, Digital Equipment Corp., Nashua, NH) Place R in standard position and let $\{x_i\}$ (resp., $\{y_j\}$) be the set of x-coordinates of vertical (resp., y-coordinates of horizontal) boundary lines of tiles. Construct an auxiliary tiling (of a possibly new rectangle R') by translating all line segments in R's tiling as follows. If a segment is on a line corresponding to an integer value of x_i or y_j, it is not moved. If it is a vertical segment lying on $x = x_i$, where x_i is not an integer, translate it rightward or leftward to the line $x = [x_i] + 1/2$. Similarly, vertical segments on $y = y_j$ are translated up or down to $y = [y_j] + 1/2$, if y_j is not an integer. This construction may reduce the number of tiles, but this is unimportant.

Now, if the conclusion is false then R' is a rectangle in standard position having both side-lengths equal to one-half of an odd integer. Hence, R' contains an odd number of squares in the (uncolored) checkerboard described in the previous proof. But the hypothesis implies that each tile in R' has an even number of squares, contradiction. ■

(5) *Polynomials* (Adrien Douady, École Normale Supérieure, Paris) Place R in standard position and construct an auxiliary tiling in a way similar to the preceding proof. Choose a parameter t and translate only those segments having a noninteger coordinate. Translate vertical segments on $x = x_i$ rightward to $x = x_i + t$, and horizontal segments upward to $y = y_j + t$. If t comes from a sufficiently small interval $[0, \varepsilon]$, this construction yields a tiling of R', with the same number of tiles as in R.

Now, if the conclusion is false then R' is an $(a + t) \times (b + t)$ rectangle whence its area is a quadratic polynomial in t. But the hypothesis implies that each $w \times h$ tile in R becomes, in R', a tile of one of the forms $w \times (h \pm t)$, $(w \pm t) \times h$, $w \times h$. In all cases the area of the modified tile is a linear or constant function of t, and, hence, the same is true of the area of R'. Since t can take on any value in an interval, this contradicts the quadratic representation of the area. ∎

(6) *Prime numbers* (Raphael Robinson, Univ. of California, Berkeley) We claim that for each prime p, either the height or width of R is within $1/p$ of an integer. It follows that one of these is an integer. To prove the claim, scale the entire tiling up by a factor of p in each direction, and consider the tiling obtained by replacing all tile-corners (x, y) in the scaled-up tiling by $([x], [y])$. This yields an integer-sided rectangle tiled by integer-sided rectangles, each of which has one side a multiple of p. Therefore, the area of the large integer-sided rectangle is a multiple of p, whence one of its sides must be a multiple of p. Moreover, the dimensions of this rectangle differ from the dimensions of the scaled-up rectangle by less than 1. It follows that R has a side that differs from an integer by less than $1/p$. ∎

(7) *Eulerian path* (Michael S. Paterson, Univ. of Warwick, Coventry, England) Let Γ be the graph whose vertices are the corners of all the tiles, with two vertices joined whenever they correspond to the ends of a horizontal side of an H-tile or the vertical side of a V-tile. Multiple edges may exist. To make the picture clearer (and to see that Γ is planar), curve the edges a little in the direction of the tile defining the edge (see Figure 3). All vertices (except the corners of the large rectangle) lie on either 2 or 4 rectangles, and hence on either 2 or 4 edges in Γ. The corner vertices lie on 1 edge. It follows that a walk along edges that begins at one corner and does not repeat any edges will not terminate until it hits another corner, thus proving Theorem 1. ∎

(8) *Bipartite graph* (variation of Eulerian path proof) Place R in standard position, let S be the set of corners of tiles having both coordinates integers, and let T be the set of tiles. Form a bipartite graph on $S \cup T$ by connecting each point in S to all tiles of which it is a corner. There is an even number of edges because the hypothesis implies that each tile has 0, 2, or 4 corners in S. But each point in S that is not a corner of R lies on either 2 or 4 tiles. Since $(0, 0)$, which lies on only one tile, is in S, there must be another point in S lying on an odd number of tiles. This can happen only if another corner of R lies in S, which means that either the width or height of R is an integer. ∎

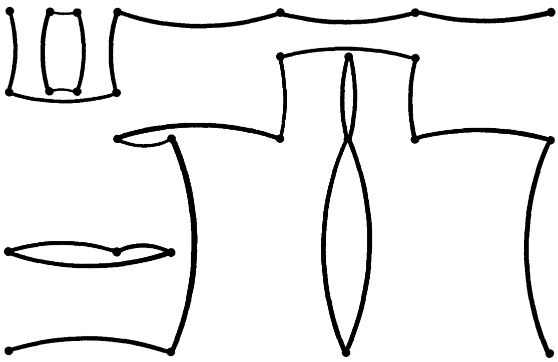


FIG. 3. The near-Eulerian graph Γ arising from the tiling in Figure 1.

(9) *Induction* (Raphael Robinson) The proof will be by induction on the number of H-tiles in a tiling in which each H-tile has width 1 and each V-tile has height 1. Since tiles may be split in their designated direction, this case suffices. Choose any H-tile T_0 (if there is none the result is immediate). If there are H-tiles whose lower border shares a segment with T_0's upper border, choose one and call it T_1. Otherwise only V-tiles share this border, and we may expand T_0 upward 1 unit. This does not increase the number of H-tiles, and the cut vertical tiles still have height 1. Continue expanding T_0 upward until either the top of the rectangle is reached, or a choice of an abutting H-tile T_1 is possible. Then continue upward similarly from T_1 to get T_2, etc. This yields a chain $T_0, T_1, \ldots, T_m$ of H-tiles from T_0 to the top of R. We can work downward from T_0 similarly, thus getting a chain

$$T_{-n}, \ldots, T_{-1}, T_0, T_1, \ldots, T_m$$

of H-tiles stretching from bottom to top. Remove these tiles and slide the rest together to get a rectangle with fewer H-tiles; induction applied to this smaller rectangle yields the result for the original rectangle. ■

(10) *Induction, variation* (Richard Bishop, Univ. of Illinois; Stan Wagon) Define a *V-link* to be a maximal vertical line segment in the tiling whose interior is not crossed by any horizontal line segment. Define *H-link* similarly. A link is *reducible* if it is a V-link (resp., H-link) having only H-tiles (resp., V-tiles) on one of its sides. In the tiling of Figure 1 there are lots of reducible links, for example, the V-link separating the large V-tile in the center from the two H-tiles on its left. It suffices to show that all tilings have a reducible link. For if we are given, say, a reducible V-link with only H-tiles bordering it on the right, let w be the width of the narrowest of these H-tiles. Then expand all tiles bordering the V-link on the left w units rightward (see Figure 4). Since heights are unchanged, V-tiles remain V-tiles; since widths are changed by the addition or subtraction of w, H-tiles remain H-tiles. But this expansion reduces the number of tiles by at least 1, as required for the induction.

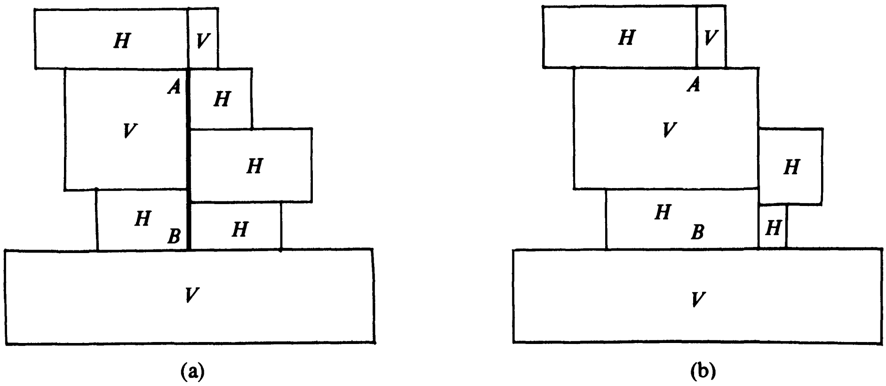


FIG. 4. In (a), AB is a reducible V-link. Figure (b) shows the tiling after the tiles adjacent to AB on the left have been expanded rightward.

A reducible link must exist, for otherwise there is a chain of H-tiles from bottom to top, each connected to the next along an H-link, and a chain of V-tiles from left to right, connected along V-links. The chains must cross, and the crossing must be an intersection of an H-link with a V-link in the interior of the links, contradicting the definition of a link. ■

(11) *Minimal cut-set* (Paul Seymour, Bell Communications Research, Morristown, NJ) Define a graph Γ as follows. The vertices are all horizontal line segments in the tiling, and two vertices are connected by m edges if there are m tiles (*either* H-tiles or V-tiles) connecting the corresponding segments. The exterior of R is considered as a tile, thus adding an additional edge connecting the top and bottom vertices. The tiling yields an embedding of Γ in the plane, since the vertical bisectors of the tiles can be used to form the edges (Figure 5). The edge corresponding to the additional tile can be drawn as in Figure 5, though it is more natural to preserve symmetry by embedding on the surface of a sphere instead.

Let Γ^* be the dual graph of Γ; the vertices of Γ^* are the faces of Γ and two vertices in Γ^* are connected by an edge if the corresponding faces in the planar embedding of Γ are incident. The faces of Γ have a simple structure: each face arises from part of a vertical segment in the tiling—a V-link, in the terminology of the preceding proof (see Figure 5)—and all tiles adjacent to the V-link. And if two faces in Γ are incident along an edge, then there is a tile whose vertical boundaries lie on the V-links corresponding to the faces.

Now let S be the set of edges in Γ corresponding to H-tiles, together with the exterior edge from top to bottom. If the removal of S does not disconnect the top vertex from the bottom vertex, then there is a top-to-bottom path with all vertical steps integers, as desired. Otherwise, let S' be a minimal subset of S whose removal disconnects the top from the bottom in Γ, and let S^* be the set of edges in Γ^* corresponding to edges in S'. By a well-known theorem for planar graphs [7, Thm.

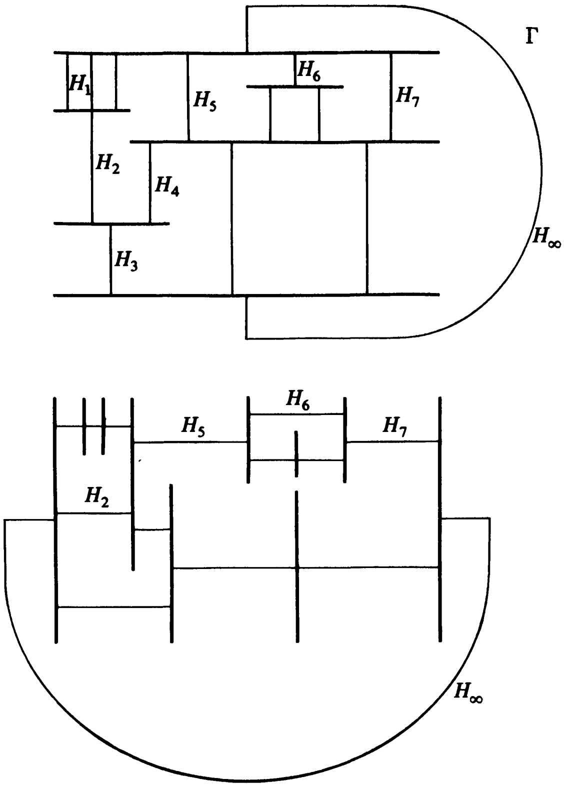


FIG. 5. The upper diagram represents the graph Γ of the minimal cut-set proof corresponding to the tiling in Figure 1. Horizontal segments are vertices, and vertical segments are edges, marked with an H if they arise from an H-tile in the tiling. The lower diagram shows the representation of the dual graph, Γ^*, using V-links in the tiling and horizontal segments as edges. The edges in S' and S^*, corresponding to a minimal cut-set in Γ, are H_2, H_5, H_6, H_7, and H_∞.

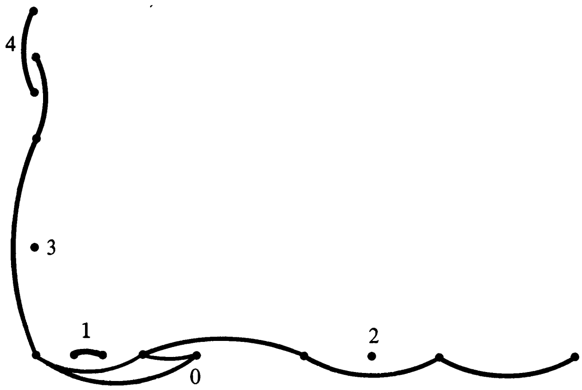
15C], S^* is a cycle in Γ^*. Moreover, since S' must contain the exterior edge, S^* induces a path from the left boundary of R to the right boundary, which has every horizontal step of integer length. Therefore, the width of the rectangle is an integer. ■

(12) *Sweep-line* (Gennady Bachman, Univ. of Illinois; Mihalis Yannakakis, Bell Labs, Murray Hill, NJ) Assume R is an $a \times b$ rectangle in standard position, and that b is not an integer. Let $\{R_i\}$ be the set of tiles, but assume that the closed segment forming the bottom border of each has been removed. Let a_i, b_i be the width and height, respectively, of R_i. Define f: $[0, b] \to [0, a]$ by setting $f(t)$ equal to the sum of all a_i such that R_i intersects the line $y = t$ and the y-coordinate of the top of R_i is not an integer. Then $f(0) = 0$ and it is easy to check that whenever f changes its value then it does so in a way that it remains an integer; as the

"sweep-line" crosses a horizontal line in the tiling the difference between f's gains and losses is an integer. Therefore, $f(b)$ is an integer. But since b is not an integer $f(b)$ is simply the sum of the widths of all tiles touching the top, that is, $f(b) = a$. ■

(13) *Step functions* (Melvin Hochster, Univ. of Michigan; Attila Máté, Brooklyn College) Place the rectangle in standard position. Then define a graph Γ whose vertices are all points on the x-axis such that some tile has a vertical boundary at that value (call these x_i, in increasing order), and all points on the y-axis that occur as top or bottom coordinates of some tile (y_i). Connect two vertices on the x-axis if some H-tile spans the interval and connect two vertices on the y-axis if some V-tile spans the interval (see Figure 6). The goal is to show that the origin lies in the same connected component of Γ as either $(a, 0)$ or $(0, b)$.

Assign, in an arbitrary way, distinct numbers to the connected components in Γ. Then define a step function on $[0, a]$ by defining f on the interval (x_i, x_{i+1}) to be



4	0	−4	0	0	8	−8	0
−4	0	4	0	0	−8	8	0
4	0	−4	0	0	8	−8	0
−3	0	3	0	0	−6	6	0
3	0	−3	0	0	6	−6	0

FIG. 6. The graph (with components numbered 0–4) and grid of the step function proof, using the tiling of Figure 1.

the number of the component that contains x_{i+1} less the number of the component that contains x_i. Note that the sum of the f-values on the intervals between two vertices connected by an edge is 0. Define g similarly on $[0, b]$. Now, refine the tiling into a grid by drawing all lines $x = x_i$ and $y = y_i$, and observe that $f(x)g(y)$ is constant in the interior of each rectangle in the grid. Moreover, the sum of these products over all grid rectangles contained in one of the tiles is 0 (see Figure 6). Therefore, the sum over all grid rectangles is 0. But this sum is just the product of $\Sigma\{f(I)$: I an interval between consecutive vertices on the x-axis$\}$ with $\Sigma\{f(J)$: J an interval between consecutive vertices on the y-axis$\}$. Therefore one of these sums vanishes, which implies that the origin and one of $(a, 0), (0, b)$ lie in the same component. ■

(14) *Sperner's Lemma* (James Schmerl, Univ. of Connecticut) Assume the conclusion is false and R is placed in standard position. Triangulate R by drawing a diagonal in each tile. Then label all vertices in the tiling as follows: (x, y) is labelled A if $x \in \mathbf{N}$, B if $x \notin \mathbf{N}$ but $y \in \mathbf{N}$, and C if neither x nor y is an integer. Then by a variation to Sperner's Lemma (see [**6**, Lemma 2]), the number of triangles labelled ABC is odd. But the hypothesis implies that no triangle is so labelled, contradiction. ■

3. Generalizations. A first reaction to these proofs might be that they are not all different, since many of them have similar ingredients. In some cases this view is valid; the real double-integral proof is a specialization of the complex double-integral proof, and the checkerboard proof is a discretization of the real double-integral proof using a $\{\pm 1\}$-valued function instead of a product of sines. Also, the two induction proofs are closely related, as are the Eulerian path and bipartite graph proofs. But an examination of various generalizations brings out differences in all the other proofs (see Appendix).

A natural generalization of Theorem 1 is to the case where the integers are replaced by other groups of reals. Consider a tiling of R where each tile has one *designated* side, not necessarily of integer length (a tile with designated width (resp., height) is called an H-tile (resp., V-tile)). The goal here is to show that R has either its width in the (additive) subgroup of R generated by widths of H-tiles or its height in the group generated by heights of V-tiles. For example, if each tile has either integer width or algebraic length, then R has either integer width or algebraic length. The Eulerian path, minimal cut-set, sweep-line, step function, and polynomial proofs, as well as the variation to the induction proof, all yield this generalization with essentially no modifications. The bipartite graph proof works if S is the set of tile-corners having both coordinates in the corresponding groups. The induction proof can be made to work in this case, if one excises only part of the chosen horizontal tiles, corresponding to the width of the narrowest member, thus reducing the number of horizontal tiles.

Note that although the Eulerian path, minimal cut-set, and step function proofs all work by finding a path in a certain graph, there are essential differences. The first two use graphs that are planar, while the step function proof might construct a

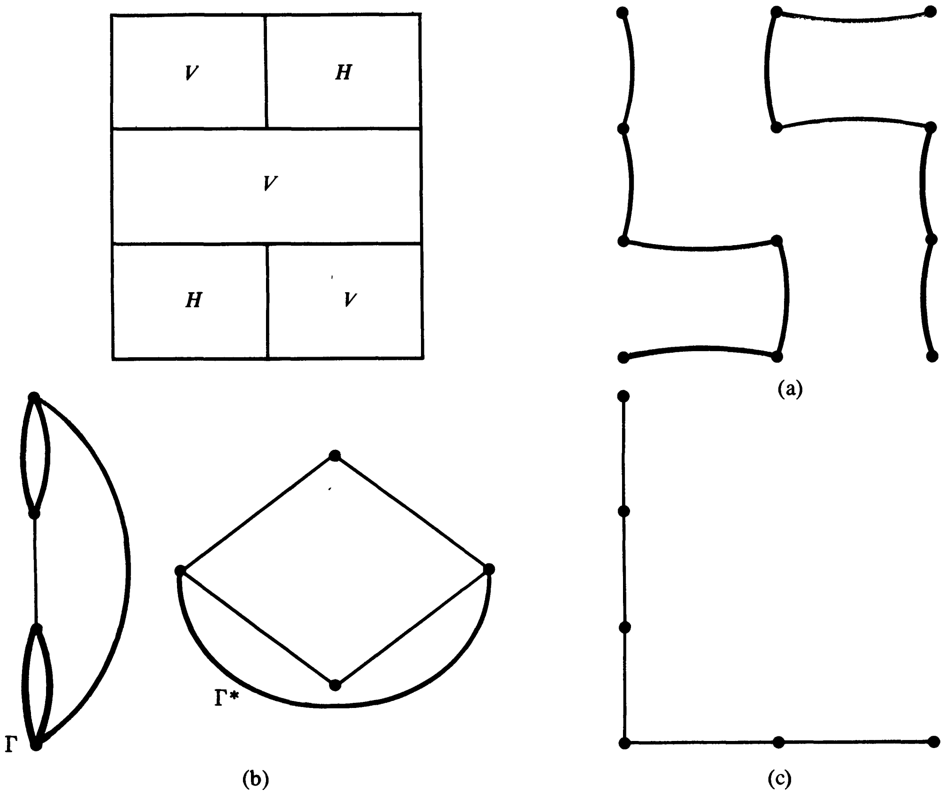


FIG. 7. Graph (a) is from the Eulerian path proof, the graphs labelled (b) are Γ and Γ* from the minimal cut-set proof, and graph (c) is from the step function proof.

nonplanar graph. The Eulerian proof is the only one of the three that shows that there is a path along integer-length sides of tiles from one side of the rectangle to the opposite side. However, the step function proof seems to have the capability of discovering "paths" that the others miss (see Figure 7).

Rusza has pointed out that Theorem 1 remains true only if it is assumed that tiles having at least one corner in $\mathbf{Z}^2$ have an integer side (here we assume R is in standard position). Another way of stating this is: Each tile has either 0, 2 or 4 corners in $\mathbf{Z}^2$. Rusza's square-counting proof, the bipartite graph proof, and the polynomial proof yield this result with no modification. The step function and Eulerian path proofs work as well (for the latter, consider only vertices lying in $\mathbf{Z}^2$), as does the Sperner Lemma proof.

As observed by several of the authors of proofs of Theorem 1, that result generalizes to higher dimensions. All the proofs, except (apparently) the minimal cut-set, sweep-line, and induction proofs, yield this generalization. Moreover, the

higher-dimensional version allows k (rather than just 1) of the sides of each tile to be "designated."

THEOREM 2. *Suppose a box R in $\mathbf{R}^n$ is tiled with n-dimensional boxes and each tile has at least k integer sides. Then R has at least k integer sides.*

Proof. The polynomial proof requires almost no modification. The tiling can be perturbed by moving hyperplanes t units, for small t, as in the proof of Theorem 1. If the conclusion is false then the volume of the modified box is a polynomial in t having degree greater than k. But the hypothesis implies that each tile in the auxiliary tiling is a polynomial of degree at most k, contradiction. ■

Several of the other proofs work as well. For the real integral proof, replace the integrand by the product of $t + \sin 2\pi x_r$, $r = 1, \ldots, n$. Then the integral over a box is a polynomial in t that is divisible by t^k if and only if the box has at least k integer sides. For the prime number proof, consider only primes p larger than any of the side-lengths. This guarantees that p^2 does not divide any of the side-lengths in the scaled-up box. The step function proof works if $f(x)$ and $g(y)$ are replaced by $t + f(x_1), t + f(x_2), \ldots,$ as in the extension of the integral proof, and a similar approach generalizes the checkerboard proof, which works easily if $k = 1$. The square counting proof works too, though if $k > 1$ one must use an odd integer when moving the boundary hyperplanes to an integer value; then the power of two dividing the number of squares in the auxiliary rectangle corresponds to the number of integer sides.

The Eulerian path and bipartite graph proofs yield Theorem 2 if $k = 1$, since each corner (except the corners of the ambient box) still lies on an even number of tiles. For larger k one can use induction, as pointed out by Andreu Mas-Colell: the $k = 1$ case yields one integer-length side; then project to the hyperplane perpendicular to this direction and use induction on the dimension. The advantage of this inductive approach is that it yields Ruzsa's extension for $k > 1$, where it is assumed only that tiles having a corner with all coordinates integers have k integer sides.

The polynomial, Eulerian path, and step function proofs of Theorem 2 show that the group-theoretical generalization to arbitrary n and k is valid. More precisely: If an n-dimensional box is tiled by boxes, each of which has at least k designated sides, then there are at least k directions in which the side-length of the ambient box lies in the subgroup of R generated by the designated side-lengths in the direction.

Another generalization comes from considering multiple tilings of the rectangle, that is, finitely many tiles that are not necessarily pairwise disjoint, but such that each point of the ambient rectangle (except for the boundaries of the tiles) is contained in the same finite number (the *multiplicity*) of tiles. The integration proofs work in the integer case, as do the checkerboard, polynomial, square counting (replace $1/2$ by $1/p$, where p is a prime larger than the multiplicity), and prime number proofs (use primes larger than the multiplicity). The Eulerian path proof will work if, as pointed out by Paterson, one makes a directed graph, with edges directed out of the lower left and upper right corners of each tile, and into the other

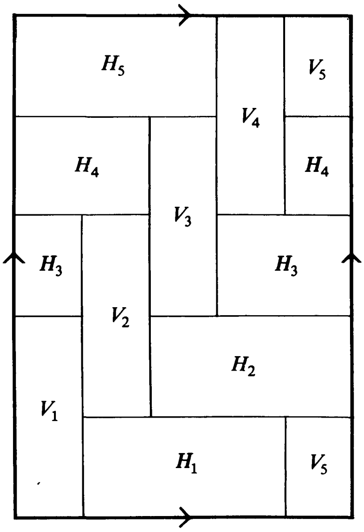


FIG. 8. H-tiles are 6×3; V-tiles are 2×6; torus is 10×15. Dividing by 6 yields a tiling of a $\frac{5}{3} \times \frac{5}{2}$ torus using $1 \times \frac{1}{2}$ and $\frac{1}{3} \times 1$ tiles.

two corners. Then the lower left corner has out-degree equal to the multiplicity, while the vertices not equal to a corner have equal in-degree and out-degree. Hence a directed walk starting from the lower left corner will end at one of the adjacent corners of R. The Eulerian path, step function, and polynomial proofs work in the case of groups as well.

Next, we can try to generalize to the case where sides of the rectangle are identified, that is, to the cylinder or torus. Consider the cylinder first, where we assume that opposite vertical sides are identified. The direct generalization of Theorem 1 is valid, as shown by either the sweep-line proof, the induction proof (which was invented for the cylinder and torus), the variation to the induction proof, or the Eulerian path proof (modified as in the proof of Theorem 3 below).

The torus is more interesting since Theorem 1 is false. Consider an $a \times b$ flat torus, that is, an $a \times b$ rectangle in the plane with opposite sides identified. The example in Figure 8, discovered independently by Solomon Golomb and Raphael Robinson, shows that the naive generalization of Theorem 1 is false.

Theorem 1 does nevertheless generalize to the torus, although the statement is more complicated. The following theorem was first proved in the integer case by Robinson, whose proof used the method of the induction proof of Theorem 1 and could be extended to the case of arbitrary subgroups. The proof of Theorem 3 given below combines ideas of the Eulerian path and induction proofs, and is due to Joan Hutchinson and the author. Note the curious situation that the original result in rectangles extends to arbitrary subgroups of $\mathbf{R}$, while the toroidal result generalizes to arbitrary subfields of $\mathbf{R}$.

THEOREM 3 (R. M. Robinson). *Suppose an a-by-b flat torus is tiled with rectangles parallel to the sides of the torus. Suppose each tile, regardless of its length or width, is designated to be either an H-tile or a V-tile and let G_H (resp., G_V) be the group generated by the widths of the H-tiles (resp., heights of the V-tiles). Then at least one of the following is true*:

(1) *a is in G_H*;

(2) *b is in G_V*;

(3) *For some relatively prime integers m and n, ma is in G_H and nb is in G_V.*

Proof. Let Γ be the graph associated with the tiling, as described in the Eulerian path proof. On the torus, loops can occur. To ensure that Γ embeds on the torus, curve the edges a little in the direction of the tile defining the edge; see Figure 9. As in the planar case, each vertex has degree 2 or 4. (In degenerate cases, such as a tiling with one tile, the corners have 2 or 4 loops.) Thus each component of Γ is Eulerian. In particular, any edge lies on a simple cycle.

The proof will be by induction on N, the total number of tiles. If $N = 1$ either (1) or (2) holds. For $N > 1$ observe that if Γ has a noncontractible cycle, then one of (1), (2) or (3) follows. For we may assume that there is a simple noncontractible cycle C. If C winds exactly once in one of the directions then (1) or (2) holds. Otherwise we may use the well-known result that if C winds more than once in one direction, then its winding numbers in the two directions are relatively prime (this is a consequence of P. Lévy's "Universal Chord Theorem" which implies that if $\gcd(m, n) = d$ then a simple curve from the origin to (m, n) has a chord that is a translate of the segment from the origin to $(m/d, n/d)$ (see [**4**, p. 23])). This yields (3). For example, the graph of the Robinson-Golomb tiling has two cycles, each winding thrice around the horizontal direction and twice vertically, so $3a$ is in G_H and $2b$ is in G_V. However, there are tilings for which Γ has no noncontractible cycles (example in Figure 10); in such cases we shall show that the tiling can be modified so that there are fewer than N tiles.

Suppose then that Γ has only contractible cycles. Then Γ must have a simple contractible cycle with no edges in its interior (called an *empty* cycle). For if C is a

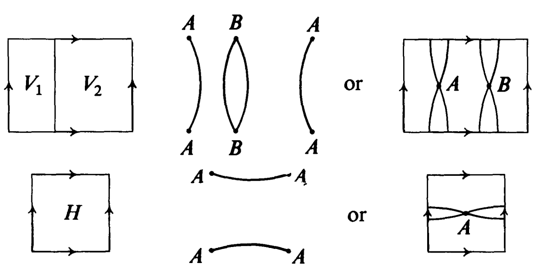


FIG. 9. Two examples—one with two tiles, one with just a single tile—of graphs associated with tilings of a torus.

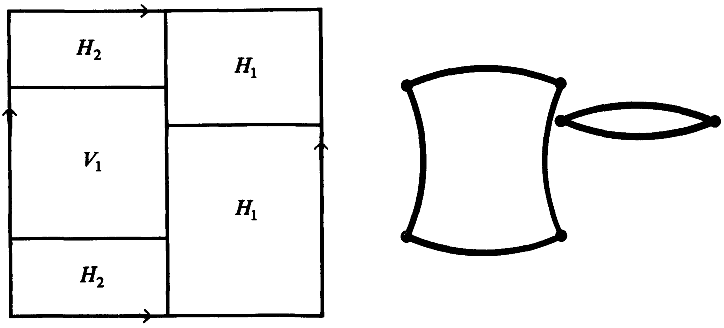


FIG. 10. An example of a toroidal tiling for which the graph has no noncontractible cycle.

simple contractible cycle with the fewest number of edges in its interior then this number must be 0; any edge inside C would lie on a cycle having fewer edges in its interior than C does.

First suppose Γ has an empty contractible cycle which, when viewed in the tiling, has no tile in its interior. Such a cycle, viewed in the tiling, must traverse each part of its boundary twice, once in each direction. Since the cycle is simple, this means it can have no right angles, and so must look like one of the cycles in Figure 11. In either case we can modify the tiles as in Figure 4 (expanding one side to absorb the narrowest—or shortest—tile on the other side), which reduces N by at least 1, as desired for the induction.

If there is no cycle as in the preceding paragraph, then Γ has an empty contractible cycle C that does have a tile, say, an H-tile, in its interior. Because C is empty, both the top and bottom of the tile correspond to edges on C. Label the tile's corners a, b, c, d starting from the upper right and going clockwise. Because C is a simple cycle, C must have the form $a \ldots bc \ldots da$. Now, adding the vertical steps in C between a and b yields that the distance from a to b lies in G_V. But then we may

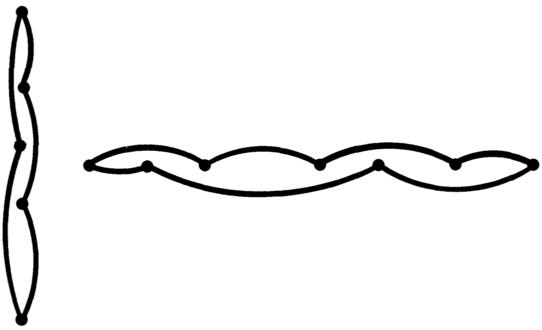


FIG. 11. A contractible cycle that, when viewed in the tiling, has no tile in its interior, must look like one of these two cycles.

switch the tile from an H-tile to a V-tile, shortening the length of the cycle. We can continue shortening the cycle in this way until it no longer has a tile in its interior. Then we are in the preceding case, where it was easy to reduce the number of tiles. ■

Not too much is known for higher-dimensional tori. Robinson has generalized his prime number proof to show that if such a torus is tiled with boxes having an integer side, then the torus has at least one rational side. This can also be proved by the step function proof, which has the advantage of working for arbitrary groups.

THEOREM 4 (Robinson, Mâté). *Suppose an n-dimensional flat torus with side-lengths a_i, $i = 1, \dots, n$, is tiled by n-dimensional boxes parallel to the sides of the torus, with each tile having one designated side. Let G_i be the group generated by the designated side-lengths in the ith direction. Then for at least one a_i there is a positive integer m such that ma_i is in G_i.*

Proof (Mâté). We give the details assuming the case of an $a \times b$ flat torus in $\mathbf{R}^2$; the extension to higher dimensions will be clear. Assume the $a \times b$ rectangle is in standard position and the origin is the corner of a tile. Extend the tiling periodically to the whole plane and define a graph Γ as in the step function proof: If (x, y) is a corner of a tile in any copy of the torus then the points $(x, 0)$ and $(0, y)$ are vertices. Connect two vertices on the x-axis if the interval they define is spanned by an H-tile in the tiling of the plane; this includes the case of tiles straddling a vertical boundary of a torus. Define edges on the y-axis likewise using V-tiles.

It is sufficient to prove the following claim, for if Γ has infinitely many vertices on, say, the x-axis that are in the same connected component, then that component must contain two vertices of the form $(x, 0), (x + ma, 0)$. This implies that ma is in G_1.

Claim. The graph Γ has an infinite connected component.

Proof of claim. To prove the claim, assume it is false. Then all components are finite and we may define a function C_1 on the points on the x-axis corresponding to vertices in Γ by letting $C_1(x)$ be the least t such that $(t, 0)$ is in Γ and in the same component as $(x, 0)$. Define C_2 for vertices on the y-axis similarly. Now define a step function f on the x-axis by letting f equal $C_1(x'') - C_1(x')$ on the interval between any two consecutive vertices x', x'' in Γ.

As before, the sum of f-values over intervals subdividing a single H-tile in the tiling of $\mathbf{R}^2$ is zero. Moreover, because of the periodicity of the tiling of the plane, $C_1(x + a) = C_1(x) + a$ and f is periodic with period a. It follows that the sum of f-values over intervals subdividing a single H-tile in the original torus in standard position is 0. These properties also hold for the step function g, defined using C_2 analogously to f. To conclude, proceed as in Theorem 1 to refine the tiling into a grid and observe that the sum of the fg values over a tile vanishes, whence the sum over the entire $a \times b$ torus vanishes. But this is a contradiction since this sum equals ab: the sum of f (resp., g) over the intervals in $[0, a]$ (resp., $[0, b]$) is simply $C_1(a) - C_1(0) = a$ (resp., $C_2(b) - C_2(0) = b$). ■

The preceding argument also works on a box in $\mathbf{R}^n$ where some, but not all, sides are identified. The result then states that either there is an "unidentified" direction of the box whose side-length is in the subgroup generated by designated lengths in that direction, or there is an "identified" direction for which an integer multiple of the side-length is in the group corresponding to that direction. In the case of the standard torus or cylinder this is not best possible, but the proofs in those cases do not generalize to higher dimensions. Unlike the proof of Theorem 3, the preceding argument works for multiple tilings, and so yields something for multiple tilings of the standard two-dimensional torus: If each tile has one integer side then at least one side of the torus is rational (and similarly in higher-dimensional multiple tilings).

4. Summary and open questions. The various generalizations considered here do a fairly complete job of distinguishing the proofs. If one calls two proofs equivalent provided they work on the same set of generalizations then, unless new modifications are found, the only equivalences are (1) ~ (2) ~ (3) and (9) ~ (10). The two most powerful proofs seem to be the Eulerian path and step function proofs. The former fails only on high-dimensional tori, multiple tilings of the standard torus, and the $k > 1$ case of Theorem 2; the latter works in all cases, except the cylinder and torus, where it does not yield the best possible result. A definitive comparison will have to wait until the true situation in higher dimensions is resolved; see Problem (a) below.

Problem (a). Can the seemingly weak statement about tilings of higher-dimensional tori or cylinders be improved, or is it best possible? The simplest unsolved case is that of a box with left and right faces identified. Can such a box having dimensions $a \times \beta \times \gamma$, where a is rational and β and γ are irrational, be tiled with boxes each of which has one side of unit length?

Problem (b) (S. Golomb). For which triples (a, b, k) can the $a \times b$ torus be tiled using copies (vertical or horizontal) of a $1 \times k$ tile?

REMARKS. De Bruijn's original result characterized the rectangles that could be tiled using copies of a $1 \times k$ tile: either the height or the width of the rectangle is a multiple of k. This is true for an $a \times b$ torus if k is a prime power. This was first proved by Robinson and Golomb using coloring techniques; it also follows from Theorem 3 above if one divides everything by k and observes that the relatively prime coefficients m and n cannot both absorb a power of the same prime. It follows that $k = 6$ is the smallest number for which there is a triple (a, b, k) as in Question (a) with neither a nor b divisible by k. An unsolved special case of Question (a) is the problem of determining which $a \times b$ tori can be tiled with copies of a 1×6 tile. Golomb has shown that the 10×15 torus is the smallest example.

Problem (c). What is the situation regarding double tilings of the standard torus where each tile has at least one integer side? Is it true that either one side of the torus is an integer or both sides are rational?

Acknowledgments. The author is grateful to David Gale, Solomon Golomb, Joan Hutchinson, Hugh Montgomery, and Raphael Robinson for their collaboration and enthusiasm and to the authors of the proofs for permission to include them.

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Appendix to justify claim that proofs are different:

Proofs:

(1) Complex double integral
(2) Real double integral
(3) Checkerboard
(4) Counting squares
(5) Polynomials
(6) Prime numbers
(7) Eulerian path
(8) Bipartite graph
(9) Induction
(10) Induction, variation
(11) Minimal cut-set
(12) Sweep-line
(13) Step functions
(14) Sperner's Lemma

Generalizations:

1. Plane
2. Plane, Ruzsa hypothesis
3. Plane, arbitrary groups
4. n-dimensions, $k = 1$
5. n-dimensions, $k > 1$
6. n-dimensions, $k > 1$, Ruzsa hypothesis
7. Cylinder
8. Torus
9. Plane, multiple tiling
10. Plane, multiple tiling, arbitrary groups
11. High dimensional torus
12. Torus, multiple tiling

Proof number	Works in cases
1, 2, 3	1, 4, 5, 9
4	1, 2, 4, 5, 9
5	1, 2, 3, 4, 5, 9, 10
6	1, 4, 5, 9, 11
7	1, 2, 3, 4, 7, 8, 9, 10
8	1, 2, 3, 4, 9, 10
9, 10	1, 3, 7, 8
11	1, 3
12	1, 3, 7
13	1, 2, 3, 4, 5, 9, 10, 11, 12
14	1, 2, 3, 4

Trees and Euler Tours in a Planar Graph and Its Relatives

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1. Introduction. Let G be a graph embedded in the plane. We describe two relatives of G: its dual and its medial. It is our purpose to show that the numbers of spanning trees of G, of spanning trees of the dual, of directed Euler tours of the medial and of arborescences of the medial are all the same. These results are known, but are not all easily accessible in the literature. Although not trivial, the proofs should be comprehensible to most upper level undergraduates—indeed, they might make an interesting introduction to planar graphs.

In this work, a graph is allowed to have loops and multiple edges. Such things are sometimes called "multigraphs" in the literature. Our terminology is based loosely on [**6**], which is an excellent text on graph theory.

A *spanning tree* of a graph G is a connected subgraph of G that has no circuits and contains all the vertices of G. For example, if G is the graph in Figure 1, then the five spanning trees of G are those exhibited in Figure 2.

Now from the drawing of G in the plane that is given in Figure 1, we can construct the *dual graph* $D(G)$. To get $D(G)$, draw a vertex in each face of G and, for each edge e of G, draw an edge $\tau(e)$ of $D(G)$ joining the vertices of $D(G)$ that lie in the faces of G on opposite sides of e. This process is shown in Figure 3—the original graph G is given by the short dashed lines, while the dual $D(G)$ is given by the long dashed lines. Observe that $D(G)$ becomes a planar graph if we draw each edge $\tau(e)$ of $D(G)$ so that it intersects e once and does not intersect any other edge of G. The map τ provides a natural bijection between the edges of G and $D(G)$;

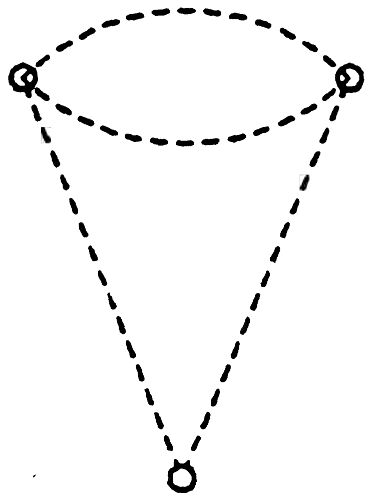

FIG. 1.

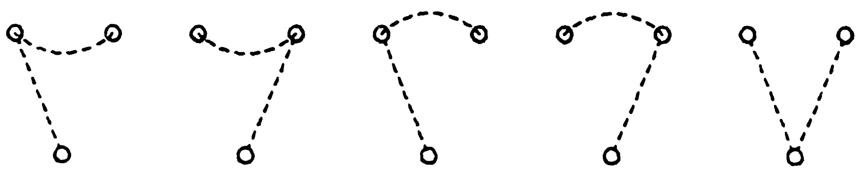

FIG. 2.

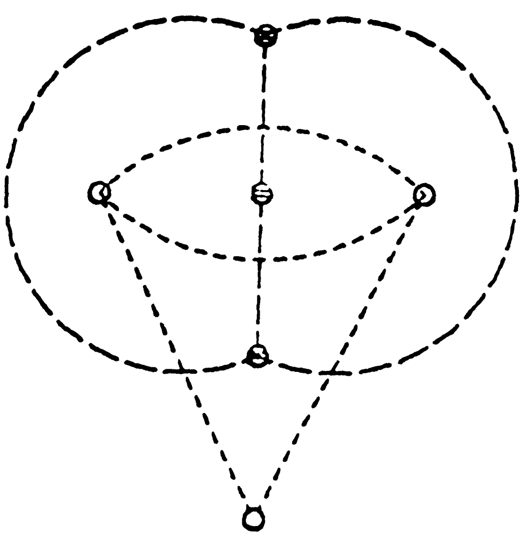

FIG. 3.

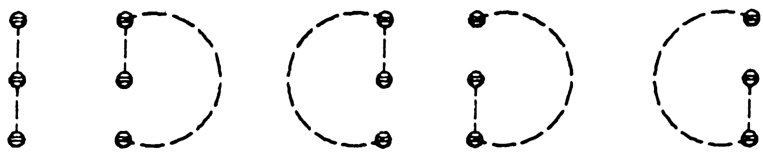

FIG. 4.

each face of $D(G)$ contains exactly one vertex of G, so that $D(D(G)) = G$. We point out that $D(G)$ depends on the particular drawing of G. There are conditions known that determine when G has only one dual graph [7], but we shall not pursue that discussion here.

In Figure 4 we have exhibited the spanning trees of $D(G)$. Coincidentally (?), the number of spanning trees of G is equal to the number of spanning trees of $D(G)$. The reader might like to convince himself/herself that the same thing happens for the graph G' and its dual drawn in Figure 5. Both G' and its dual have 11 spanning trees.

In Section 2, we shall see that if G is any planar graph, then G and its dual have the same number of spanning trees.

For a planar graph G, there is another graph that can be constructed from G. The *medial graph* $M(G)$ is obtained by inserting a vertex in the middle of each edge of G. For each pair of vertices v and v' of $M(G)$, from the edges e and e', respectively, and for each face F of G that has e and e' consecutive in the boundary of F, we draw an edge of $M(G)$ joining v and v'. Thus, v and v' will have as many edges joining them as there are faces that have e and e' consecutive. See Figure 6(a), where the medial is constructed for the graph G of Figure 1. In this case G is the dashed graph while $M(G)$ is the solid graph. The reader might wish to reassure

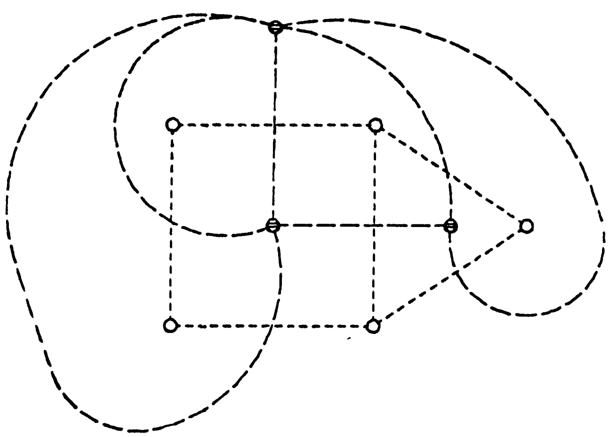

FIG. 5.

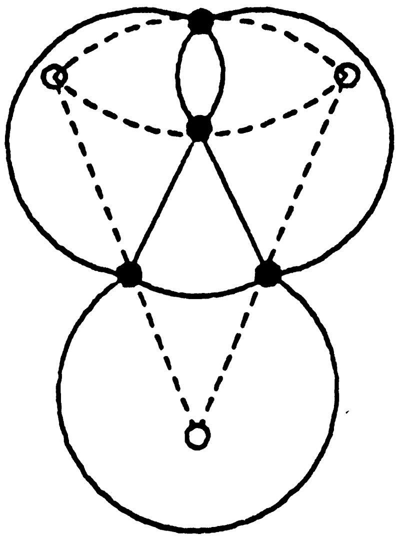

FIG. 6(a).

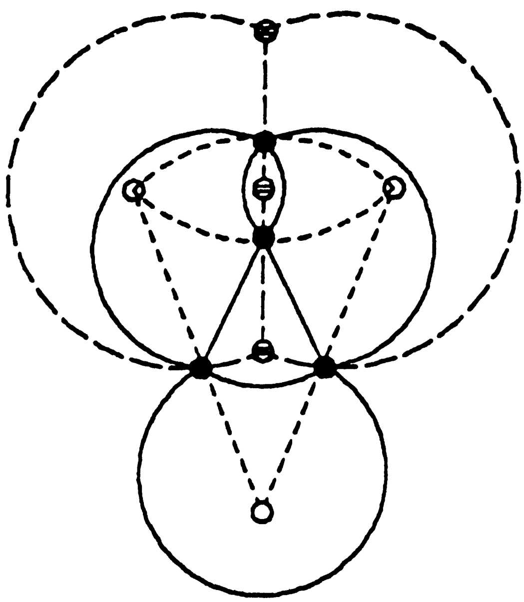

FIG. 6(b).

himself that $M(G)$ does not have the same number of spanning trees that G has. Note that $M(G)$ is a planar graph, has one vertex for each edge of G and each vertex of $M(G)$ is incident with four edges of $M(G)$. The medial is related to, but different from, the line graph $L(G)$ of G (cf. [3, p. 71]). If G has no vertices of degree 1 or 2, then $M(G)$ is a subgraph of $L(G)$. However, $M(G)$ is always 4-regular, whereas $L(G)$ will normally have vertices with larger degrees. Like the dual, the medial depends on the particular drawing of G in the plane. Note also that if G is connected, then $M(G)$ is also connected.

Observe that we could have constructed $M(G)$ using $D(G)$ and inserting a vertex of $M(G)$ in each edge $\tau(e)$ of $D(G)$ at the point where it crosses e. That is, $M(G) = M(D(G))$. We illustrate this in Figure 6(b) and leave the proof of the general result to the reader (see [5]). Here G is the short dash graph, $D(G)$ consists of the long dashes and $M(G)$ is the solid graph.

We now set out to orient the edges of $M(G)$ so as to make it *balanced*, i.e., the number of edges pointing in to a vertex v is equal to the number pointing out. This condition, in a connected graph, is equivalent to the existence of a directed Euler tour. The reader might guess, correctly, that the number of directed Euler tours in $M(G)$ is equal to the number of spanning trees of G (and hence to the number of spanning trees of $D(G)$). This relationship is explained in Section 3.

Note that each face of $M(G)$ contains a vertex of either G or $D(G)$, but not both. Around a vertex of $M(G)$, there are four faces. If we go through these faces in order, they will alternate containing a vertex of G and a vertex of $D(G)$. To put it

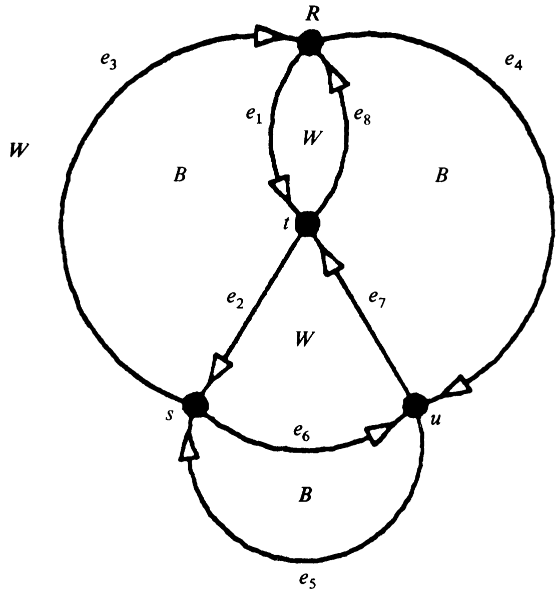

FIG. 7.

another way, if we color black the faces of $M(G)$ that contain vertices of G and white those faces of $M(G)$ that contain vertices of $D(G)$, then every edge of $M(G)$ separates a white face from a black one. Thus, an edge of G (or of $D(G)$) goes through a vertex of $M(G)$ to the diagonally opposite face.

Fix the black and white face coloring of $M(G)$ just described. Now orient all the edges of $M(G)$ so that as we go clockwise around the perimeter of a black face, the edges are pointing in the direction of travel. See Figure 7. Thus, an insect crawling along an edge, in the direction of the edge, will always have a white region on its left and a black on its right.

It should be apparent to the reader that, at each vertex of $M(G)$, the directed edges alternate entering and leaving—see Figure 8. Since each vertex has two edges entering and two leaving, there is a directed Euler tour in $M(G)$. This is a walk along $M(G)$ that starts and ends at the same vertex and traverses each edge exactly once and the traversal is in the same direction as the edge. The sequence $(R, e_1, t, e_2, s, e_6, u, e_5, s, e_3, R, e_4, u, e_7, t, e_8, R)$ is an example of such a tour in $M(G)$ as depicted in Figure 7. To have a standard representation for a directed Euler tour, we will specify the vertex R at which the tour starts and also the last edge of the tour, which necessarily points in to R. In the example in Figure 7, we designate e_8 to be the "last edge."

For the $M(G)$ we have been considering, the Euler tours are

$$(R, t, s, u, s, R, u, t, R), \quad (R, t, s, R, u, s, u, t, R),$$
$$(R, u, t, s, u, s, R, t, R), \quad (R, u, s, R, t, s, u, t, R),$$
$$\text{and} \quad (R, u, s, u, t, s, R, t, R).$$

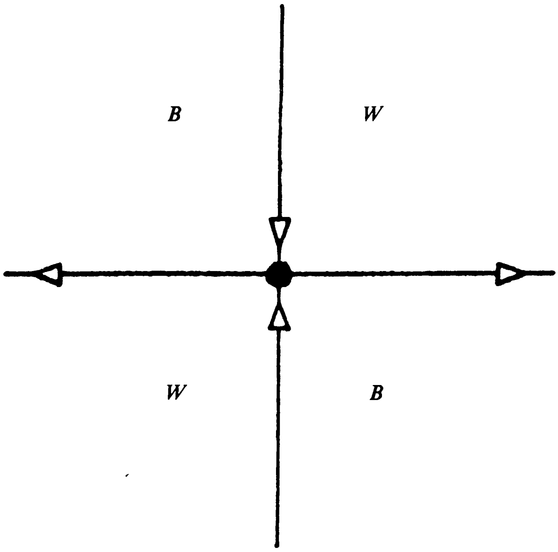

FIG. 8.

(We have omitted the edge-terms since, in this case, there is no ambiguity. In general, there may be two edges pointing from one vertex to another. In such an event, it is important to know in which order the edges are traversed.)

Believe it or not, we have yet to finish. An *arborescence* in $M(G)$ rooted at R is a spanning tree A of $M(G)$ such that, for each vertex other than R, there is exactly one edge of A pointing in, while no edge of A points in to R. The botanical imagery is appropriate since one can view this as the (directed) flow of water from the base (root) of a tree to the leaves. An example of an arborescence is given in Figure 9. The reader is encouraged to verify that the number of arborescences in $M(G)$ rooted at R is five. This is the same number as the number of spanning trees in G. In Section 4 we show that the number of arborescences rooted at R is equal to the number of directed Euler tours in $M(G)$.

2. Spanning trees in the dual. In this section, we shall see how to relate the spanning trees of the planar graph G to the spanning trees of its dual $D(G)$. Consider the tree T specified in Figure 10 by the short dashed lines. The long dashed edges of $D(G)$ that correspond to edges not in T form a spanning tree of $D(G)$. It is our goal to prove that this always happens.

Earlier, we introduced the map τ: $E(G) \to E(D(G))$ from the edges of G to the edges of $D(G)$. We can extend this to a map on subsets of edges by: $\tau(E') = \{\tau(e) | e \in E'\}$ for any subset E' of $E(G)$. Likewise, we can obtain a complementary map $\tau^*(E') = \{\tau(e) | e \notin E'\}$ for any subset E' of $E(G)$. Evidently, $\tau(E') \cup \tau^*(E') = E(D(G))$ for any subset E' of $E(G)$.

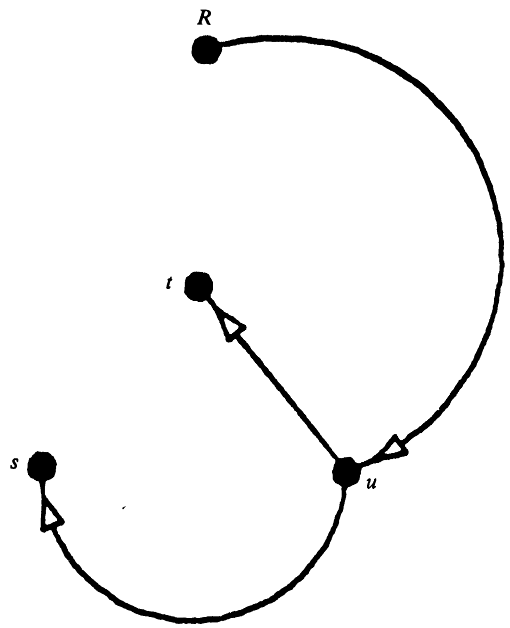

FIG. 9.

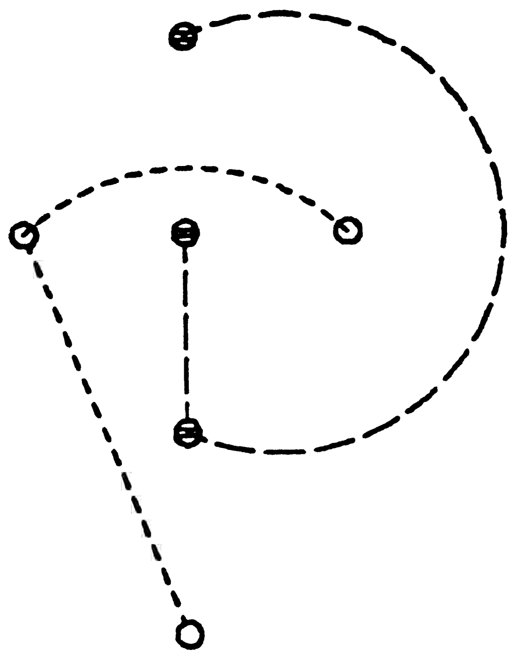

FIG. 10.

THEOREM 1. *Let T be a spanning tree in G and let $\tau^*(T)$ be the subgraph of $D(G)$ containing the edges of $D(G)$ corresponding to edges of G not in T. Then $\tau^*(T)$ is a spanning tree of $D(G)$.*

Proof. If $\tau^*(T)$ contains a circuit P, then some vertex v of G lies within P, while some other, w, lies outside P. By the Jordan Curve Theorem, the path in T joining v and w crosses P, which contradicts the definition of $\tau^*(T)$.

Now let v and w be any vertices of $D(G)$. There is no circuit in T, so v and w are still connected in the plane after we delete the edges of T. Thus, we can find a polygonal curve in the plane that joins v and w, but avoids the edges of T. Suppose this curve takes us through the faces $F_1, \dots, F_k$ of G. For $j = 2, 3, \dots, k$, let $\tau(e_j)$ be the edge of $D(G)$ that joins the vertices of $D(G)$ in the faces F_{j-1} and F_j. These edges contain a path in $D(G)$ that joins v and w—evidently these edges are in $\tau^*(T)$. Thus, $\tau^*(T)$ is connected and contains all the vertices of $D(G)$. ■

The map τ^* from the set of spanning trees of G to the spanning trees of $D(G)$ is clearly injective. Hence, G has at most as many spanning trees as does $D(G)$. The equality follows from the fact that $D(D(G)) = G$.

This correspondence between the spanning trees of G and $D(G)$ has been known for a long time. The result is an exercise in [**1**, p. 143, #9.2.5]. We do not know who is responsible for the first proof of this fact.

3. Euler tours in the medial graph. In this section, we describe how the spanning trees of G correspond to Euler tours in the medial $M(G)$. Let E be a directed Euler tour of $M(G)$. At each vertex of $M(G)$, E enters and leaves on adjacent edges. Slightly separate the edges at each vertex of $M(G)$ as determined by E. In Figure 7, we have a medial graph $M(G)$, with a directed Euler tour $(R, t, s, u, s, R, u, t, R)$. Figure 11 is the result of splitting each vertex of $M(G)$ in accordance with E.

This splitting at each vertex results in a closed curve C that does not intersect itself. Thus, the Jordan Curve Theorem applies. Evidently, the black faces will all be on the right side of the curve (as we traverse in the direction of the edges) and the white faces will all be on the left side. As arranged earlier, the black faces are those that contain the vertices of G and the white faces are those that contain the vertices of $D(G)$. Let T be the subgraph of G that contains the edges of G that do not cross C and let $\tau^*(T)$ be the subgraph of $D(G)$ consisting of the edges corresponding to edges not in T. Note that $\tau^*(T)$ consists of those edges of $D(G)$ that do not cross C. In Figure 12 we have the dashed edges yielding T.

The claim is that T is a connected subgraph of G that meets every vertex of G. As argued in the proof of Theorem 1, staying on the black side of C we can get from any vertex of G to any other. Analogously, $\tau^*(T)$ is a connected subgraph of $D(G)$ that contains every vertex of $D(G)$. By Theorem 1, it must be the case that T and $\tau^*(T)$ are spanning trees of G and $D(G)$ respectively. Thus, every directed Euler tour of $M(G)$ produces a spanning tree of G.

Conversely, suppose T is a spanning tree of G. To produce a directed Euler tour of $M(G)$, we split the vertices of $M(G)$ according to T: if edge e of G is in T, split

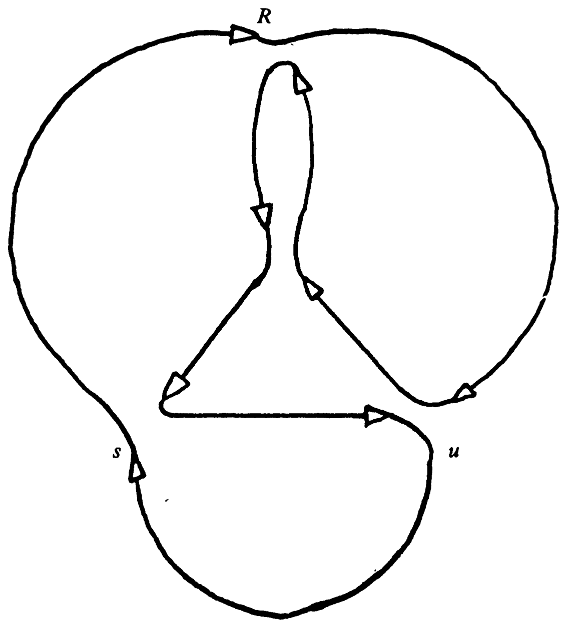

FIG. 11.

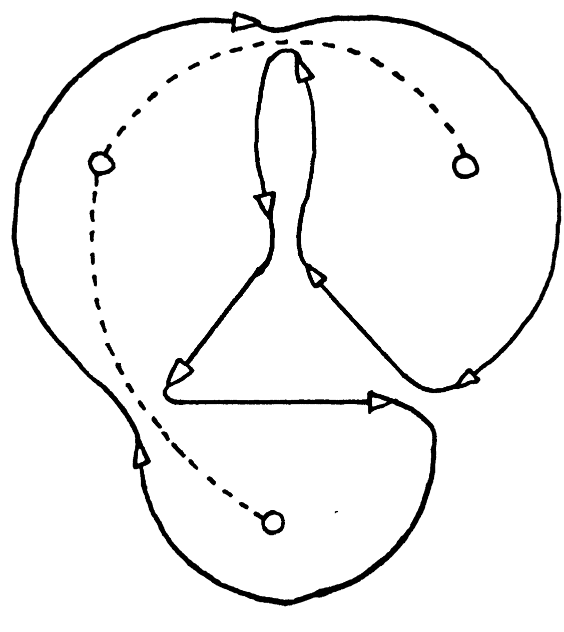

FIG. 12.

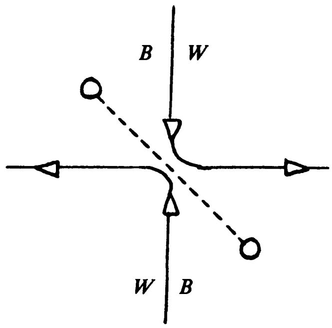

FIG. 13(a).

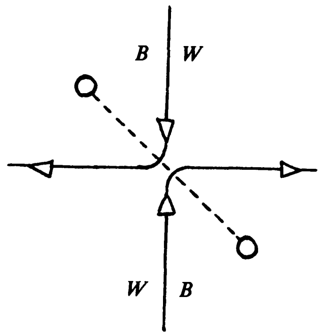

FIG. 13(b).

the corresponding vertex of $M(G)$ as in Figure 13(a), while if e is not in T, then split the vertex of $M(G)$ as in 13(b).

Evidently, if C is the resulting collection of closed nonselfintersecting curves of $M(G)$, the edges of T are precisely those that do not cross C. Looking at this from the dual perspective, we see that the edges of $\tau^*(T)$ are precisely those edges of $D(G)$ that do not cross C. Since both T and $\tau^*(T)$ are spanning trees, it follows that C creates exactly two complementary regions in the plane. Therefore, C is a single closed curve. Translating C into $M(G)$, we get a directed Euler tour of $M(G)$. See Figures 14(a) and 14(b) for an example of this operation.

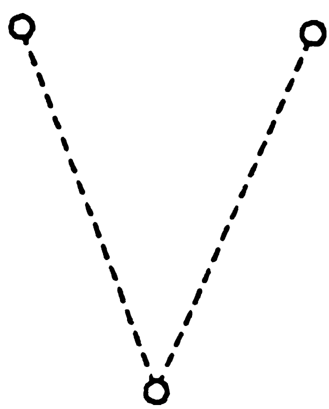

FIG. 14(a).

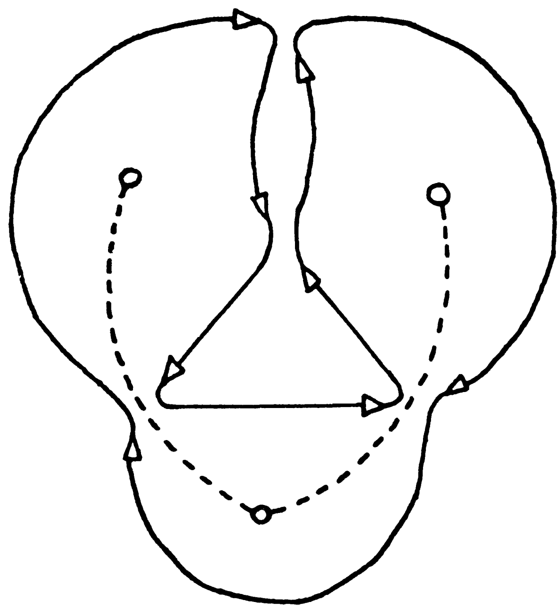

FIG. 14(b).

This result can be summarized in the following statement.

THEOREM 2. *The number of directed Euler tours in $M(G)$ is equal to the number of spanning trees of G.*

The first proof of this result is due to Kotzig [4].

4. Euler tours and arborescences in $M(G)$. Our last correspondence is between the directed Euler tours and arborescences in $M(G)$. In fact, the correspondence we give here applies to any directed graph (not necessarily planar) such that each vertex has exactly two edges pointing in and two pointing out. As discussed in the introduction, we specify a root vertex R and an edge e^* pointing in to R to be the last edge of any Euler tour. Observe that this last edge is not in any arborescence of $M(G)$, since no edge of the arborescence points in to R. The arguments in this section are somewhat more technical than their predecessors.

The correspondence we shall use is the following. Let E be any directed Euler tour. Label the edges of the tour in order, so that $E = (R, e_1, \ldots, e_n, R)$, with $e_n = e^*$. Grow the corresponding arborescence by starting with $A_0 = R$ and then letting A_{j+1} be $A_j \cup e_k$ where k is the least index such that e_k takes us to a vertex not in A_j. Continue this process for as long as possible. (Note that, since the number of edges in an arborescence is one less than the number of vertices, the number of iterations is equal to one less than the number of vertices in $M(G)$.) For the Euler tour

$$(R, e_1, t, e_2, s, e_3, R, e_4, u, e_5, s, e_6, u, e_7, t, e_8, R)$$

of the graph in Figure 7 we have the sequence A_0, A_1, A_2, A_3 shown in Figure 15. It is quite clear that we construct an arborescence in this manner. What is not clear is that different directed Euler tours produce different arborescences and that all arborescences can be produced in this manner.

To show that this is indeed the case, we exhibit the inverse construction. Given an arborescence A, construct the directed Euler tour E inductively. We work backwards, starting with $E_0 = (v^*, e^*, R)$, where e^* goes from v^* to R. Let E_j have initial vertex v, which we assume, for the moment, is not R. Now there are always

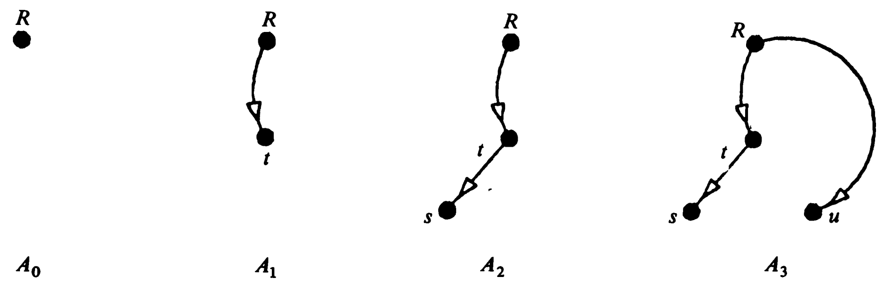

FIG. 15.

two edges, say e and f, pointing in to v. Exactly one of these two is in A; let us say this is e. If f does not already appear in E_j, then let E_{j+1} be (f, E_j). Otherwise, let E_{j+1} be (e, E_j). In other words, the first time we back up to v, we back out on the edge not in A and the second (and last) time we back in to v, we back out on the edge in A. If the initial vertex of E_j is R, then back out along the edge which is not e^*. The arborescence in Figure 9 produces the directed Euler tour $(R, u, s, u, t, s, R, t, R)$. Note that e_8 is always the last edge of the tour.

It is not immediately apparent that this algorithm necessarily produces an Euler tour. Let $W = (R, e_1, v_1, \dots, v_{n-1}, e_n, R)$ be the directed closed walk of $M(G)$ obtained from the algorithm. Since we stopped at this point, all the edges in to (and hence out from) R must be in W. Now, let x be any vertex of $M(G)$. In A there is a unique directed walk from R to x; let it be $(R, f_1, x_1, \dots, f_k, x_k)$, where $x = x_k$.

By the preceding comments, every edge incident with R is in W, and so f_1 occurs in W. Inductively, then, we show that f_j is in W. If f_{j-1} is in W, then the algorithm implies that x_{j-1} occurs twice in W. Therefore, every edge incident with x_{j-1} occurs in W. This applies, in particular, to f_j.

It follows that x occurs twice in W and this is true for every vertex of $M(G)$. Consequently, W is a directed Euler tour of $M(G)$.

Finally, we must show that the two constructions are inverses of one another. Let E be a directed Euler tour, let A be the arborescence constructed from E and let E' be the Euler tour obtained from A. We must show that $E = E'$. Both E and E' end on e^*. Assume, inductively, that E and E' have the common ending

$$\left(v_j, e_{j+1}, v_{j+1}, \dots, v_{n-1}, e^*, R\right).$$

Suppose, first, that $v_j \neq R$. If this is the first time we have backed in to v_j, then the unique edge e from A pointing in to v_j is not e_j in E; otherwise we would have chosen the other inpointing edge when we created A out of E. Also, we do not choose e as e_j when we create E' from A—recall we choose the edge not in A the first time we back in to a vertex. If this is the second time we have backed in to v, then the preceding sentences show both E and E' back out along e; there is no choice left. On the other hand, if $v_j = R$, then there is a unique edge pointing in to R that is not e^*. This edge must be e_j. Therefore E and E' have the same ending $(v_{j-1}, e_j, \dots, e^*, R)$ and the fact that $E = E'$ follows by induction.

Now suppose A is an arborescence, E is the Euler tour constructed from A and A' is the arborescence obtained from E. We must show $A = A'$. Write $E = (R, e_1, v_1, \dots, v_n, e^*, R)$. Clearly, e_1 is in A' if and only if e_1 points in to some vertex other than R. Now, in constructing E from A, unless $v_1 = R$, we know that e_1 is also in A. Therefore, e_1 is in A if and only if e_1 points in to some vertex other than R. Therefore, e_1 is in A if and only if e_1 is in A'.

We now proceed inductively, and assume that for all $1 \leq j \leq k$ $e_j \in A$ if and only if $e_j \in A'$. We know that $e_{k+1} \in A'$ if and only if v_{k+1} is the first occurrence of the vertex $v = v_{k+1}$ in E, which happens if and only if this is the second occurrence of v as we go backward in E. This last condition is equivalent to $e_{k+1} \in A$. Therefore, $A = A'$ as required.

What have we accomplished? We have exhibited a map f from the directed Euler tours to the arborescences and a map g from the arborescences to the Euler tours such that $f \circ g$ and $g \circ f$ are both identity maps. Consequently, f and g are bijections and, therefore, the following holds.

THEOREM 3. *The number of directed Euler tours in $M(G)$ is equal to the number of arborescences of $M(G)$ rooted at R.*

This last result is a special case of the so-called BEST Theorem (for de Bruijn, Ehrenfest, Smith and Tutte), which relates the number of Euler tours in an oriented graph having an Euler tour to the number of arborescences in the graph. A proof can be found, for example, in Berge [**2**, Thm. 4, p. 169]. In fact, the proof there is a straightforward generalization of the one given here.

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A Rose is a Rose...

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PETER M. MAURER received his B.A. in mathematics in 1969 from St. Benedict's College in Atchison, Kansas. After three years of writing computer programs for the U.S. Army and five more years writing programs for the state government of Iowa, he returned to school at Iowa State University. He received his M.S. in 1979 and his Ph.D. in 1982, both in Computer Science. Since 1982 he has been employed by AT & T Bell Laboratories and AT & T Information Systems. His main professional interest is using mathematics to solve problems in the design of microprocessors.

1. Introduction. The past two decades have seen massive advances in computer graphics. Computer-generated pictures have progressed from simple line drawings [1] to breathtakingly beautiful full-color displays [2]. However, sometimes even a simple line-drawing program can exhibit interesting, and unexpected, mathematical behavior. "The Rose" is an example of such a program. This program has been used as a demo for AT & T's "DMD 5620" terminal [3] and other sophisticated equipment. It gets its name from the polar-coordinate graph of the function $r = \sin(n\theta)$, where n is a positive integer. The graph of this function is an n-petaled rose if n is odd, and a $2n$-petaled rose if n is even, as demonstrated by Figure 1.

2. The basic algorithm. "The Rose" uses the following algorithm, called algorithm-A, to display polygons inscribed in n-petaled and $2n$-petaled roses.

1. Choose integers n, d such that $1 \leqslant n \leqslant 359$ and $1 \leqslant d \leqslant 359$.
2. Set θ equal to zero, and set (*oldx*, *oldy*) to $(0,0)$.
3. Set θ equal to $\theta + \mathrm{d}$. If $\theta \geqslant 360$ replace θ by the remainder obtained when dividing θ by 360. (That is, reduce mod 360.)
4. Compute $n\theta$, reduce it mod 360, convert the result from degrees to radians, and set x equal to the final result.
5. Set r equal to the sin of x.
6. Convert θ from degrees to radians, and set t equal to the result.
7. Convert the point (t, r) from polar to rectangular coordinates to obtain the point (*newx*, *newy*).
8. Draw a line from (*oldx*, *oldy*) to (*newx*, *newy*).
9. If θ is equal to zero then stop, else set (*oldx*, *oldy*) to (*newx*, *newy*) and go back to step 3.

Algorithm-A computes the points $(\theta, \sin(n\theta))$ for $\theta = 0°, d°, 2d°, \ldots,$ and draws lines between each pair of successively computed points. The first computed point and the last computed point always coincide, so the figure drawn is always a closed polygon. The values of n and d can be chosen at random or supplied by the

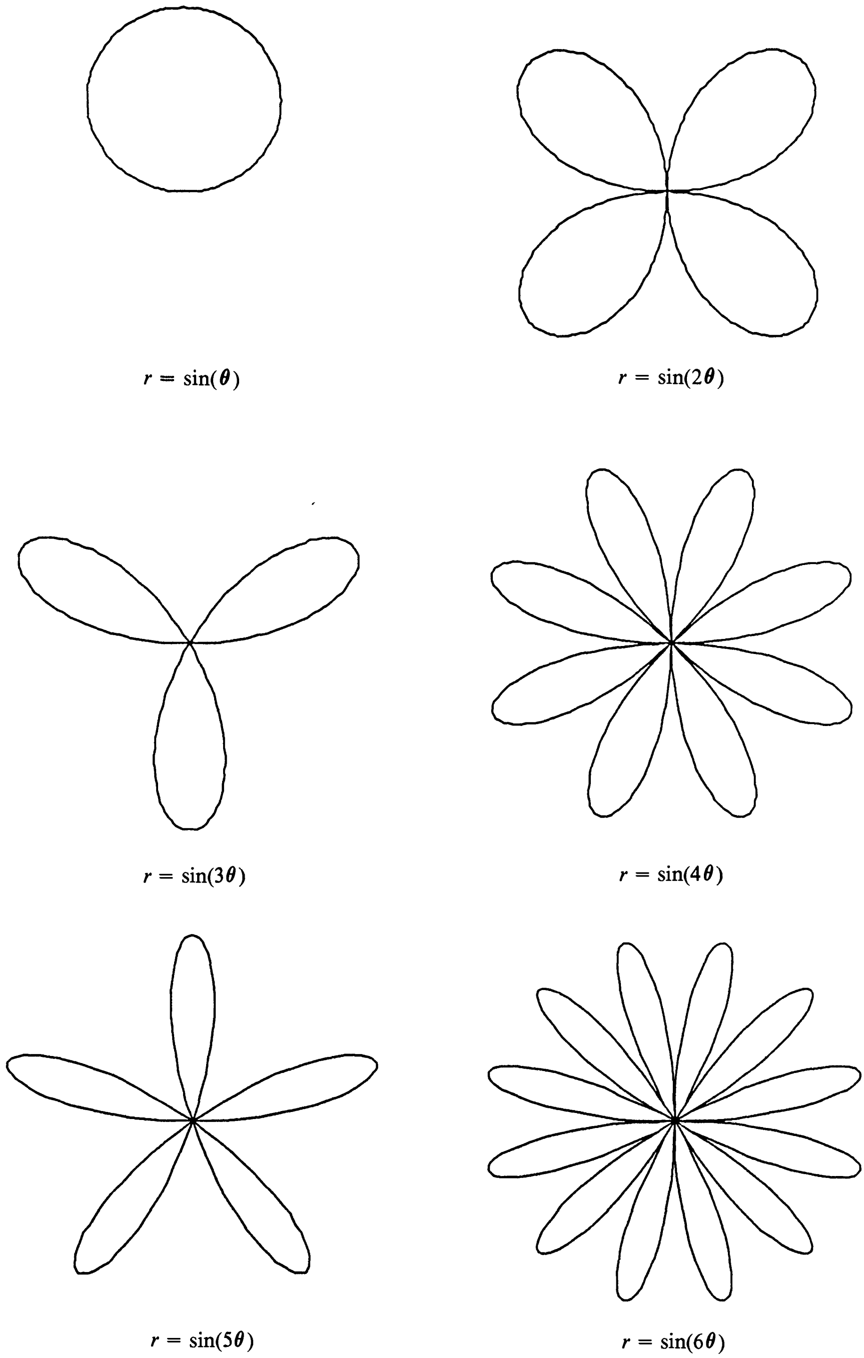
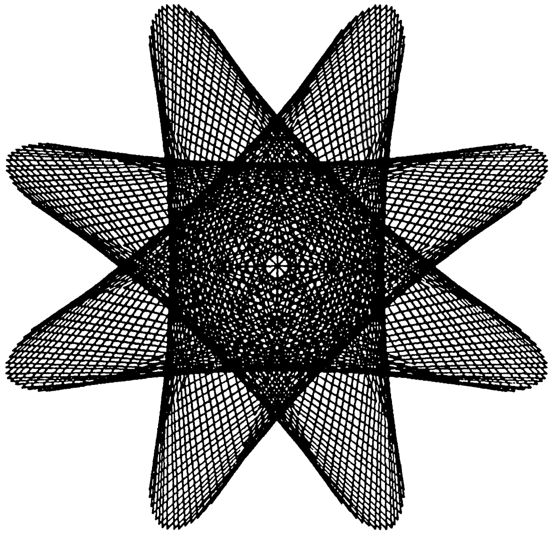
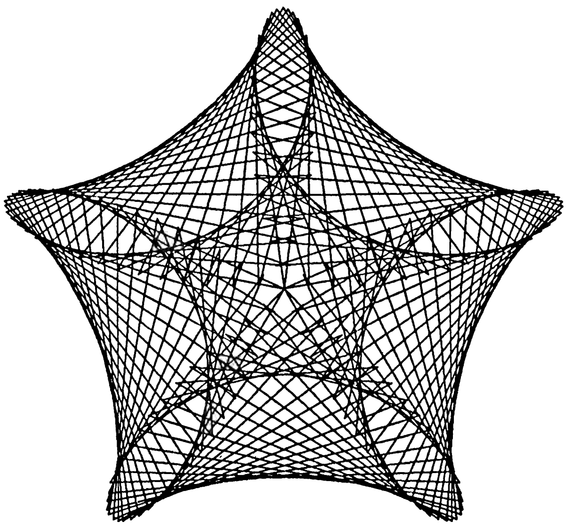


FIG. 1. The graphs of $r = \sin(n\theta)$ for $\theta = 1$–6.



$n = 4,\ d = 43$



$n = 5,\ d = 97$

FIG. 2. "The Rose" drawings for random n and d.

user of the program. Figure 2 gives examples of pictures drawn with randomly-chosen values for n and d.

Unfortunately, not all of the drawings generated by algorithm-A are as beautiful as those shown in Figure 2. Many of the drawings contain only a few lines, and many consist of a single dot. It would be esthetically pleasing to get rid of these degenerate figures but first it is necessary to understand why they occur.

Let G be the additive group of integers mod 360. Examination of steps 3 and 9 of algorithm-A makes it obvious that the number of lines in a drawing is equal to the order of d in G. (Some of the lines may be degenerate with starting and ending points coinciding.) It is a simple matter to show that the order of d in G is equal to $360/k$, where k is the greatest common divisor of d and 360. Now, let H be the subgroup of G generated by d. H has k distinct cosets in G of the form $H, H+1, H+2, \ldots, H+k-1$. A degenerate drawing is produced when the order of H is less than 360. However, when this is the case, a drawing can also be produced for each of the cosets $H+1, H+2, \ldots$. Furthermore, a different drawing will be produced for each distinct coset. The degenerate figures can be eliminated by superimposing the drawings for the cosets $H+1, H+2, \ldots$ over the drawing for H. Algorithm-B does exactly this:

1. Choose integers n, d such that $1 \leqslant n \leqslant 359$ and $1 \leqslant d \leqslant 359$.
2. Set T and c equal to zero.
3. Set θ equal to T. Compute the point $(2\pi\theta/360, \sin(2\pi n\theta/360))$, convert it to rectangular coordinates and set (*oldx*, *oldy*) to the result.
4. Set θ equal to $\theta + d$. If $\theta \geqslant 360$ replace θ by the remainder obtained when dividing θ by 360.
5. Compute $n\theta$, reduce it mod 360, convert the result from degrees to radians, and set x equal to the final result.
6. Set r equal to the sin of x.
7. Convert θ from degrees to radians, and set t equal to the result.
8. Convert the point (t, r) from polar to rectangular coordinates to obtain the point (*newx*, *newy*).
9. Draw a line from (*oldx*, *oldy*) to (*newx*, *newy*).
10. Add 1 to c.
11. If θ is equal to T then go to step 12, else set (*oldx*, *oldy*) to (*newx*, *newy*) and go back to step 4.
12. If $c \geqslant 360$ stop, else add 1 to T and go back to step 3.

Figure 3 gives drawings for various cosets with $n = 3$ and $d = 72$. Figure 4 demonstrates the difference between algorithm-A and algorithm-B for $n = 4$ and $d = 120$.

With this modification, it is possible to study the evolution of a drawing for a fixed n, as d ranges from 1–360, without worrying about degenerate drawings. Figures 5a and 5b illustrate this evolution for $n = 2$. First the apparently smooth line of the drawing becomes wider and more "lacy" until the space between the loops disappears. At the same time a squarish figure appears in the center.

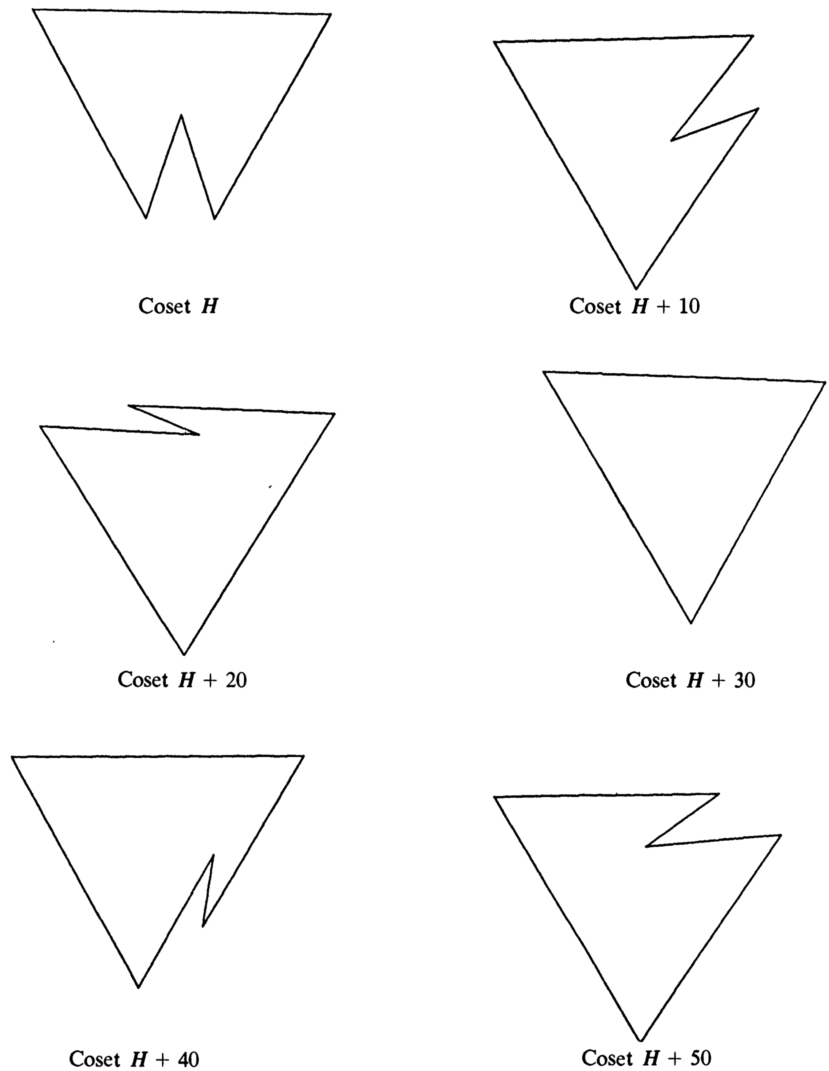
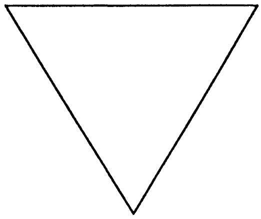


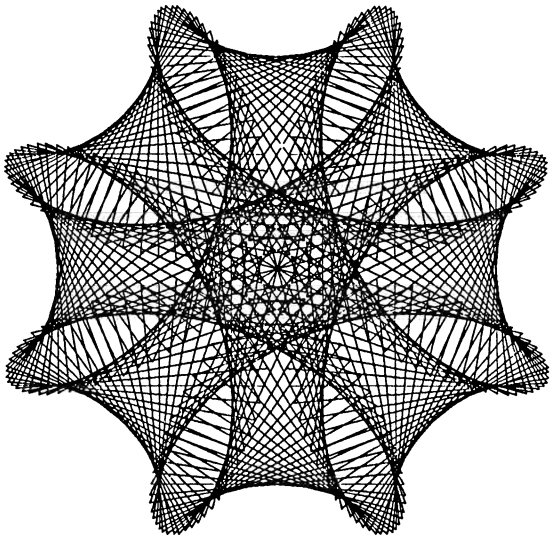
FIG. 3. Coset drawings for $n = 3$, $d = 72$.

Eventually the squarish figure grows until it overwhelms the entire figure leaving holes for the original petals. Then the petals become more "hairy" looking and the squarish shape begins to degenerate into filigree between the petals, until the shape disappears entirely leaving only the hairy petals. This form of evolution takes place (at different rates) for all figures with small n.

3. Very large *n*. All of the examples given so far have used fairly small values of n, even though step 1 of both algorithms allows n to range from 1 to 359. As n



$n = 4$, $d = 120$ (Algorithm-A)



$n = 4$, $d = 120$ (Algorithm-B)

FIG. 4. The difference between algorithms A and B.

becomes very large (say, greater than 60) the structure of the underlying rose disappears, but other puzzling phenomena begin to occur. Figure 6 gives examples of some of these phenomena.

Consider the drawing for $n = 181$, $d = 2$. This looks suspiciously like two copies of the drawing for $n = 1$, $d = 1$, rotated 180 degrees from one another. Further experimentation with the program will show that the drawing for $n = 121$, $d = 3$ is three circles whose centers are 120 degrees apart, and the drawing for $n = 91$, $d = 4$ is four circles whose centers are 90 degrees apart. Furthermore, the drawing for $n = 183$, $d = 2$ resembles two copies of the drawing for $n = 3$, $d = 1$, rotated 180 degrees from each other.

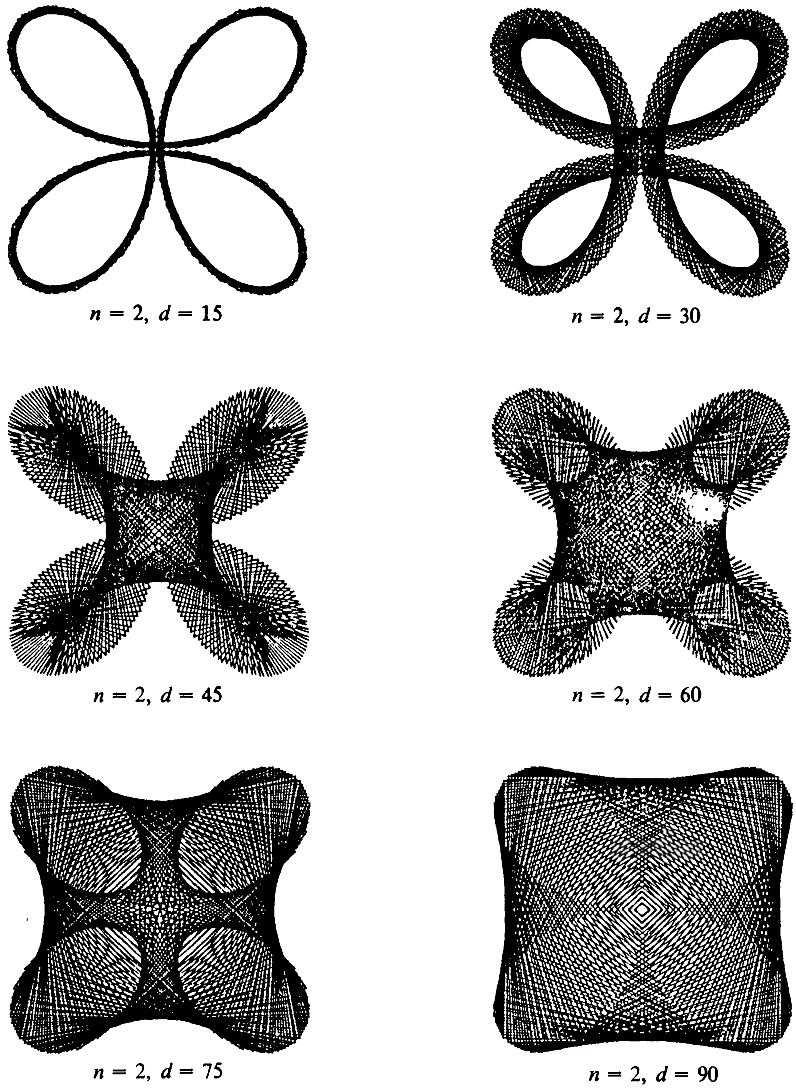
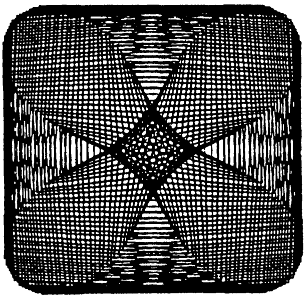


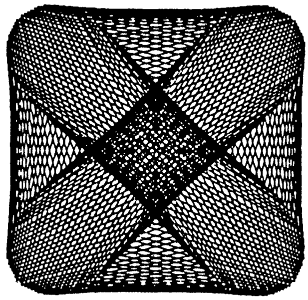
FIG. 5a. The evolution of $n = 2$ for $d = 1$–90.

These resemblances are not superficial as the following theorem shows. We will call this theorem the *zero* + theorem because it involves adding an integer to a zero-divisor in the ring of integers mod 360. The proof is a simple calculation and is omitted.

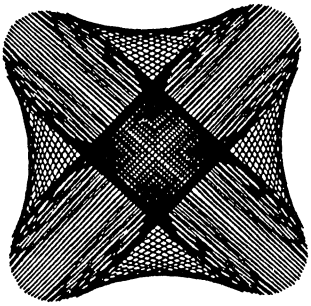
THE ZERO + THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees and let the points* $(\theta, \sin(\theta))$ *represent the angle and radius of points in the usual polar coordinate system.*



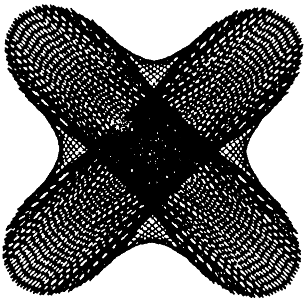
$n = 2,\ d = 105$



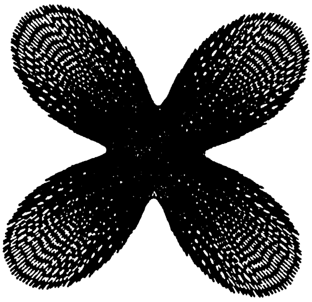
$n = 2,\ d = 120$



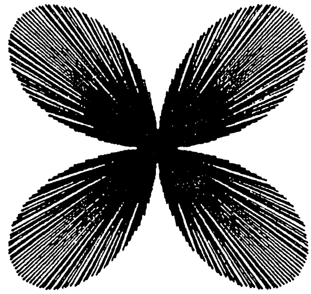
$n = 2,\ d = 135$



$n = 2.\ d = 150$



$n = 2,\ d = 165$



$n = 2,\ d = 180$

FIG. 5b. The evolution of $n = 2$ for $d = 90$–180.

Let θ, k, n, and m be integers and let $nm = 360$. If θ is of the form $mj + i$ where i and j are integers, then the point $(\theta, \sin((n + k)\theta))$ lies on the curve defined by $(\theta, \sin(k\theta))$ rotated ni degrees clockwise about the origin.

Note that the *zero* + theorem assumes that figures are drawn using a d that divides 360. This implies that each copy of the "small n" figure is generated by a distinct coset of the subgroup H of G generated by d. It turns out that the points

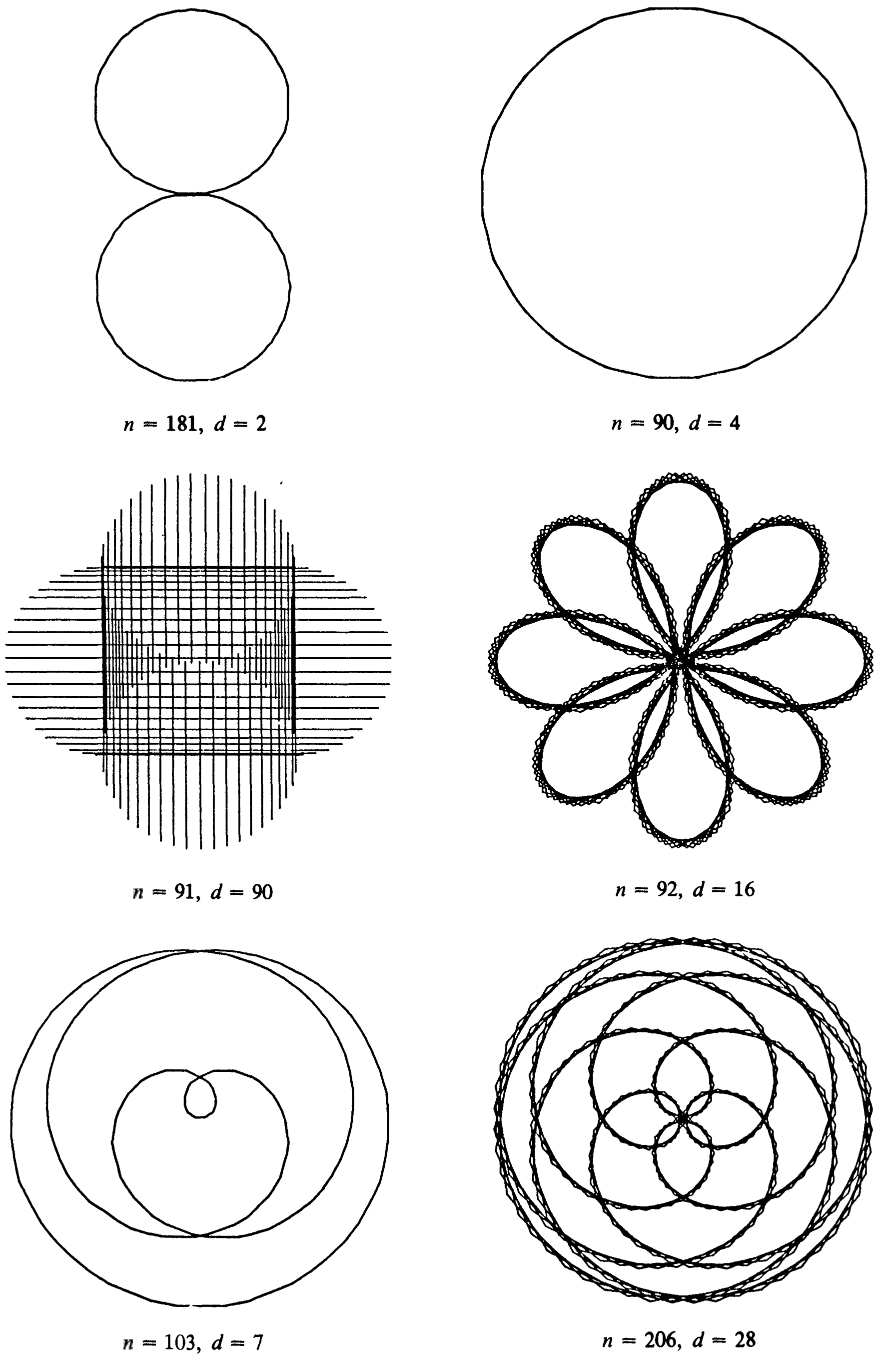


FIG. 6. Drawings with very large n.

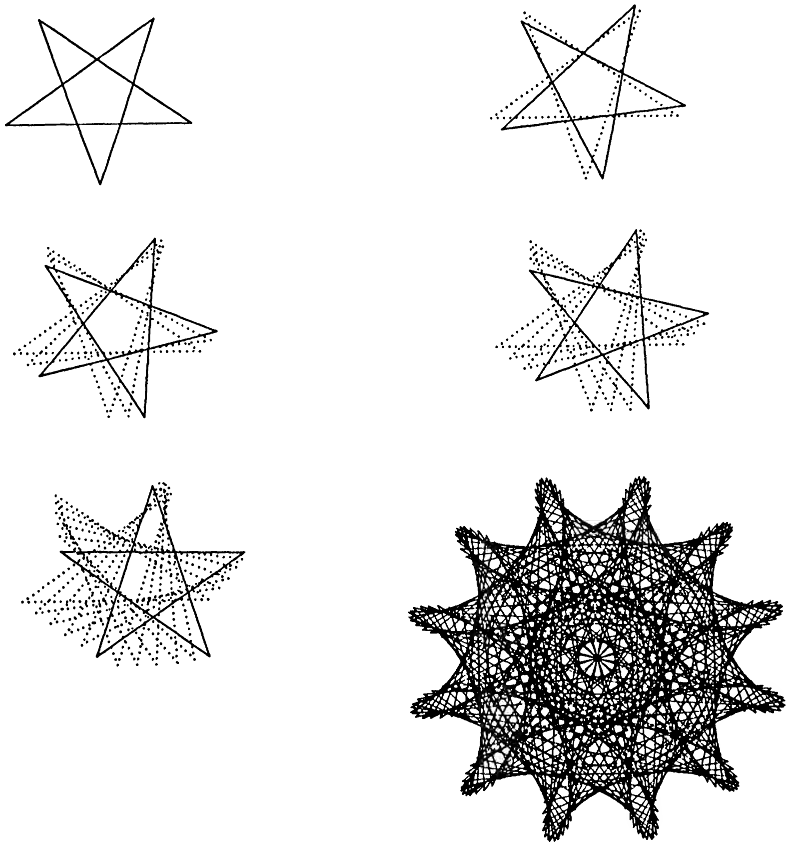


FIG. 7. The rotating pentangle.

computed from each successive coset are ni degrees further along the graph of the "small n" figure as well as being rotated ni degrees about the origin. For certain values of n and d, this produces an amazing visual effect when the figure is drawn. For example when the polygon for $n = 5 + 1 = 6$ and $d = 72$ is drawn, the observer sees a rotating pentangle which eventually produces a figure which looks nothing like a pentangle (see Figure 7).

The *zero* + theorem can be used to explain the appearance of the drawings for $n = 181$, $d = 2$ (two copies of $r = \sin(\theta)$); $n = 91$, $d = 90$ (four underlying circles joined by horizontal and vertical lines); and $n = 92$, $d = 16$ (four "evolved" copies of $r = \sin(2\theta)$, that overlap in pairs).

The *zero* + theorem can be extended to negative offsets from zero-divisors by observing that the graph of $r = \sin((-n)\theta)$ is identical to the graph of $r = \sin(n\theta)$ rotated 180 degrees about the origin. (Proof: $\sin(n\theta + 180) + \sin(n\theta)\cos(180) + \cos(n\theta)\sin(180) = -\sin(n\theta) = \sin(-n\theta)$.) It is a consequence of the *zero* + theo-

rem that the figures for $n \geqslant 360$ are identical to those for $0 \leqslant n \leqslant 359$. Therefore, selecting n in this range provides the richest possible set of drawings.

Now consider the drawing for $n = 90$, $d = 4$. This drawing consists of two coinciding circles of radius 1 (as opposed to the radius .5 circle generated by $n = 1, d = 1$) and two coinciding dots in the center (the dots may not be visible in Figure 6). Similar drawings are generated for $n = 120, d = 3$ and for $n = 72, d = 5$. The following theorem explains the appearance of these drawings. We will call this theorem the *zero* theorem, because it concerns the zero-divisors of the ring of integers mod 360.

THE ZERO THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees, and let* θ *and* n *be integers and let* n *divide* 360. *Then all points of the form* $(\theta, \sin(n\theta))$ *lie on* $m = 360/n$ *concentric circles centered on the origin (some of the circles may be of equal radius, and some may be of radius zero).*

Proof. Let m be the integer such that $nm = 360$. Consider all points θ of the form $mk + a$ with a constant. Then $\sin(n\theta) = \sin(n(mk + a)) = \sin(nmk + na)$ $= \sin(360k + na) = \sin(360k)\cos(na) + \cos(360k)\sin(na) = \sin(na)$. Since both n and a are constant, so is $\sin(na)$. As a ranges from zero to $m - 1$, m distinct sets of values are produced. The graph of $r = k$ is a circle of radius $|k|$ about the origin. □

For $n = 90$, there are four distinct sets of values which produce the curves $r = \sin(0)$, $r = \sin(90)$, $r = \sin(180)$, and $r = \sin(270)$, which are circles of radius 0, 1, 0, and 1, respectively. When the polygon for $n = 90$ and $d = 4$ is drawn, one can see the second big circle being plotted, since the plotting points don't coincide. The dot in the center is, obviously, the plot of the zero-radius circles.

Lastly, consider the two most puzzling drawings in Figure 6, namely, those for $n = 103, d = 7$ and $n = 206, d = 28$. Experimentation with the parameters $n = 103, d = 7$ will show that even a slight change in n or d will make the spiral disappear. Furthermore, there are only a few combinations of n and d that give rise to spirals in the first place. It turns out that the curve for $n = 103$, and $d = 7$ coincides exactly with the graph of $r = \sin(\theta/7)$ plotted in 7-degree increments from 0 to 2520 degrees. The critical factor is that the product of 103 and 7 is equivalent to 1 mod 360. The following theorem, which we will call the *unity* theorem, explains this phenomenon.

THE UNITY THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees, and let* n *and* m *be integers such that* $nm \equiv 1 \bmod 360$. *Let* α *be an arbitrary integer. Then the points* $(\alpha, \sin(n\alpha))$ *all lie on the graph of the equation* $r = \sin(\theta/m)$. *Furthermore, if* $r = \sin(n\alpha)$ *is evaluated in* m *degree increments, an approximation to the graph of* $r = \sin(\theta/m)$ *will be produced.*

Proof. Given an integer $\alpha \sin(n\alpha) = \sin((m/m)n\alpha) = \sin(nm\alpha/m)$. Since $nm\alpha \equiv \alpha \bmod 360$, the points $(\alpha, \sin(n\alpha))$ and $(nm\alpha, \sin(nm\alpha/m))$ coincide. To show that $0, m, 2m, 3m, \ldots$ produce successive points along the curve, observe that $\sin(nim) = \sin(i)$, $i = 0, 1, 2, 3, \ldots$. □

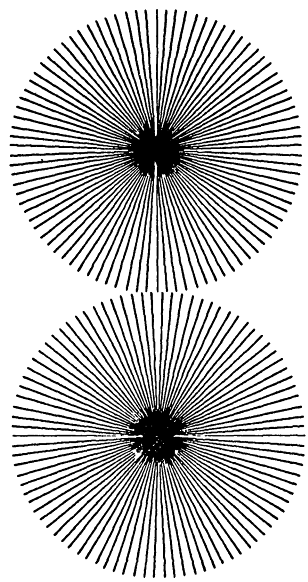
The *unity* theorem has the following corollary: Let m and n be two integers, $1 \leqslant m \leqslant 359$, $1 \leqslant n \leqslant 359$, such that n has a multiplicative inverse mod 360. Then it is possible to produce an approximation to the graph of $r = \sin(m\theta/n)$ using algorithm-B and an appropriate selection of the parameters n and d. The final drawing of Figure 6, with $n = 206$ and $d = 28$ is an "evolved" graph of $r = \sin(2\theta/7)$.

4. Dividing the circle into an arbitrary number of parts. Although dividing the circle into 360 equal parts is a time-honored tradition, there is no reason why some other number of subdivisions cannot be used. In fact, sometimes a small change in the number of circle subdivisions can make a profound difference in the drawings generated for a given n and d. The following algorithm, called "algorithm-C" allows the circle to be divided into z parts, where z is an arbitrary positive integer.

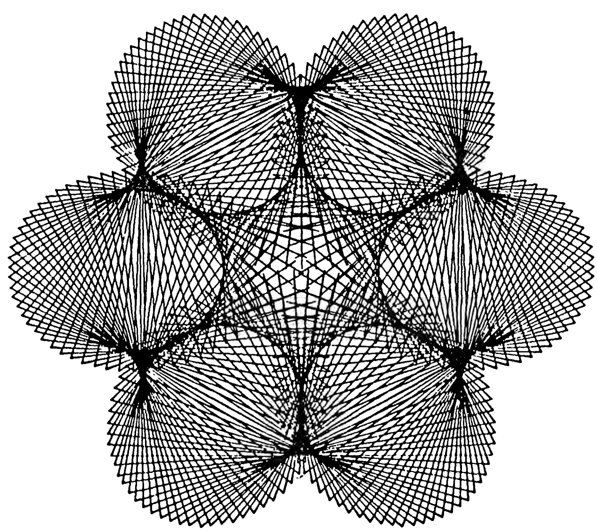
1. Choose integers z, n, d such that $1 \leqslant n < z$ and $1 \leqslant d < z$.
2. Set T and c equal to zero.
3. Set θ equal to T. Compute the point $(2\pi\theta/z, \sin(2\pi n\theta/z))$, convert it to rectangular coordinates and set (*oldx*, *oldy*) to the result.
4. Set θ equal to $\theta + d$. If $\theta \geqslant z$ replace θ by the remainder obtained when dividing θ by z.
5. Compute $n\theta$, reduce it mod z, multiply by $2\pi/z$, and set x equal to the final result.
6. Set r equal to the sin of x.
7. Set t equal to $2\pi\theta/z$.
8. Convert the point (t, r) from polar to rectangular coordinates to obtain the point (*newx*, *newy*).
9. Draw a line from (*oldx*, *oldy*) to (*newx*, *newy*).
10. Add 1 to c.
11. If θ is equal to T then go to step 12, else set (*oldx*, *oldy*) to (*newx*, *newy*) and go back to step 4.
12. If $c \geqslant z$ stop, else add 1 to T and go back to step 3.

Figure 8 demonstrates the effect of changing the number of circle-subdivisions from 360 to 359. Since 359 is prime, there are no analogs of the *zero* and *zero* + theorems, but the *unity* theorem still applies. Figure 9 gives some examples of drawings created with 359 circle subdivisions. None of these drawings could have been created with 360 subdivisions.

5. Conclusion. Because they are static, the drawings presented in this article cannot do justice to "The Rose" program. Readers with access to high-speed computer graphics equipment are encouraged to implement their own versions of "The Rose" and view the construction of the drawings first-hand. Many of the drawings exhibit apparent motion as they are being drawn, and in some cases, such as the rotating pentangle described above, the visual effect is quite stunning.

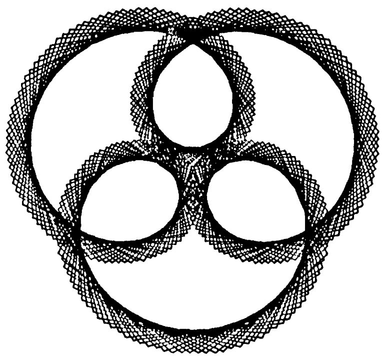


$n = 181,\ d = 90,\ z = 360$

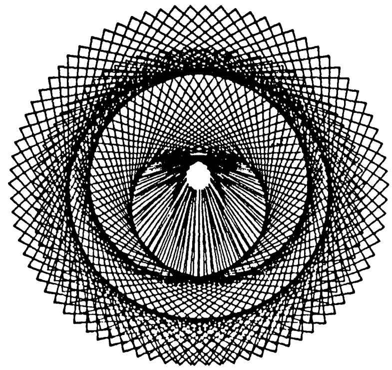


$n = 181,\ d = 90,\ z = 359$

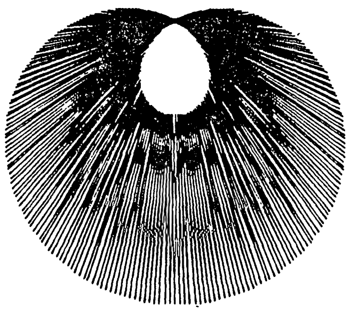
FIG. 8. The effect of changing circle subdivisions.



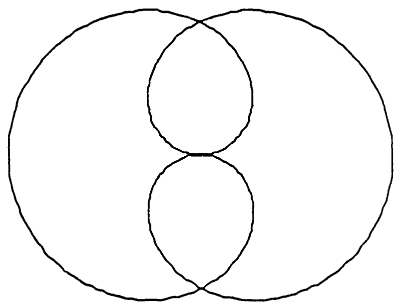
$n = 216,\ d = 50$



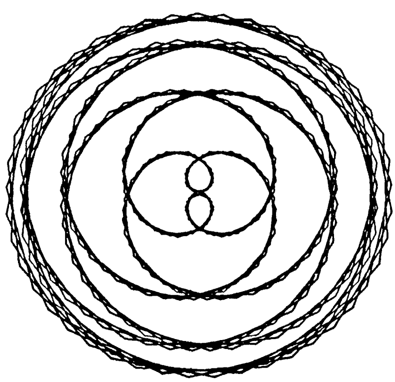
$n = 40,\ d = 89$



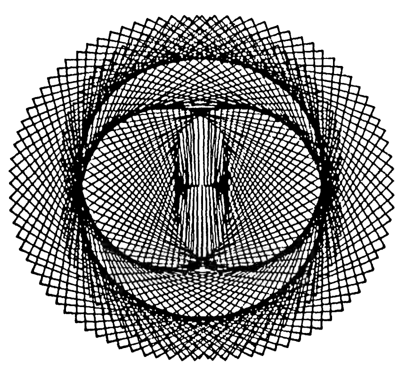
$n = 120,\ d = 92$



$n = 180,\ d = 2$



$n = 45,\ d = 32$



$n = 90,\ d = 90$

FIG. 9. Drawings created with $z = 359$.

There are a number of interesting, and probably not too difficult, problems that remain to be solved. Among them are:

1. What are the rules that govern the shape of the individual cosets drawn by algorithm-B?
2. Many of the drawings have apparent curves that are generated by intersections of lines. What are the parametric equations for these curves? Are any of these curves well known? How do the parametric equations evolve as the figure evolves?
3. Algorithms A and B both draw illustrations of the additive group of integers mod 360 and its subgroups. What about the multiplicative group and its subgroups?
4. When n is odd, each line drawn by algorithms A and B is drawn twice. Can any use be made of this?
5. In this article, considerable use was made of the fact that $\sin(n\theta)$ is periodic in $360°$. In fact, the periods of these functions are usually much smaller than $360°$. Is this important?
6. What about sums and products of sin and cos functions?

There are undoubtedly many other interesting questions that could be asked. The reader is encouraged to try his hand at discovering them.

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The Mathematician's Dictionary

What mathematical concept can be defined as:
Its value on any bowlfull is the length of the longest floating noodle?

(See page 702)

Letters to the Editor

Editor:

In their paper, *Homomorphisms on* $C(R)$, (this MONTHLY, Aug.-Sept. 1986, p. 555), Aron and Fricke give an elementary proof that if ϕ: $C(R) \to R$ is a nonzero, algebra homomorphism, then ϕ is an evaluation, i.e., for some c in R, $\phi(f) = f(c)$ for all f in $C(R)$. It is sufficient to assume that ϕ is only a ring homomorphism. This follows from a basic property of the real number field which, apparently, is not as well known as it ought to be, namely:

THEOREM. *The only nonzero homomorphism of the real field into itself is the identity mapping.*

Now let $F(X)$ be any ring of real-valued functions on a nonempty set X which contains the ring $\hat{R}(X)$ of real constants on X. Let ϕ: $F(X) \to R$ be a nonzero ring homomorphism. If r is a real number, then we denote the corresponding constant function on X by $\hat{r}$, i.e. $\hat{r}(x) = r$ for all x in X. Then (as Aron and Fricke show using only the ring properties) $\phi(\hat{1}) = 1$. And since $\hat{R}(X)$ is an isomorphic copy of R, by the theorem above, $\phi(\hat{r}) = r$ for all r in R. Hence, $\phi(rf) = \phi(\hat{r}f) = \phi(\hat{r})\phi(f) = r\phi(f)$ for all r in R and f in $F(X)$. Therefore, our ring homomorphism is a real algebra homomorphism.

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Editor:

I would like to bring to the reader's attention an inaccurate statement appearing in my article [1] and kindly pointed out to me by Professor Mary L. Boas, Department of Physics, DePaul University. On p. 67, I claim that "if any two of the principal moments of inertia are equal, then (3) immediately reduces to a simpler linear system which is easily seen to yield only stable rotations." In fact, stable rotations occur only about the axis corresponding to the unique principal moment of inertia, although the instability of the other rotations is degenerate and exhibits different behavior from the wobble of a body having three distinct principal moments of inertia, the main subject of my note. Furthermore, the wobble in this case is more difficult to observe experimentally than that which occurs when all the principal moments of inertia are distinct.

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NOTES

EDITED BY CAROL G. CRAWFORD, RICHARD LIBERA, AND ANITA E. SOLOW

For instructions about submitting Notes for publication in this department see the inside front cover.

Generalization of Some Algorithms of Euler

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Euler [2] demonstrated four cases of the general theorem proven below. Euler's intent was to simultaneously approximate the $L-1$ irrationals

$$r^{1/L},\quad r^{2/L},\ldots,\quad r^{(L-1)/L}$$

by rational numbers, r a suitable integer. Euler gave algorithms for $L = 2,3,4,5$. We will define an $L \times L$ matrix $E_L(r)$ and construct a complete set of eigenvalues and eigenvectors. Taking powers of this matrix and multiplying by a vector embraces Euler's algorithms. Rates of convergence are evident. This matrix provides an instructive example for Perron's theorem on positive matrices, anticipated in this instance by Euler.

Let $\mathbb{Z}$ denote the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers.

Let $r > 1$ be an integer. Define the $L \times L$ matrix

$$E_L(r) = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ r & 1 & 1 & \ldots & 1 & 1 \\ r & r & 1 & \ldots & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r & r & r & \ldots & 1 & 1 \\ r & r & r & \ldots & r & 1 \end{bmatrix}.$$

In other words $E_L(r) = (e_{ij})$, $e_{ij} = 1$ if $1 \leqslant i \leqslant j \leqslant L$, $e_{ij} = r$ if $1 \leqslant j < i \leqslant L$. Then $\det E_L(r) = (1-r)^{L-1}$ and $E_L(r)^{-1} = (d_{ij})$, $d_{i,i} = 1/(1-r)$ if $1 \leqslant i \leqslant L$, $d_{i,i+1} = 1/(r-1)$ if $1 \leqslant i < L$, $d_{L,1} = r/(r-1)$, and $d_{i,j} = 0$ otherwise.

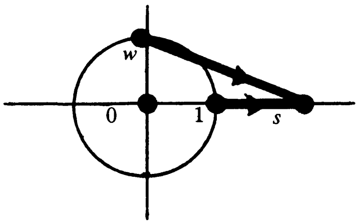
Let s satisfy $s^L = r > 1$ and set

$$v(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \ldots \\ s^{L-2} \\ s^{L-1} \end{bmatrix} \in \mathbb{C}^L.$$

Then $E_L(r)v(s) = v(s)(r-1)/(s-1)$. Let $w^L = 1$ and choose s real, $s^L = r$, $s > 1$. Then for L distinct w's the vectors $v(ws)$ and scalars $(r-1)/(ws-1)$ form a linearly independent set of L distinct eigenvectors and eigenvalues respectively for

$E_L(r)$. Notice that $|ws - 1| = |s - w|$ and that

$$|s - w| > |s - 1| \quad \text{if } w \neq 1 = |w|.$$



Therefore $(r - 1)/(s - 1)$ is the dominant eigenvalue, the remaining eigenvalues having smaller absolute values.

Let $b \in \mathbb{C}^L$ where

$$b = \sum_w b_w v(ws)$$

for appropriate $b_w \in \mathbb{C}$; the summation is over L distinct w's, $w^L = 1$. Then

$$E_L(r)^k b = ((r-1)/(s-1))^k \sum_w b_w v(ws)((s-1)/(ws-1))^k. \tag{1}$$

From the strict inequality $s - 1 < |ws - 1|$ noticed above it follows that

$$\lim_{k \to \infty} E_L(r)^k b(s-1)^k/(r-1)^k = b_1 v(s).$$

If v is a nonzero vector, $\mathbb{C}v$ is the complex line generated by v. We summarize the above in the following result. This is a special case of Perron's theorem, e.g., Theorem 4, page 292, in Bellman [**1**].

THEOREM. *Let $L, r \geqslant 2$ be integers, s the positive Lth root of $r = s^L$. Then for $0 \neq b \in \mathbb{C}^L$,*

$$\lim_{k \to \infty} \mathbb{C} E_L(r)^k b = \mathbb{C} v(s).$$

In particular, the lines generated by the columns of $E_L(r)^k$ converge (as $k \to \infty$) to the line generated by the above vector of Lth roots of r.

COROLLARY (Euler [**2**]; $L = 2, 3, 4, 5$). *Let $0 \neq b \in \mathbb{Z}^L$. Set*

$$E_L(r)^k b = \begin{bmatrix} q \\ p_1 \\ p_2 \\ \cdots \\ p_{L-2} \\ p_{L-1} \end{bmatrix},$$

where $q > 0$ for large enough k. Then

$$p_j/q \to r^{j/L}, \quad 1 \leqslant j < L, \quad \text{as } k \to \infty.$$

Specifically,

$$|r^{j/L} - p_j/q| \leqslant constant\ \varepsilon^k/q.$$

The constant is independent of k *and* $0 < \varepsilon < 1$.

As an example, Euler [2, p. 193] takes $L = 2$, $r = 3$ and seeks rational approximations to

$$\sqrt{3} = 1.73205080756887729352744634150587\ldots.$$

He writes down the first five matrix powers

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}, 2\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, 2\begin{pmatrix} 5 & 3 \\ 9 & 5 \end{pmatrix}, 4\begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}, 4\begin{pmatrix} 19 & 11 \\ 33 & 19 \end{pmatrix}.$$

G.c.d.'s of columns of these matrix powers are always powers of two. He then takes $b = \binom{1}{2}$ and multiplies

$$4\begin{pmatrix} 19 & 11 \\ 33 & 19 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 41 \\ 71 \end{pmatrix}.$$

Actually, Euler is working a century before matrices as we know them were invented so he is without the convenience of matrix notation. That doesn't stop him however: instead of multiplying matrices he makes the corresponding linear substitutions. Considering the ratio $71/41 = 1.731\ldots$ he takes a new $b = \binom{41}{71}$, multiplies,

$$\begin{pmatrix} 19 & 11 \\ 33 & 19 \end{pmatrix}\begin{pmatrix} 41 \\ 71 \end{pmatrix} = \begin{pmatrix} 1560 \\ 2702 \end{pmatrix},$$

and reduces $2702/1560 = 1351/780 = 1.732051\ldots$. This approximation, as Euler observes, first differs in the sixth place from the decimal string for $\sqrt{3}$. He stops at this point, but we continue on and get further successive stages

$$\begin{pmatrix} 19 & 11 \\ 33 & 19 \end{pmatrix}\begin{pmatrix} 780 \\ 1351 \end{pmatrix} = \begin{pmatrix} 29681 \\ 51409 \end{pmatrix}$$

and $51409/29681 = 1.732050806\ldots$,

and next $978122/564719 = 1.732050807569\ldots$,

and next $37220045/21489003 = 1.732050807568876\ldots$.

Euler's algorithm corresponds to taking approximation vectors

$$E_L(r)^{km}b,$$

where in the case of $L = 2$, $r = 3$ he took $m = 5$ and $k = 1, 2$. We know in this case from the Corollary above that accuracy at worst increases by powers of $\varepsilon = (\sqrt{3} - 1)/(\sqrt{3} + 1) = 0.26\ldots$. Then $\varepsilon^5 = 0.0013\ldots$ and $\varepsilon^{10} = 0.0000019\ldots$ which explains the improvement in accuracy.

For a geometric example, rather than a numerical example, consider the case of $r = 2$. Recall that $\det E_L(2) = (-1)^{L-1}$ so that all of the L integers of any column of each power of $E_L(2)$ will be relatively prime (not so for integers $r > 2$).

Geometrically, $E_L(2)$ and its powers are automorphisms of $\mathbb{Z}^L$. Notice that

$$E_L(2)^{-1} = (d_{ij}), \quad d_{ii} = -1, \quad 1 \leqslant i \leqslant L, \quad d_{i,i+1} = 1, \quad 1 \leqslant i < L,$$
$$d_{L,1} = 2 \quad \text{and } d_{ij} = 0 \quad \text{otherwise}$$

The columns of $E_L(2)^k$ form a basis for $\mathbb{Z}^L$, $E_L(2)^k$ is invertible, its inverse has integer entries, $E_L(2)^k \in \mathrm{GL}(L, \mathbb{Z})$, any $k \in \mathbb{Z}$. $\mathrm{GL}(L, \mathbb{Z})$ are the $L \times L$ integral matrices with det ± 1. Note that $\mathrm{GL}(1, \mathbb{Z}) = \{\pm 1\}$. We consider $\mathrm{GL}(1, \mathbb{Z})^L$ to be embedded in $\mathrm{GL}(L, \mathbb{Z})$ as the diagonal matrices in $\mathrm{GL}(L, \mathbb{Z})$, diagonal matrices with diagonal entries ± 1.

Denote by $C_{L,k,j}$ the jth column of $E_L(2)^k$, $1 \leqslant j \leqslant L$. Consider the simplex $\Delta_L^{(k)}$, $k \in \mathbb{Z}$, defined by the convex hull of the origin and the columns of $E_L(2)^k$. That is for

$$\Delta_L^{(0)} = \left\{ t \in \mathbb{R}^L \colon t = \begin{pmatrix} t_1 \\ \cdots \\ t_L \end{pmatrix}, 0 \leqslant t_j \quad \text{for } 1 \leqslant j \leqslant L \quad \text{and} \quad \sum_{1 \leqslant i \leqslant L} t_i \leqslant 1 \right\},$$

then

$$\Delta_L^{(k)} = \left\{ x \in \mathbb{R}^L \colon x = \sum_{1 \leqslant j \leqslant L} t_j C_{L,k,j}, \quad t \in \Delta_L^{(0)} \right\}$$
$$= E_L(2)^k \Delta_L^{(0)}.$$

Define the cross-polytope $\nabla_L^{(k)}$ by the union of the 2^L signed images of $\Delta_L^{(k)}$. That is

$$\nabla_L^{(k)}(2) = \bigcup_{T \in \mathrm{GL}(1, \mathbb{Z})^L} E_L(2)^k T \Delta_L^{(0)}.$$

We will call the following proposition an exclusion proposition. It states that rational approximations to the line can come from lattice points which are $\pm$columns of $E_L(2)^k$. Any better rational approximations must lie outside the cross-polytope $\nabla_L^{(k)}(2)$. As k increases without bound the constant volume cross-polytopes $\nabla_L^{(k)}(2)$ engulf the line $\mathbb{R} v_L(2)$.

Proposition. *The intersection of the set of lattice points $\mathbb{Z}^L$ and the cross-polytope $\nabla_L^{(k)}(2)$, $k \in \mathbb{Z}$, is exactly the set of vertices of $\nabla_L^{(k)}(2)$ and the origin.*

Proof. If $x \in \nabla_L^{(k)}(2) \cap \mathbb{Z}^L$, then $x = E_L(2)^k T t$ where $t \in \nabla_L^{(0)}$, $T \in \mathrm{GL}(1, \mathbb{Z})^L$. Since $E_L(2)^{\pm k} \in \mathrm{GL}(L, \mathbb{Z})$, $t = T E_L(2)^{-k} x \in \mathbb{Z}^L$. Then

$$0 \leqslant t_j \leqslant \sum_{1 \leqslant i \leqslant L} t_i \leqslant 1$$

implies at most one $t_j = 1$ and the rest zero or all $t_j = 0$. Therefore $x = \pm C_{L,k,j}$ for some j, $1 \leqslant j \leqslant L$ or $x = 0$. Thus if $0 \neq x \in \mathbb{Z}^L \cap \nabla_L^{(k)}(2)$, then x is $\pm$ a

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Applications of a Product Identity to Sums of Squares and Partitions

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The following identity, ascribed to Gauss [**2**, p. 23], is usually derived from the Jacobi triple product identity.

$$\prod_{m=1}^{\infty}(1-x^m)=\prod_{m=1}^{\infty}(1+x^m)\sum_{k=-\infty}^{\infty}(-1)^k x^{k^2}. \tag{1}$$

Recent papers [**5**], [**6**] have used the triple product identity to prove results about sums of two squares. In 1877 G. H. Halphen used (1) to prove that primes congruent to 1 (mod 4) and to 3 (mod 8) are sums of 2 and 3 squares, respectively [**3**, pp. 244, 266]. Of course the first result, along with the identity

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2,$$

quickly leads to the usual characterization of sums of 2 squares.

Although a fairly compact analytic proof of the triple product identity is possible [**4**], identity (1) can also be proved without analysis in the ring of formal power series, as developed in [**9**] or [**10**]. The proof I give is a nonanalytic adaptation of one sometimes given for the Triple Product Identity [**7**, p. 188; **11**, p. 186]. Later I will show how (1) applies to sums of squares and partitions with an odd number of parts.

The first part of the proof of (1) involves only integral polynomials in the variables x and y. We define

$$F(y)=\prod_{i=1}^{n}(1+x^{2i-1}y)(y+x^{2i-1})=\sum_{i=-n}^{n}A_i y^{n+i}, \tag{2}$$

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The first part of the proof of (1) involves only integral polynomials in the variables x and y. We define

$$F(y)=\prod_{i=1}^{n}(1+x^{2i-1}y)(y+x^{2i-1})=\sum_{i=-n}^{n}A_i y^{n+i}, \tag{2}$$

where the A_i are integral polynomials in x. Then

$$(y + x^{2n-1})F(x^2y) = x^{2n-1}(1 + x^{2n+1}y)F(y).$$

If we substitute the formula involving the A_i's into the latter equation and equate the coefficients of y^{n+i}, we get

$$A_{i-1} = (1 - x^{2n+2i})A_i/x^{2i-1}(1 - x^{2n-2i+2}). \tag{3}$$

Now $A_n = x^{1+3+\cdots+(2n-1)} = x^{n^2}$, and it is straightforward from this and (3) to prove

$$A_{n-k} = (1 - x^{4n})(1 - x^{4n-2})\cdots(1 - x^{4n-2k+2})x^{(n-k)^2}$$
$$/(1 - x^2)(1 - x^4)\cdots(1 - x^{2k})$$

by induction on k. If we let $j = n - k$, this can be written as

$$A_j = x^{j^2}\prod_{i=1}^{n-j}(1 - x^{4n-2i+2})(1 - x^{2i})^{-1}.$$

Now if we set $y = -1$ in (2) we get

$$\prod_{i=1}^{n}(1 - x^{2i-1})^2 = \sum_{i=-n}^{n}(-1)^i A_i, \tag{4}$$

where

$$A_j = x^{j^2}\prod_{i=1}^{n-j}(1 - x^{4n-2i+2})(1 + x^{2i} + x^{4i} + \cdots),$$

the infinite sum in this expression being our first use of formal power series. Now fix r. The coefficient of x^r in the left side of (4) is the same as that in $\Pi_{i\geqslant 1}(1 - x^{2i-1})^2$ for n sufficiently large. Note that the coefficient of x^r in A_j is 0 unless $j^2 \leqslant r$, in which case it is the same as the coefficient of x^r in $x^{j^2}\Pi_{i\geqslant 1}(1 - x^{2i})^{-1}$ for n sufficiently large. We see that

$$\prod(1 - x^{2i-1})^2 = \prod(1 - x^{2i})^{-1}\sum_j(-1)^j x^{j^2},$$

and so

$$\begin{aligned}\prod(1 + x^i)\sum_j(-1)^j x^{j^2} &= \prod(1 - x^{2i-1})^2(1 - x^{2i})(1 + x^i)\\ &= \prod(1 - x^i)(1 - x^{2i-1})(1 + x^i)\\ &= \prod(1 - x^{2i})(1 - x^{2i-1}) = \prod(1 - x^i),\end{aligned}$$

where i runs from 1 to infinity in the products.

The application of identity (1) to sums of squares depends on the function $s(n)$, which is defined for $n > 0$ to be the sum of those positive divisors d of n such that n/d is odd. We take $s(n) = 0$ for $n \leqslant 0$. If A is a formal power series, we denote its (formal) derivative by $D(A)$ and its logarithm by $L(A)$. It is proved in [9] that these functions obey the usual rules, including $D(L(A)) = A^{-1}D(A)$.

LEMMA. *Let*

$$B = \prod_{i \geq 1} (1 - x^i)(1 + x^i)^{-1}$$

and

$$S = \sum_{n \geq 1} s(n) x^{n-1}.$$

Then $-D(B) = 2BS$.

Proof. Let

$$B_k = \prod_{1 \leqslant i \leqslant k} (1 - x^i)(1 + x^i)^{-1}.$$

Then

$$\begin{aligned} -D(L(B_k)) &= -D\Big(\sum_{1 \leqslant i \leqslant k} \{ L(1 - x^i) - L(1 + x^i) \} \Big) \\ &= \sum_{1 \leqslant i \leqslant k} \left\{ (1 - x^i)^{-1} i x^{i-1} + (1 + x^i)^{-1} i x^{i-1} \right\} \\ &= 2 \sum_{1 \leqslant i \leqslant k} (i x^{i-1} + i x^{3i-1} + i x^{5i-1} + \cdots) \\ &= 2 \sum_{n \geqslant 1} f(n) x^{n-1}, \end{aligned}$$

where $f(n)$ is the sum of those divisors d of n with $0 < d \leqslant k$ and n/d odd. Note that $f(n) = s(n)$ for $n \leqslant k$.

Now

$$-D(L(B_k)) = -B_k^{-1} D(B_k),$$

so

$$-D(B_k) = 2B_k \Big\{ \sum_{1 \leqslant n \leqslant k} s(n) x^{n-1} + \sum_{n > k} f(n) x^{n-1} \Big\}.$$

Thus if r is fixed and k sufficiently large, the coefficients of x^r in $-D(B)$, $-D(B_k)$, and $2BS$ are all the same.

THEOREM 1 (G. H. Halphen). *For n any positive integer*

$$\begin{aligned} & s(n) - 2s(n-1) + 2s(n-4) - 2s(n-9) + \cdots \\ & \quad = \begin{cases} (-1)^{n+1} n & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{5}$$

Proof. From (1) $B = 1 + 2\sum_{i \geqslant 1} (-1)^i x^{i^2}$. Plugging this into the equation of the Lemma and equating the coefficients of x^{n-1} produces the desired identity.

Halphen's application of (5) to sums of squares is quite direct. If n is not a square, then (5) implies that $s(n)$ is even. Likewise if n is not the sum of 2 squares,

1966 Hedrick lecture and will appear in his book *Basic Hypergeometric Series and Applications*.

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The Construction of Matrices with Required Properties Over the Integers

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Prompted by a paper [1] that deals with matrices having integer elements and integer eigenvalues, a description is given of methods used for some time by the author, in producing such matrices for examination questions and classroom examples. In the process matrices having certain properties such as symmetry, defectiveness, and nonsingularity over the integers, are considered. The emphasis throughout is on ease of construction and control of element size.

1. Elementary ideas. If a matrix has eigenvalue λ, then the matrix $C = aA + bI$, where a and b are scalars, has eigenvalue $a\lambda + b$.

This follows immediately since

$$|C - (a\lambda + b)I| = |(aA + bI) - (a\lambda + b)I| = |a(A - \lambda I)| = 0.$$

If a and b are integers this provides a means of forming a new integer matrix with

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This follows immediately since

$$|C - (a\lambda + b)I| = |(aA + bI) - (a\lambda + b)I| = |a(A - \lambda I)| = 0.$$

If a and b are integers this provides a means of forming a new integer matrix with

integer eigenvalues from any such one. Note that the eigenvectors of C are unchanged from those of A. This is of course a particular case of a more general theorem showing that if A has eigenvalue λ then $f(A)$ has eigenvalue $f(\lambda)$, where $f(A)$ is a polynomial in A or a rational function of A. See Theorems 7.3.1 and 7.3.3 of §7.3 of [**2**].

To provide an initial matrix a quick and easy model, illustrated for the 3×3 case, is for arbitrary integral values of a, b and c

$$A = \begin{bmatrix} a & b & c \\ a-\lambda_2 & b+\lambda_2 & c \\ a-\lambda_3 & b & c+\lambda_3 \end{bmatrix}, \tag{1.1}$$

which by inspection of $|A - \lambda I|$ is seen to have eigenvalues λ_2 and λ_3 and, thus, by considering the trace of A, has a third eigenvalue $\lambda_1 = a + b + c$.

A more general form which has some extra flexibility is

$$\begin{bmatrix} a & b & c \\ k_2(a-\lambda_2) & k_2 b+\lambda_2 & k_2 c \\ k_3(a-\lambda_3) & k_3 b & k_3 c+\lambda_3 \end{bmatrix} \tag{1.2}$$

with eigenvalues λ_2, λ_3 and $a + k_2 b + k_3 c$, for all nonzero integers k_2 and k_3. Both types may be transposed if required.

The advantage of these forms is that the relationship between the chosen eigenvalues and the final form is very direct, and the numbers involved can be kept small and easy to handle. The extension to higher orders is obvious.

2. Repeated eigenvalues. The case where all the eigenvalues, except perhaps for one, are equal gives rise to a simple general form, which then results in some easily derived relationships.

2.1 *A general form*

Consider the n-square matrix $A = B + \lambda I$, where $B = [k_i\beta_j]$, $i, j = 1, 2, \ldots, n$. Note that $B = \mathbf{k}\boldsymbol{\beta}^T$, $\mathbf{k}^T = [k_1 k_2 \cdots k_n]$, $\boldsymbol{\beta}^T = [\beta_1\beta_2 \cdots \beta_n]$ and that all the rows of B are multiples of $\boldsymbol{\beta}^T$. Thus B has only one independent row and the rank (r) of B is unity.

Since $|A - \lambda I| = |B| = 0$, λ is an eigenvalue of A. If $\mathbf{x}$ is an eigenvector corresponding to λ then $A\mathbf{x} = \lambda\mathbf{x}$ or $(A - \lambda I)\mathbf{x} = 0$ that is $B\mathbf{x} = 0$ and $\mathbf{x}$ lies in the null space of B. Thus the number of independent eigenvectors corresponding to the eigenvalue λ, known as the geometric multiplicity of λ, equals the dimension of the null space of B, i.e. the nullity (ν) of B. Now by the dimension theorem $\nu = n' - r$ equal to $n - 1$ in this case, and so the geometric multiplicity of λ is $n - 1$; but the number of times an eigenvalue occurs, i.e., its algebraic multiplicity $\geqslant$ its geometric multiplicity (see Theorem 2.2.1 p. 57 [**3**] or Theorem 7.6.1 p. 214 of [**2**]) and thus the eigenvalue λ must occur at least $(n - 1)$ times. Since there are n eigenvalues in all and their sum equals the trace of A the remaining eigenvalue μ

must be given by

$$\mu + (n-1)\lambda = \text{Trace}\, A = \text{Trace}\, B + n\lambda.$$
$$\therefore \mu - \lambda = \text{Trace}\, B.$$

So starting with an $n \times n$ matrix B each of whose rows is a multiple of some vector, the matrix $A = B + \lambda I$ has eigenvalue λ occurring $(n-1)$ times and the remaining eigenvalue μ is given by

$$\mu = \lambda + \text{Trace}\, B = \lambda + \mathbf{k}^T\boldsymbol{\beta} = \lambda + k_1\beta_1 + k_2\beta_2 + \cdots + k_n\beta_n.$$

2.2 *The eigenvectors*

Corresponding to the eigenvalue λ we have $A\mathbf{x} = \lambda\mathbf{x}$ or

$$(\mathbf{k}\boldsymbol{\beta}^T + \lambda I)\mathbf{x} = \lambda\mathbf{x},$$

i.e., $\mathbf{k}\boldsymbol{\beta}^T\mathbf{x} = 0$, which is satisfied by

$$\boldsymbol{\beta}^T\mathbf{x} = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 0.$$

Possible eigenvectors associated with λ are thus

$$\mathbf{x}_j^T = \Big[\beta_j, 0, 0, \ldots, \underset{j\text{th position}}{-\beta_1}, 0, \ldots, 0\Big], \qquad j = 2, 3, \ldots, n.$$

Corresponding to μ we have $A\mathbf{x} = \mu\mathbf{x}$ which gives

$$(\mathbf{k}\boldsymbol{\beta}^T + \lambda I)\mathbf{x} = (\lambda + \mathbf{k}^T\boldsymbol{\beta})\mathbf{x},$$

i.e., $\mathbf{k}\boldsymbol{\beta}^T\mathbf{x} = \mathbf{k}^T\boldsymbol{\beta}\mathbf{x}$.

A solution $\mathbf{x}_1$ to this is given by $\mathbf{k} = [k_1, k_2, \ldots, k_n]^T$ because

$$\mathbf{k}\boldsymbol{\beta}^T\mathbf{k} = \mathbf{k}(\boldsymbol{\beta}^T\mathbf{k}) = \mathbf{k}\,\text{Trace}\, B = \text{Trace}\, B\mathbf{k} = (\mathbf{k}^T\boldsymbol{\beta})\mathbf{k} = \mathbf{k}^T\boldsymbol{\beta}\mathbf{k}.$$

2.3 *The minimal polynomial*

The case of Section 2.1 is also of special interest because it gives a minimal polynomial of the second degree. Consider

$$B^2 = (\mathbf{k}\boldsymbol{\beta}^T)(\mathbf{k}\boldsymbol{\beta}^T) = \mathbf{k}(\boldsymbol{\beta}^T\mathbf{k})\boldsymbol{\beta}^T = \mathbf{k}\,\text{Trace}\, B\boldsymbol{\beta}^T = \text{Trace}\, B(\mathbf{k}\boldsymbol{\beta}^T) = (\text{Trace}\, B)B.$$

Since $\text{Trace}\, B = \mu - \lambda$ we have

$$B^2 = (\mu - \lambda)B$$

and

$$(A - \lambda I)(A - \mu I) = B[B + (\lambda - \mu)I] = B^2 + (\lambda - \mu)B = 0$$

so that

$$A^2 - (\lambda + \mu)A + \lambda\mu I = 0. \tag{2.3.1}$$

This result of course follows directly from the fact that the minimal polynomial of a simple (i.e., diagonable) matrix has only simple zeros, see Theorem 2.4.2 and Corollary 2 of Theorem 4.13.2 of [**3**].

An example may help at this stage.

Let

$$\mathbf{k}^T = [1, 1, -2], \qquad \boldsymbol{\beta}^T = [1, -2, -1] \text{ and } \lambda = 1;$$

then

$$A = \mathbf{k}\boldsymbol{\beta}^T + \lambda I = \begin{bmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ -2 & 4 & 3 \end{bmatrix}$$

has eigenvalues 2, 1, 1 with corresponding eigenvectors $[1, 1, -2]^T$, $[-2, -1, 0]^T$ and $[-1, 0, -1]^T$, and furthermore

$$A^2 - 3A + 2I = 0.$$

2.4 *Integer inverse matrices*

Using the form of (2.1) where $A = B + \lambda I$, $B = \mathbf{k}\boldsymbol{\beta}^T$ and where Trace $B = \mathbf{k}^T\boldsymbol{\beta} = \mu - \lambda$, then after rearrangement of result (2.3.1), assuming $\lambda\mu \neq 0$, we get

$$A^{-1} = \frac{(\lambda + \mu)I - A}{\lambda\mu} = \frac{\mu I - B}{\lambda\mu}. \tag{2.4.1}$$

We thus have an integer inverse matrix if A is an integer and $\lambda, \mu = \pm 1$. We consider two cases.

Case I. $\lambda\mu = -1$ and $\lambda + \mu = 0$, $(\lambda = \pm 1, \mu = \mp 1)$

From (2.4.1) we have $A^{-1} = A$, i.e., A is involutory, and thus $B \pm I$, where $B = \mathbf{k}\boldsymbol{\beta}^T$, and Trace $B = \mp 2$ is involutory.

For example,

$$\begin{bmatrix} 5 & -3 & -4 \\ 5 & -3 & -4 \\ 5 & -3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 & -4 \\ 5 & -2 & -4 \\ 5 & -3 & -3 \end{bmatrix}$$

is its own inverse.

Case II. $\lambda\mu = +1$, $\lambda + \mu = \pm 2$, $(\lambda = \mu = \pm 1)$

Here

$$A = B + \lambda I, \qquad \lambda = \pm 1,$$

and

$$A^{-1} = \mu I - B = \lambda I - B \quad \text{with} \quad \text{Trace } B = \mu - \lambda = 0.$$

For example,

$$A = B + I = \begin{bmatrix} 6 & -3 & -2 \\ 5 & -2 & -2 \\ 5 & -3 & -1 \end{bmatrix} \text{ and } I - B = \begin{bmatrix} -4 & 3 & 2 \\ -5 & 4 & 2 \\ -5 & 3 & 3 \end{bmatrix} = A^{-1}$$

(note $B^2 = 0$).

These forms of integer inverse matrices are ones which are quick and easy to construct, and whose inverse is also easy to write down. More generally an integer inverse matrix may be formed by starting with a matrix whose determinant is ± 1

(e.g., a triangular or diagonal matrix with diagonal elements ± 1) and adding multiples of rows (columns) to other rows (columns).

3. Similarity Transformation Methods

3.1 *General results*

As a particular case of Theorem III.12 in **[4]** it follows that every integer matrix with integer eigenvalues is similar over the integers to an integer triangular matrix. This can be directly demonstrated using a process of deflation **[5]**, akin to Schur's method, Theorem 2.10.1 in **[3]** but using integer similarity matrices which can always be constructed—see Corollary II.1 **[4]**.

3.2 *Particular methods*

Now that we have a simple method for constructing integer matrix 'inverse pairs,' we can use them in a similarity transformation with triangular or diagonal matrices to construct matrices with specified eigenvalues. For example if $A = SDS^{-1}$ where D is diagonal then, as usual, the eigenvalues of A are the diagonal elements of D with corresponding eigenvectors given, in order, by the columns of S.

The following variation on a similarity transformation will be useful later in the case of symmetric matrices. If A and B are $n \times n$ matrices then AB and BA have the same eigenvalues; for if $(AB)\mathbf{x} = \lambda\mathbf{x}$ then $B(AB)\mathbf{x} = B\lambda\mathbf{x}$ which gives $(BA)\mathbf{Bx} = \lambda\mathbf{Bx}$. If in particular we have $AB = D$, a diagonal matrix, then the eigenvalues of BA are the diagonal elements of D and since $(BA)B = B(AB) = BD$ we see that the corresponding eigenvectors of BA are the columns of B in order. Such a product as $AB = D$ is easily realised by taking a pair of inverse matrices as in §2.4 and multiplying the rows of the first matrix *or* the columns of the second matrix by the required eigenvalues.

Renaud's method **[1]** also seems to be related to this. Putting his statement in matrix form it runs as follows. If U and V are $(n-1) \times n$ matrices with rows $\mathbf{u}_i$ and $\mathbf{v}_i$, respectively, then a construction for an integer matrix A with integer eigenvalues is

$$A = U^T V + kI, \quad \text{where } UV^T \text{ is upper triangular.}$$

Since as he states, his condition on UV^T is difficult to achieve for higher values of n, it would seem easier in practice to begin with an upper triangular matrix T, an invertible integer matrix U_n $(n \times n)$ and form $V_n^T = U_n^{-1}T$; in which case

$$U_n^T V_n = U_n^T\left(U_n^{-1}T\right)^T = U_n^T T^T \left(U_n^T\right)^{-1}$$

has the same eigenvalues as the diagonal elements of T.

4. Symmetric matrices with integer eigenvalues

4.1 *General case*

With symmetry in mind and using the idea of §3.2 we note that BB^T and B^TB are symmetric and have the same eigenvalues. Thus if we pick an integer matrix B such that B^TB is a diagonal matrix with integer elements, then BB^T will be a symmetric matrix with integer elements and integer eigenvalues.

Following §2.2 the eigenvectors corresponding to λ are given by

$$\mathbf{x}_j = \Big[k_j, 0, 0, \ldots, \underset{j\text{th position}}{-k_1}, 0, 0, \ldots, 0\Big]^T, \qquad j = 2, 3, \ldots, n$$

and corresponding to μ the eigenvector is

$$\mathbf{x}_1 = \mathbf{k} = [k_1, k_2, \ldots, k_n]^T.$$

As an example with $\mathbf{k} = [1, -1, 2]^T$, $a = 2$ and $\lambda = -7$ we get

$$A = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}[1, -1, 2] - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 4 \\ -2 & -5 & -4 \\ 4 & -4 & 1 \end{bmatrix}$$

a matrix with eigenvalues $5, -7, -7$ and corresponding eigenvectors $[1, -1, 2]^T$, $[1, 1, 0]^T$ and $[2, 0, -1]^T$.

Some general results relating to symmetric matrices with specified eigenvalues and eigenvectors may be found in **[6]**.

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An Elementary Proof of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$

BOO RIM CHOE
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Euler's formula given in the title is one of those formulas in mathematics that can be proved in many different ways. For example, one can prove it by using Cauchy's Residue Calculus (**[1]**, **[2]**, **[3]**, **[4]**) or Weierstrass' Product Theorem (**[5]**) in the theory of functions of one complex variable. Also, many advanced calculus books present it as an exercise so that one can prove it by means of Parseval's Theorem (**[6]**, **[7]**) or Fourier series expansions (**[6]**, **[7]**, **[8]**, **[9]**). In addition, there are many papers that contain elementary proofs; several are referenced in the article **[10]**.

The purpose of this note is to give another elementary proof of this formula. Our proof is more elementary, as the following calculations show.

Following §2.2 the eigenvectors corresponding to λ are given by

$$\mathbf{x}_j = \Big[k_j, 0, 0, \ldots, \underset{j\text{th position}}{-k_1}, 0, 0, \ldots, 0\Big]^T, \qquad j = 2, 3, \ldots, n$$

and corresponding to μ the eigenvector is

$$\mathbf{x}_1 = \mathbf{k} = [k_1, k_2, \ldots, k_n]^T.$$

As an example with $\mathbf{k} = [1, -1, 2]^T$, $a = 2$ and $\lambda = -7$ we get

$$A = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}[1, -1, 2] - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 4 \\ -2 & -5 & -4 \\ 4 & -4 & 1 \end{bmatrix}$$

a matrix with eigenvalues $5, -7, -7$ and corresponding eigenvectors $[1, -1, 2]^T$, $[1, 1, 0]^T$ and $[2, 0, -1]^T$.

Some general results relating to symmetric matrices with specified eigenvalues and eigenvectors may be found in **[6]**.

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The purpose of this note is to give another elementary proof of this formula. Our proof is more elementary, as the following calculations show.

First of all, since $\sum_{n=1}^{\infty} 1/n^2 = \sum_{n=0}^{\infty} 1/(2n+1)^2 + \sum_{n=1}^{\infty} 1/(2n)^2$ (by absolute convergence), we only need verify that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{1}$$

Let us consider the Taylor series expansion of arcsin x near $x = 0$, which we can easily derive from the binomial series expansion of $(1 - x^2)^{-1/2}$ near $x = 0$,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1}. \tag{2}$$

The series in (2) converges at $x = 1$ by Raabe's test, hence converges uniformly on $[-1, 1]$ by Weierstrass' M-test. Accordingly, the series in (2) actually represents arcsin x on $[-1, 1]$ by Abel's Theorem.

Substitute $x = \sin t$ into both sides of (2) to obtain

$$t = \sin t + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \sin^{2n+1} t, \tag{3}$$

where $-\pi/2 \leqslant t \leqslant \pi/2$. Integrating both sides of (3), from 0 to $\pi/2$, term by term, we find

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \int_0^{\pi/2} \sin^{2n+1} t \, dt. \tag{4}$$

Moreover, by Wallis' formula (**[11]**, **[12]**) we have

$$\int_0^{\pi/2} \sin^{2n+1} t \, dt = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{for } n = 1, 2, 3, \ldots. \tag{5}$$

Finally, (4), together with (5), shows (1).

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An Invariant of Finite Abelian Groups*

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Let G be a finite abelian group. Then

$$G \cong \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \cdots \oplus \mathbf{Z}_{m_l}, \tag{1}$$

where the m_i are prime powers. The (unordered) sequence of numbers $m_1, m_2, \ldots, m_l$ completely determines G up to isomorphism, so any invariant of G is some symmetric function of the m_i. The product of the m_i is simply $|G|$, the order of G: in this note we consider another invariant, the sum of the m_i. We show that it behaves fairly nicely with respect to extensions, and that it can be interpreted as the smallest degree of any symmetric group in which G imbeds.

For any abelian group G with decomposition (1), let

$$T(G) = \sum_{i=1}^{l} m_i.$$

(Put $T(G) = 0$ if G is the trivial group.) We call $T(G)$ the trace of G: it should be thought of as having the same relation to the order of G as the trace of a matrix has to its determinant. Clearly $T(G' \oplus G'') = T(G') + T(G'')$ for any finite abelian groups G' and G''. Now suppose the abelian group G is an extension of G' by G'' (i.e., G' is a subgroup of G and $G'' = G/G'$), not necessarily their direct sum. Is $T(G) = T(G') + T(G'')$? It is easy to see that the answer is negative, since $\mathbf{Z}_8$ is an extension of $\mathbf{Z}_4$ by $\mathbf{Z}_2$. Nevertheless, the following result holds.

PROPOSITION 1. *If the finite abelian group G is an extension of G' by G'', then $T(G) \geqslant T(G') + T(G'')$.*

Proof. Since trace distributes over direct sums, it is enough to consider only the case where G, G' and G'' are p-groups for some prime p. Let

$$G \cong \bigoplus_{i \geqslant 1} \mathbf{Z}_{p^i}^{n_i},$$

and denote the corresponding invariants of G' and G'' by n_i', n_i'' respectively. Then,

$$T(G) = \sum_{i \geqslant 1} |\mathbf{Z}_{p^i}^{n_i}| = \sum_{i \geqslant 1} p^i n_i,$$

and similarly for G', G'', so we must show

$$\sum_{i \geqslant 1} p^i(n_i' + n_i'') \leqslant \sum_{i \geqslant 1} p^i n_i. \tag{2}$$

For $r \geqslant 0$, let $F_r(G)$ be the subgroup of elements of order p^r or less in G ($F_0(G)$ is

*Research partially supported by a grant from the Naval Academy Research Council.

the trivial group). Now the natural homomorphism $G \to G/G' = G''$ restricts to a homomorphism $F_r(G) \to F_r(G'')$ (not necessarily onto) with kernel $F_r(G) \cap G' = F_r(G')$. Thus $F_r(G'')$ contains $F_r(G)/F_r(G')$ as a subgroup, and so

$$|F_r(G)| \leqslant |F_r(G')|\,|F_r(G'')|. \tag{3}$$

Now

$$F_r(G) \cong \bigoplus_{i=1}^{r-1} \mathbf{Z}_{p^i}^{n_i} \oplus \mathbf{Z}_{p^r}^{n_r + n_{r+1} + \cdots},$$

so (3) implies

$$\sum_{i=1}^{r-1} i n_i + \sum_{i \geqslant r} r n_i \leqslant \sum_{i=1}^{r-1} i(n_i' + n_i'') + \sum_{i \geqslant r} r(n_i' + n_i'').$$

Subtract this inequality from

$$\sum_{i \geqslant 1} i n_i = \sum_{i \geqslant 1} i(n_i' + n_i''),$$

which follows from $|G| = |G'|\,|G''|$, to get

$$\sum_{i > r} (i - r) n_i \geqslant \sum_{i > r} (i - r)(n_i' + n_i'').$$

Now multiply this inequality by s_r, a nonnegative number to be determined, and sum over r to obtain (after rearrangement)

$$\sum_{i \geqslant 1} \sum_{r=0}^{i-1} s_r (i - r) n_i \geqslant \sum_{i \geqslant 1} \sum_{r=0}^{i-1} s_r (i - r)(n_i' + n_i'').$$

This will be (2), provided

$$\sum_{r=0}^{i-1} s_r (i - r) = p^i$$

for all $i \geqslant 1$, which happens if and only if $s_0 = p$, $s_1 = p(p-2)$, and $s_r = p^{r-1}(p-1)^2$ for $r \geqslant 2$.

It follows from Proposition 1 that the trace of a group is at least as large as the trace of any of its subgroups. We use this fact in proving the next result.

PROPOSITION 2. *If an abelian group G is imbedded in the symmetric group Σ_n, then $T(G) \leqslant n$.*

Proof. Let $\Omega_1, \Omega_2, \ldots, \Omega_s$ be the orbits of $\{1, 2, \ldots, n\}$ under the action of G. Then G is isomorphic to a subgroup of $G_1 \oplus G_2 \oplus \cdots \oplus G_s$, where G_i is the restriction of G (thought of as a group of bijections of $\{1, 2, \ldots, n\}$) to Ω_i (Cf. [**1**, p. 16]). Since each G_i is a transitive permutation group, and an abelian transitive permutation group has order equal to its degree [**1**],

$$n = \sum_{i=1}^{s} |\Omega_i| = \sum_{i=1}^{s} |G_i|.$$

Now it follows easily from the definition that any group has trace less than or equal to its order, so

$$n \geqslant \sum_{i=1}^{s} T(G_i) = T(G_1 \oplus \cdots \oplus G_s) \geqslant T(G).$$

Remark. It is evident that any finite abelian group G can be imbedded in $\Sigma_{T(G)}$ by sending each generator of an indecomposable summand to a cycle of length equal to its order.

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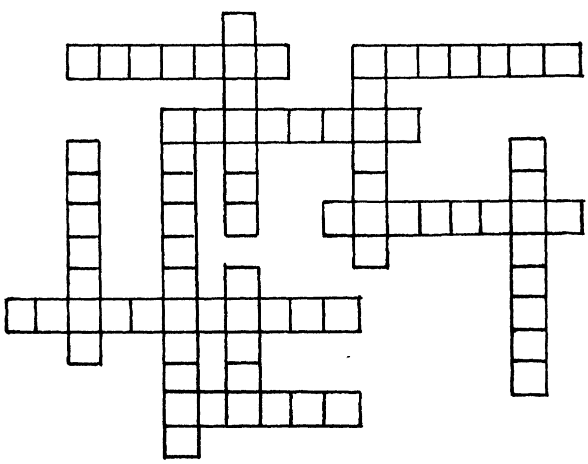
A NEW TYPE OF CROSSWORD PUZZLE

Can you solve the following *self-enumerating* crossword puzzle? Instead of receiving a list of clues, here you are told that each of the six horizontal and six vertical entries has the form: UMP..TEEN #S

Here *umpteen* is a variable standing for a normal English number-word (of whatever length) correctly counting the total appearances of #, one of the twelve different letters that will turn out to occur in the *completed* puzzle. A blank cell separates *umpteen* from #, the latter followed by plural *s*, where requisite. All entries read from left to right or from top to bottom. The top left horizontal entry might thus be:

NINE QS (but isn't).

The self-enumeration will be found to be complete and inpeccable, the finished pattern comprising a two-dimensional *self-intersecting reflexicon* (from reflexive lexicon, or self-determining word-list).



(The Solution will appear in the October issue)—LEE SALLOWS

In the figure, we have the parabola with equation $y = x^2/4c$, with focus $F(0,c)$ and directrix $y = -c$. $P(x_0, y_0)$ is a point on the parabola, l_1 is the line through P and parallel to the axis of the parabola, and l_2 is the line through P and tangent to the parabola. $D(x_0, -c)$ is the intersection of l_1 with the directrix and E is the intersection of l_2 with the line FD. Line l_2 has slope $x_0/2c$ (using calculus) while FD has slope $-2c/x_0$, so that these are perpendicular. Since $|\overline{FP}| = |\overline{PD}|$, the right triangles FEP and DEP are congruent. Thus $\beta = \angle FPE = \angle DPE = \alpha$.

It is also easy to verify that the point E lies on the x-axis.

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An Application of Random Walk to a Problem in Population Genetics

HARRY GONSHOR
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

In [**1**] the author begins by saying "If the answer to a problem turns out to be simple, there is probably a good explanation for it!" An excellent example of this arose in a course I recently taught called "Mathematical Models in the Social Sciences."

This course, which used [**2**] as a text, covered topics such as graph theory and Markov chains. This is a welcome change from routine courses in calculus. It is a place where students learn that mathematics does not consist only of dull tedious computation but includes material that even a person who thinks that he hates mathematics would regard as fun!

The particular problem referred to above arose in connection with an application of the theory of absorbing Markov chains to genetics. The Markov chain found at the bottom of [**2**, p. 316] is

	DD	RR	DH	DR	HH	HR
DD	1	0	0	0	0	0
RR	0	1	0	0	0	0
DH	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$	0
DR	0	0	0	0	1	0
HH	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
HR	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$

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HR	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$

The rows and columns stand for pairs of genotypes (D stands for dominant, H for heterozygous, and R for recessive). The transition probabilities are given by the probability distribution of offspring given a certain state, e.g., given state HH which corresponds to a pair of heterozygotes the probability of a given offspring being D is $1/4$, hence, assuming independence, the probability of two given offspring both being D is $1/16$. All the other entries in the matrix are obtained similarly.

The two absorbing states are DD and RR. By the usual theory of absorbing Markov chains we obtain the absorption probabilities as follows: Q, the lower left block, is

$$\begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{vmatrix};$$

R, the lower right block, is

$$\begin{vmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{4} \end{vmatrix}.$$

The absorption probabilities are then given by $(1 - Q)^{-1}R$, which in this case is

$$\begin{array}{c|cc|} & DD & RR \\ \hline DH & \frac{3}{4} & \frac{1}{4} \\ DR & \frac{1}{2} & \frac{1}{2} \\ HH & \frac{1}{2} & \frac{1}{2} \\ HR & \frac{1}{4} & \frac{1}{4} \end{array}.$$

Note that the rows of $(1 - Q)^{-1}R$ correspond to the transient states (i.e., the nonabsorbing states) and the columns to the absorbing states. Also the entry in a given row and column gives the probability that the process will be absorbed by the state corresponding to the column if it starts in the state corresponding to the row, e.g., if we begin at state DH the probability of being absorbed at DD is $3/4$. Now the pair DH has altogether 3 dominant genes out of a total of 4. Thus we have the "coincidence" as also pointed out in [2] that the probability of eventual absorption into a state of pure dominant is precisely equal to the proportion of dominant genes in the starting state. This is true in all other cases. Furthermore on page 319, problem 9, the student is asked to do a similar study in the case of set linkage and the claim is that a similar result holds.

We now refer to the quote at the beginning of this article. The above results should follow from general principles that do not require tedious computations such

Unfortunately for the purpose of our main application we need to generalize the above result to a form which is somewhat awkward to state although the proof remains the same. We allow the possibility that some states i are subdivided into several states, but the game is still fair if in states corresponding to i, player X has i dollars, e.g., we may have states i_1 and i_2 such that i_1 surely goes into i_2, but i_2 may go to $i + 1$ with probability $1/2$ and $i - 1$ with probability $1/2$. We still want 0 and n to be the only absorbing states. It is clear that the above proof still works. What is not so clear is the point of making such a generalization. In fact, from the point of view of the gambling situation it seems silly to distinguish between i_1 and i_2. However, the distinction is significant when one looks at the corresponding genetic situation as we will now see.

In fact, let us look back at the genetic example mentioned earlier. Let us say that a pair of genotypes is in state i if the total number of dominant genes contained in the pair is i. Then $0 \leqslant i \leqslant 4$. For example, the pair DH corresponds to 3. The pairs DR and HH both correspond to 2 so that this example indicates the need for the generalization above. If we regard this in a natural way as a gambling situation it follows from elementary reasoning with expectations that this is a fair game. Hence the so-called coincidences follow from the above result.

The same argument would apply to various other situations such as set linkage or polyploidy. Note, for example, that a direct computation in the case of tetraploidy using the expression $(1 - Q)^{-1}R$ requires inverting a matrix of order 13. I would certainly hate to do this and this is not because I'm superstitious!

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An Alternative Approach to the Dimension Theorem for Inner Product Spaces

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Introduction. On many occasions, I have been in the position of discussing some applications of linear algebra to combinatorics with students who have only studied vector spaces over the real numbers. Since my applications usually involve vector spaces over $\mathbb{Z}_2$ or some other finite field, I try to give a short introduction to vector spaces over arbitrary fields. After giving a few examples, I usually say something like this: "Almost all of the results that you proved over the reals are true in this

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more general setting. In fact, the proofs you gave for those results will usually translate directly into proofs in this more general setting. Almost all of the exceptions occur with the inner product. First off, the condition that the inner product of a nonzero vector with itself is positive does not even make sense unless the field is an ordered field. In the general setting, this condition is replaced by the requirement that no nonzero vector is orthogonal to every vector. In this general setting, some nonzero vectors may be self-orthogonal and orthogonal sets of vectors may not extend to orthogonal bases. Hence, proofs which depend upon extending orthogonal sets to orthogonal bases do not generalize. And, in some cases, the results themselves do not hold in the more general setting. However, there is one very important result with a proof that is traditionally based upon the extension of an orthogonal set to an orthogonal basis which, nevertheless, holds in the general setting. Namely:

$$\dim(S) + \dim(S^{\perp}) = \dim(V)$$

(where S is a subspace of the vector space V and $S^{\perp}$ denotes the orthogonal complement of S).

There are elementary proofs of this important result which are valid for general inner product spaces. Yet, to the best of my knowledge, no text for a first-year course in linear algebra gives a generalizable proof. The most accessible references to the general result are some of the "second level" linear algebra texts such as: *Linear Algebra and Geometry, a Second Course* by Irving Kaplansky **[2]**, or *Metric Affine Geometry* by Ernst Snapper and Robert Troyer **[4]**. I think that our students would be better served if a generalizable proof for this result was presented—even in a course which considers only the usual inner product over the reals. It is the purpose of this note to indicate just how this could be done. Specifically, in Section 3, we will outline the inner product portion of a course on real vector spaces which is designed to separate the inner product results into two groups: those results which have statements and proofs which do not depend on the fact that the reals are an ordered field and those that do depend on this fact. Thus, we will be able to make the blanket statement: "All results except those explicitly stated for the usual inner product over the reals are valid for finite-dimensional vector spaces over an arbitrary field with an arbitrary inner product; furthermore, the proofs given for these results are also valid in the general setting."

1. Definitions. By an inner product $*$ for a finite-dimensional vector space V, we mean a nonsingular symmetric bilinear form. Specifically, for any two vectors v and w in V, $v * w$ is a scalar and the following conditions are satisfied:

1. $v * w = w * v$, for all $v, w \in V$ (symmetric);
2. $(\lambda u + \mu v) * w = \lambda(u * w) + \mu(v * w)$ and $w * (\lambda u + \mu v) = \lambda(w * u) + \mu(w * v)$, for all $u, v, w \in V$ and all scalars λ and μ (bilinear);
3. $v * w = 0$, for all $w \in V$, implies $v = \mathbf{0}$ (nonsingular).

For the usual inner product over the reals, the nonsingularity condition is replaced

by the stronger condition:

3′. $v * v \geqslant 0$, for $v \in V$, with equality only if $v = \mathbf{0}$ (positive definite).

Since this usual inner product gives the Euclidean concept of orthogonality, we call this inner product the Euclidean inner product; and, a real finite-dimensional vector space with this inner product will be called a Euclidean vector space.

My favorite example of an inner product which is not Euclidean involves a vector space over the field of the integers modulo 2 ($\mathbb{Z}_2$). Let U be a finite set. If X and Y are subsets of U, $X + Y$ is defined to be the "symmetric difference," i.e., the set of elements that belong to X or to Y, but not both. This addition, along with the obvious scalar multiplication ($0X = \varnothing$(the empty set), $1X = X$), gives $P(U)$ (the power set or set of all subsets of U) the structure of a $|U|$-dimensional vector space over $\mathbb{Z}_2$. It is indeed very easy to check the vector space axioms. One sees that the empty set is the zero vector and that the 1-element subsets form a basis. This vector space also has a natural inner product:

$$X * Y = |X \cap Y|_{\text{mod}\,2}.$$

Again, it is easy to check that conditions 1, 2, and 3 are satisfied. In this example, the stronger condition 3′ does not even make sense. This vector space contains self-orthogonal vectors in profusion: all subsets of even cardinality are self-orthogonal.

The interest in general inner products is not restricted to vector spaces over fields other than the reals. Consider $\mathbb{R}^4$, the space of 4-tuples of real numbers. This vector space with the Minkowski inner product is central to the theory of special relativity. The Minkowski inner product is given by:

$$(x, y, z, t) * (x', y', z', t') = xx' + yy' + zz' - tt'.$$

Again, one easily checks that conditions 1, 2, and 3 are satisfied. In this example, it makes sense to ask if the stronger condition 3′ holds. That it does not hold is demonstrated by the vectors $v = (1, 2, 2, 3)$ and $u = (1, 1, 1, 3)$; here, v is self-orthogonal and $u * u = -6$.

2. A Typical Approach. To set the stage, assume that we are teaching a course on finite-dimensional real vector spaces. Assume further that the first two fundamental dimension theorems have been proved:

THEOREM 1. *If S and T are subspaces of the finite-dimensional vector space V, then*

$$dim(S + T) + dim(S \cap T) = dim(S) + dim(T).$$

THEOREM 2. *If V and W are finite-dimensional vector spaces and if $\Lambda: V \to W$ is a linear transformation, then*

$$dim(ker(\Lambda)) + dim(im(\Lambda)) = dim(V).$$

And now we are at the point where we introduce the usual (Euclidean) inner product. After defining the terms "orthogonal," "orthonormal," and "orthogonal

complement," we proceed to prove the following results:

THEOREM 3. *If S is any subspace of the Euclidean vector space V and if $b_1, \ldots, b_k$ is any basis for S, then*:
a) $S^\perp$ *is a subspace of* V;
b) $S \subseteq (S^\perp)^\perp$;
c) $S^\perp = \{v \mid v * b_i = 0$ *for* $i = 1, \ldots, k\}$.

THEOREM 4. *Any set, $b_1, \ldots, b_k$, of pairwise orthogonal, nonzero vectors in the Euclidean vector space V is independent and may be extended to an orthogonal basis for the whole space V.*

Let V be a Euclidean vector space of dimension at least one. Then V contains a nonzero vector which, by the preceding theorem, implies that V has an orthogonal basis. One easily checks that, if S is a subspace of V, then the restriction of the inner product to S gives S the structure of a Euclidean vector space. Thus, we have:

COROLLARY 5. *Every subspace of a Euclidean vector space has an orthogonal basis. Furthermore, every orthogonal basis of a subspace extends to an orthogonal basis for the whole space.*

If each vector in an orthogonal set of nonzero vectors is multiplied by the scalar equal to one over its length, we get an orthonormal set. Thus, we may strengthen Corollary 5:

COROLLARY 6. *Every subspace of a Euclidean vector space has an orthonormal basis. Furthermore, every orthonormal basis of a subspace extends to an orthonormal basis for the whole space.*

Let S be a subspace of the Euclidean vector space V. By Corollary 6, it has an orthonormal basis, $b_1, \ldots, b_k$ which may be extended to an orthonormal basis $b_1, \ldots, b_n$ for V. Now let v be any vector in V. It has a unique linear decomposition in terms of the above basis:

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n.$$

Since $v * b_i = \lambda_i$, we have $v * b_i = 0$ if and only if $\lambda_i = 0$. Thus, $v \in S^\perp$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. We have proved that $b_{k+1}, \ldots, b_n$ span $S^\perp$, which gives:

THEOREM 7. *If S is a subspace of the Euclidean vector space V, then $V = S \oplus S^\perp$.*

Combining Theorems 1 and 7 then gives:

THEOREM 8. *If S is a subspace of the Euclidean vector space V, then* $dim(\mathrm{S}) + dim(\mathrm{S}^\perp) = dim(\mathrm{V})$.

Combining this theorem with Theorem 3b yields:

COROLLARY 9. *If S is a subspace of a Euclidean vector space, then* $(\mathrm{S}^\perp)^\perp = \mathrm{S}$.

Let us examine the proofs of these results to see just where positive definiteness is used. Only conditions 1 and 2 are used in the proof of Theorem 3; hence Theorem 3

and its proof are valid for general inner products. However, all of the usual proofs given for Theorem 4 rely heavily on condition 3′. For example, in a typical proof of the first part of this theorem, one assumes that $b_1, \dots, b_k$ is a set of pairwise orthogonal, nonzero vectors and supposes that they satisfy a relation:

$$\lambda_1 b_1 + \cdots + \lambda_k b_k = 0.$$

Taking the inner product of both sides of this equality with b_i yields $\lambda_i(b_i * b_i) = 0$. Now, by condition 3′, $b_i * b_i$ is not zero and so λ_i must be zero. Thus the only relation possible among $b_1, \dots, b_k$ is the trivial relation. In the general case, as we have seen above, we may well have nonzero vectors which are self-orthogonal. Hence, this proof does not generalize. Nor do any of the usual proofs of the extension part of Theorem 4.

Consider the popular proof of the extension that is based on the orthogonalization of a basis $a_1, \dots, a_n$. One proceeds by induction and at the *ith* stage one has a basis $b_1, \dots, b_{i-1}, a_i, \dots, a_n$, where $b_1, \dots, b_{i-1}$ are pairwise orthogonal. This induction step consists of replacing a_i by

$$b_i = a_i + \lambda_1 b_1 + \cdots + \lambda_{i-1} b_{i-1},$$

where,

$$\text{for } j = 1, \dots, i-1, \qquad \lambda_j = -\frac{a_i * b_j}{b_j * b_j}.$$

Again, this proof does not generalize for arbitrary inner products since $b_j * b_j$ could be zero.

While Theorem 4 has valid generalizations for arbitrary inner products, their proofs require the consideration of special cases unrelated to the Euclidean inner product. I suspect that a commitment to the above sequencing of results and the fact that Theorem 4 is difficult to generalize is the reason that general inner products have been neglected in elementary texts. However, as we will see in the next section, one can avoid this difficulty by reversing the sequence of results.

3. An Alternative Approach. Again, we assume that our course is a course in real linear algebra and that we have already progressed beyond the first two dimension theorems (Theorems 1 and 2). We define inner product and give a few examples including both the Euclidean and the Minkowski inner products. We note that the Euclidean inner product satisfies condition 3′ while the Minkowski inner product does not. Next, we prove Theorem 3 for general inner products. In fact, we continue to avoid Condition 3′ until its use would appreciably simplify the proofs. In this approach, all results (including Theorems 1, 2, and 3) that are not explicitly stated for Euclidean spaces are valid for an arbitrary inner product space over an arbitrary field and can be verified using the "real" proof with the reals replaced by an arbitrary field.

In this alternative approach, our next result is:

LEMMA 10. *Let V be a finite-dimensional vector space with inner product $*$. Let $b_1, \dots, b_k$ be any independent set of vectors in V and let $\lambda_1, \dots, \lambda_k$ be any scalars. Then there exists a vector v in V so that $v * b_i = \lambda_i$, for $i = 1, \dots, k$.*

Proof. Extend $b_1, \ldots, b_k$ to a basis $b_1, \ldots, b_k, \ldots, b_n$ for V. Define the map $\Lambda : V \to \mathbb{R}_n$ by $\Lambda(v) = (v * b_1, \ldots, v * b_n)$. It follows from bilinearity that Λ is a linear transformation. It follows from Theorem 3c, that, if $v * b_1 = v * b_2 = \cdots = v * b_n = 0$, then v is orthogonal to every vector in V and, by condition 3, is the zero vector. Thus, if $\Lambda(v) = (0, \ldots, 0)$, then $v = \mathbf{0}$. Hence, the kernel of Λ is zero; and, by Theorem 2, Λ is onto. We conclude that there exists a vector v in V such that $\Lambda(v) = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$.

We may now move directly to the Dimension Theorem (old Theorem 8).

THEOREM 11. *Let V be an inner product space and let S be a subspace of V. Then, $dim(S) + dim(S^\perp) = dim(V)$.*

Proof. Let $b_1, \ldots, b_k$ be a basis for S and define $\Lambda : V \to \mathbb{R}^k$ by $\Lambda(v) = (v * b_1, \ldots, v * b_k)$. As before, this is a linear transformation. By Theorem 3c, $S^\perp = \ker(\Lambda)$. By Lemma 10, Λ is onto; therefore, $\dim(\mathrm{im}(\Lambda)) = \dim(S)$. The result now follows at once from Theorem 2.

As above, combining this result with Theorem 3b yields the following corollary (old Corollary 9):

COROLLARY 12. *If S is a subspace of an inner product space, then $(S^\perp)^\perp = S$.*

If one has not proved Theorem 2, alternate proofs of these results based on the properties of systems of linear equations can be given. (See Snapper and Troyer [**4**].) Another approach to the proofs of these results was pointed out to me by my colleague David Lissner. This approach is available if you have introduced the dual space V^* and proved the fundamental dimension result about a subspace and its annihilator. Any symmetric bilinear form $*$ induces a linear transformation:

$$\Lambda : V \to V^* \quad \text{by } [\Lambda(v)](w) = v * w.$$

Condition 3 is equivalent to requiring Λ to be an isomorphism. One easily checks that $\Lambda(S^\perp)$ is the annihilator of S and Theorem 11 follows.

Yet another approach, this one brought to my attention by Lawrence Lardy and Gerhard Gnannt, is based on pairs of "biorthogonal" bases. A development of biorthogonal bases can be found in [**3**].

Still working our way backwards through the old sequence of theorems, we come to Theorem 7 and then Theorem 4. In each of these cases, the simple substitution of "inner product space" for "Euclidean vector space" produces a statement that is not a theorem. We will describe and prove the appropriate generalizations in the next section. Here, in our introductory course, we will simply focus on the Euclidean case for these two results.

THEOREM 7. *If S is a subspace of the Euclidean vector space V, then $V = S \oplus S^\perp$.*

Alternative Proof. Suppose that v belongs to the subspace $S \cap S^\perp$. Then $v * v = 0$ and, by condition 3′, $v = 0$. We conclude that $S \cap S^\perp = \{0\}$. Then, from Theorems 1 and 11, we have that $\dim(S + S^\perp) = \dim(V)$. The result follows at once.

THEOREM 4. *Any set, $b_1, \ldots, b_k$, of pairwise orthogonal, nonzero vectors in the Euclidean vector space V is independent and may be extended to an orthogonal basis for the whole space V.*

Alternative Proof. We prove the independence of $b_1, \ldots, b_k$ as before. To prove the extension property, let S be the space spanned by $b_1, \ldots, b_k$. If $S = V$, we are done; otherwise, $S^{\perp} \neq \{0\}$ and we may choose b_{k+1} to be any nonzero vector in $S^{\perp}$. Thus $b_1, \ldots, b_{k+1}$ is a set of pairwise orthogonal, nonzero vectors. Now repeat the above argument to produce $b_{k+2}, \ldots, b_n$, where $n = \dim(V)$. Since this set is independent, it is a basis.

There is no change in the proofs of Corollaries 5 and 6. Of course, we could have simply used the old proofs of Theorems 4 and 7; it is a matter of taste. However, I do believe that the alternate proofs give a little more insight into how to generalize these results to arbitrary inner products. We discuss these generalizations next.

4. Extensions. In our typical course on real linear algebra, we may choose to pursue general inner products no further. But, if we should decide to go on or if we return to general inner products in a later course, just how do we proceed? Another way to understand what "goes wrong" with arbitrary inner products, is to recall that, in the Euclidean case, all subspaces are also Euclidean vector spaces. Specifically, suppose the real vector space V is given with the Euclidean inner product. Suppose further that S is a subspace of V. Then, one easily sees that the restriction of $*$ to S also satisfies conditions 1, 2, and 3′. On the other hand, if $*$ is an arbitrary inner product on V and if S is a subspace of V, the restriction of $*$ to S will, of necessity, satisfy conditions 1 and 2; but, it need not satisfy condition 3. For example, in Minkowski space, the subspace S spanned by the vectors $v = (1,1,1,1)$ and $w = (1,1,-1,-1)$ contains the nonzero vector $u = (0,0,1,1)$ which is orthogonal to every vector in S. Thus, in extending our results to general inner product spaces, we must deal with such "singular" subspaces and we may expect that they are not well behaved.

Let V be a finite-dimensional vector space and let $*$ be a symmetric bilinear form on V (i.e., $*$ satisfies conditions 1 and 2). Since we have not imposed condition 3, vectors orthogonal to all vectors in V may exist. If u and v are such vectors (i.e., if $u * w = v * w = 0$ for all $w \in V$), then, for any scalars λ and μ, we have $(\lambda u + \mu v) * w = 0$, for all $w \in V$. Thus, the set of vectors in V which are orthogonal to all vectors in V is a subspace. It is natural to denote this subspace by $V^{\perp}$; however, it is usually called the radical of V and is usually denoted by $\operatorname{Rad}(V)$. The properties of the $\operatorname{Rad}(V)$ that we will need throughout this section are collected in the next lemma.

LEMMA 13. *Let V be a finite-dimensional vector space and let $*$ be a symmetric bilinear form on V. Then:*

a) *$Rad(V)$ is a subspace;*

b) *V contains a subspace S such that $V = Rad(V) \oplus S$;*

c) *If S is any subspace of V such that $V = Rad(V) \oplus S$, then the restriction of $*$ to S is an inner product.*

Proof. Part a) was proved above.

Part b). Take any basis $b_1, \ldots, b_k$, for Rad(V) and extend it to a basis $b_1, \ldots, b_n$ for V. Let S be the space spanned by $b_{k+1}, \ldots, b_n$.

Part c). Clearly the restriction of $*$ to S satisfies conditions 1 and 2. To prove that condition 3 also holds, let $b_1, \ldots, b_n$ be a basis for V so that $b_1, \ldots, b_k$ is a basis for Rad(V) and $b_{k+1}, \ldots, b_n$ is a basis for S. Next, suppose that v is a vector in S that is orthogonal to every vector in S. Since v is also orthogonal to every vector in Rad(V), v is orthogonal to every basis vector. Thus, by Theorem 3c, v is orthogonal to every vector in V, i.e., v is in Rad(V). But, $S \cap \mathrm{Rad}(V) = \{0\}$ and we conclude that $v = 0$.

The next result is an immediate consequence of the definition of the radical.

LEMMA 14. *If S is any subspace of the inner product space V, then $Rad(S) = S \cap S^{\perp}$.*

Let V be an inner product space and let S be a subspace. We say that S is a nonsingular subspace if $\mathrm{Rad}(S) = \{0\}$ or, equivalently, if the restriction of the inner product to S is also an inner product; otherwise, we say that S is a singular subspace. The next result follows at once from Lemma 14 and Theorem 11.

THEOREM 15. *Let S be a subspace of the inner product space V. Then:*
a) *$Rad(S^{\perp}) = Rad(S)$.*
b) *$S^{\perp}$ is nonsingular if and only if S is nonsingular.*
c) *$V = S \oplus S^{\perp}$ if and only if S is nonsingular.*

Note that Theorem 15c is the appropriate reformulation of Theorem 7. The appropriate reformulation of the first part of Theorem 4 is:

THEOREM 16. *Any set, $b_1, \ldots, b_k$, of pairwise orthogonal vectors of an inner product space, none of which are self-orthogonal, is independent.*

Proof. Since $b_i * b_i \neq 0$, the usual proof is still valid.

I cannot resist the temptation to include here a combinatorial consequence of this theorem.

COROLLARY 17. *Let U be an n-element set and let $X_1, \ldots, X_k$ be subsets of U of odd cardinality whose pairwise intersections have even cardinalities. Then $k \leq n$.*

Proof. As vectors in $P(U)$, $X_1, \ldots, X_k$ are pairwise orthogonal but none is self-orthogonal. Hence, they are independent and $k \leq \dim(P(U)) = n$.

I know of no "purely combinatorial" proof of this result. Computing the best upper bound on the number of even sets with odd pairwise intersections makes a more complicated use of the inner product space $P(U)$. These and related results can be found in [**1**].

We return now to the generalization of the second part of Theorem 4. Here a very surprising thing occurs: the characteristic of the underlying field becomes important. Specifically, we will assume that the underlying field has characteristic different from two!

THEOREM 18. *Let V be an inner product space whose underlying field has characteristic different from two and let S be a subspace of V. Then S admits an orthogonal basis. Furthermore, if S is nonsingular, each orthogonal basis for S extends to an orthogonal basis for V.*

Proof. Let S be a nonsingular subspace of V. By Theorem 15, $S^{\perp}$ is also nonsingular and $V = S \oplus S^{\perp}$. One easily sees then that concatenating an orthogonal basis for S and an orthogonal basis for $S^{\perp}$ yields an orthogonal basis for V. Thus, extendability will follow once we can show that every nonsingular subspace admits an orthogonal basis.

Let S be any subspace of V. Then, by Lemma 13, $S = \mathrm{Rad}(S) \oplus T$ where T is a nonsingular subspace. Concatenating an orthogonal basis for T with any basis for $\mathrm{Rad}(S)$ gives an orthogonal basis for S.

Thus, all parts of this theorem will follow once we prove that every nonsingular subspace of V admits an orthogonal basis. But, by definition, every nonsingular subspace is an inner product space in its own right. Hence, we need only prove that every inner product space admits an orthogonal basis. This is easily checked to be true if the space has dimension 0 or 1. We proceed by induction on the dimension of the inner product space.

Assume that V has dimension at least two and that all proper nonsingular subspaces of V have an orthogonal basis. Suppose that we can find one vector v in V which is not self-orthogonal then $S = \langle v \rangle$ is a proper nonsingular subspace and so is $S^{\perp}$. By the induction hypothesis, $S^{\perp}$ has an orthogonal basis; adding v to this basis gives an orthogonal basis for V. Therefore, it remains only to show that V contains at least one vector which is not self-orthogonal.

Let v be a nonzero vector of V. If $v * v \neq 0$, we are done; hence, we assume that $v * v = 0$. Since $*$ is nonsingular, there exists another vector w in V so that $v * w \neq 0$. Again, if $w * w \neq 0$, we're done; hence, we assume also that $w * w = 0$. Now let $u = v + w$ and observe that

$$u * u = v * v + 2(v * w) + w * w = 2(v * w).$$

Since $v * w$ is not zero and since the field has characteristic different from 2, $u * u$ is not zero and we are done.

To see that the condition which is imposed on the characteristic of the field cannot be avoided, consider the inner product space $P(U)$. It is easy to see that $E(U)$, the set of all subsets of U with even cardinality, is a subspace of $P(U)$ that contains only self-orthogonal vectors and, therefore, admits no orthogonal basis. However, there is a version of Theorem 18 which is true for inner product spaces over fields of characteristic two (see Kaplansky **[2]**).

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2. I. Kaplansky, Linear Algebra and Geometry: A Second Course, Allyn and Bacon, Boston, MA, 1969.
3. W. Nef, Linear Algebra, McGraw-Hill, NY, 1966.
4. E. Snapper and R. Troyer, Metric Affine Geometry, Academic Press, NY, 1971.

E 3221. *Proposed by P. A. Batnik, University of Illinois at Urbana-Champaign.*

Characterize the integers p for which there exists a polynomial $P(x) = x^3 + px + q$ such that both P and P' have distinct integral zeros.

E 3222. *Proposed by Alexandru Lupas, Facultatea de Mecanica, Sibiu, Romania.*

Suppose that n is an integer greater than 1 and that Σ^* denotes summation over all pairs (i, j) of integers such that

$$1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant n, \qquad i \neq j.$$

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers such that $\Sigma^* a_i a_j > 0$, show that

$$\left(\Sigma^* a_i b_j\right)^2 \geqslant \left(\Sigma^* a_i a_j\right)\left(\Sigma^* b_i b_j\right).$$

E 3223. *Proposed by the editors. (A modification of a problem proposed by the late William F. Eberlein.)*

Given real numbers $\varepsilon > 0$ and $p > 1$, show that for all sufficiently large positive real numbers K the curve $x^p + y^p = K^p$ comes within distance ε of a point with positive integral coordinates.

E 3224. *Proposed by Bruce Reznick, University of Illinois at Urbana-Champaign and John Todd, California Institute of Technology.*

Let $n \geqslant 3$ be an integer. Determine all real polynomials p of degree n having n real zeros $a_1 < a_2 < \cdots < a_n$ such that

$$\int_{a_k}^{a_{k+1}} |p(t)| at \qquad (1 \leqslant k \leqslant n-1)$$

is independent of k.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Formula for $\lim_{x \to 0} \left(\frac{d}{dx}\right)^k \left(\frac{f(x)}{x}\right)$

E 3031 [1984, 57]. *Proposed by Gengzhe Chang, University of Science and Technology of China, Hefei, Anhui, China.*

is analyzed in W. Gautschi, Computation of Successive Derivatives of $f(z)/z$, *Math Comp*, 20 (1966) 209–214. See also W. Gautschi, Zur Numerik rekurrenter Relationen, *Computing*, 9 (1972) 107–125, in particular Example 5.2.

Also solved by 48 other readers.

Rational Solutions of a Riccati Equation

E 3055 [1984, 515]. *Proposed by Mark F. Kruelle (student), West Germany.*

Find all solutions of the Riccati equation $u' = u^2 + (a/x) - b$, $a, b \neq 0$, which are real rational functions of x.

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. Let us write $u(x) = p(x)/q(x)$, where p and q are real polynomials that have no common (complex) zeros, the leading coefficient of q being 1. From the Riccati equation it follows that

$$p'(x)q(x) - q'(x)p(x) = p^2(x) + \frac{a}{x}q^2(x) - bq^2(x). \tag{1}$$

Hence, $q(x)$ contains the factor x. By comparing the degrees of both members of (1), one easily verifies that the assumption $\mathrm{degr}(p) \neq \mathrm{degr}(q)$ leads to the contradiction

$$2\max(\mathrm{degr}(p), \mathrm{degr}(q)) \leqslant \mathrm{degr}(p) + \mathrm{degr}(q) - 1.$$

Therefore, $\mathrm{degr}(p) = \mathrm{degr}(q)$ ($= n$, say), and the leading coefficients of p^2 and bq^2 must be equal. So

$$b = \beta^2 \quad \text{with } \beta = \lim_{x\to\infty} \frac{p(x)}{q(x)}.$$

If $q(\alpha) = 0$ for a complex number α, then in view of (1) we have

$$p(\alpha)(p(\alpha) + q'(\alpha)) = 0.$$

Since $p(\alpha) \neq 0$, it follows that $q'(\alpha) = -p(\alpha) \neq 0$. So q has n different zeros $x_1, x_2, \ldots, x_n$, and

$$\frac{p(x)}{q(x)} = \beta - \sum_{i=1}^{n} \frac{1}{x - x_i} = \beta - \frac{q'(x)}{q(x)}. \tag{2}$$

Now, we introduce the function $\nu(x) = e^{-\beta x}q(x)$.

Due to (2) one has $\nu'/\nu = -u$. Substituting ν'/ν in the Riccati equation we obtain

$$\nu''(x) + \left(\frac{a}{x} - b\right)\nu = 0.$$

Consequently the polynomial q must satisfy the differential equation

$$q'' - 2\beta q' + \frac{a}{x}q = 0. \tag{3}$$

Let $q(x) = a_n x^n + \cdots + a_1 x$, where $a_n = 1$. From (3) it follows that

$$\sum_{k=2}^{n} k(k-1)a_k x^{k-2} - 2\beta \sum_{k=1}^{n} k a_k x^{k-1} + a \sum_{k=1}^{n} a_k x^{k-1} = 0.$$

Hence, it is necessary to have $a = 2\beta n$, and

$$k(k+1)a_{k+1} - 2\beta k a_k + a a_k = 0 \qquad (k = 1,2,\ldots,n-1).$$

Consequently,

$$a_k = \frac{k(k+1)}{2\beta(k-n)} a_{k+1} \qquad (k = n-1, n-2, \ldots, 1),$$

so we can state that

$$q(x) = \frac{(n-1)!}{(2\beta)^n} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!} \binom{n}{k} (2\beta x)^k. \tag{4}$$

Finally, we may conclude that there exist rational solutions of the Riccati equation only in the cases that $b > 0$ and $a/(2\sqrt{b})$ is an integer. Moreover, if $\beta = \sqrt{b}$ and $n = |a/2\sqrt{b}\,|$, then the rational function u is uniquely determined and given by (cf. (2)) $u(x) = \beta - (q'(x)/q(x))$.

Also partially solved by the proposer. O. P. Lossers provided two solutions.

Bound for a Sum

E 3070 [1985, 57]. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.*

Let a and $(t_n)_{n \geqslant 1}$ be strictly positive numbers and let the sequence $(t_n)_{n \geqslant 1}$ be bounded. Prove

$$t_1^{-a} + \sum_{n=1}^{\infty} t_1 t_2 \cdots t_n t_{n+1}^{-a} \geqslant \sum_{n=0}^{\infty} (a/(a+1))^{n-a}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. We shall make use of the following notation:

$$S_1(t) := \sum_{n=1}^{\infty} t_1 t_2 \cdots t_n^{-a}; \qquad S_2(t) := \sum_{n=2}^{\infty} t_2 t_3 \cdots t_n^{-a}.$$

Let β be a positive number, and consider the class C_β of all sequences (t) for which $0 < t_n \leqslant \beta$ $(n \in \mathbb{N})$. Assume that C_β contains an element $(t) = (t_n)$ for which $S_1(t) < \infty$ (otherwise, the proof is finished already).

Furthermore, let

$$g_\beta := \operatorname{Inf}\{S_1(t) | (t) \in C_\beta\}.$$

Clearly $g_\beta > 0$ because $S_1(t) \geqslant \beta^{-a} > 0$ for all $(t) \in C_\beta$.

Now let ε be a positive number, $0 < \varepsilon < \beta^{-a}$, and $(t) \in C_\beta$ such that $S_1(t) < g_\beta + \varepsilon$. Obviously $S_2(t) \geqslant g_\beta$, and $S_1(t) = t_1^{-a} + t_1 S_2(t)$ for some t_1, $0 < t_1 \leqslant \beta$. Combining these results we get

$$t_1^{-a} + t_1 g_\beta < g_\beta + \varepsilon,$$

i.e.,

$$g_\beta(t_1 - 1) < \varepsilon - t_1^{-a}$$

for some $t_1 \in (0, \beta]$. Obviously $t_1 < 1$ because the RHS of this inequality is negative.

Hence, we can state that for all $\varepsilon, 0 < \varepsilon < \beta^{-a}$ we have

$$g_\beta > \min_{0<t_1<1} \frac{t_1^{-a} - \varepsilon}{1 - t_1},$$

or

$$g_\beta \geqslant \min_{0<t_1<1} \frac{t_1^{-a}}{1 - t_1} = \frac{(a+1)^{a+1}}{a^a}.$$

As β was chosen arbitrarily this proves the assertion.

Remark. The inequality is best possible. This can be seen by substituting $t_n = a/(a+1)$ $(n \in \mathbb{N})$.

Also solved by W. Cat (Canada), N. Elkies, P. Fitzsimmons and A. Naj-jafar, V. Hernández and R. Vélez (Spain), E. Hertz, C. Hurd, W. Janous (Austria), J.-C. Leccia and T.-S. Than (France), E. Levine, D. Neuenschwander (Switzerland), W. Newcomb, and the proposer.

$$\sum_{k=1}^{n} \frac{k \cdot k!}{n^k}\binom{n}{k} = n$$

E 3088 [1985, 359]. *Proposed by Solomon W. Golomb, University of Southern California.*

Show that, for every positive integer n,

$$\sum_{k=1}^{n} \frac{k \cdot k!}{n^k}\binom{n}{k} = n.$$

Solution I by Paul Harper (student), Hope College, Holland, MI. Consider the problem of tossing a fair, n-sided die until we have a repetition. The probability of the first repetition occurring on the kth toss after the first $(1 \leqslant k \leqslant n)$ is

$$\left[\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)\right]\left[\left(\frac{k}{n}\right)\right] = \frac{(n-1)!k}{n^k(n-k)!}.$$

Obviously, the probability that a repetition occurs before or when $k = n$ is one, so

Now let ε be a positive number, $0 < \varepsilon < \beta^{-a}$, and $(t) \in C_\beta$ such that $S_1(t) < g_\beta + \varepsilon$. Obviously $S_2(t) \geqslant g_\beta$, and $S_1(t) = t_1^{-a} + t_1 S_2(t)$ for some t_1, $0 < t_1 \leqslant \beta$. Combining these results we get

$$t_1^{-a} + t_1 g_\beta < g_\beta + \varepsilon,$$

i.e.,

$$g_\beta(t_1 - 1) < \varepsilon - t_1^{-a}$$

for some $t_1 \in (0, \beta]$. Obviously $t_1 < 1$ because the RHS of this inequality is negative.

Hence, we can state that for all $\varepsilon, 0 < \varepsilon < \beta^{-a}$ we have

$$g_\beta > \min_{0<t_1<1} \frac{t_1^{-a} - \varepsilon}{1 - t_1},$$

or

$$g_\beta \geqslant \min_{0<t_1<1} \frac{t_1^{-a}}{1 - t_1} = \frac{(a+1)^{a+1}}{a^a}.$$

As β was chosen arbitrarily this proves the assertion.

Remark. The inequality is best possible. This can be seen by substituting $t_n = a/(a+1)$ $(n \in \mathbb{N})$.

Also solved by W. Cat (Canada), N. Elkies, P. Fitzsimmons and A. Naj-jafar, V. Hernández and R. Vélez (Spain), E. Hertz, C. Hurd, W. Janous (Austria), J.-C. Leccia and T.-S. Than (France), E. Levine, D. Neuenschwander (Switzerland), W. Newcomb, and the proposer.

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Obviously, the probability that a repetition occurs before or when $k = n$ is one, so

Hence,

$$\sum_{k=1}^{n} \frac{k \cdot k!}{n^k}\binom{n}{k} = n \sum_{k=1}^{n} \left(f^{(k)}(0) - f^{(k+1)}(0)\right)$$
$$= n\left(f^{(1)}(0) - f^{(n+1)}(0)\right) = n(1-0) = n.$$

Also solved by 135 other readers and the proposers.

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H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972, p. 13, p. 78.

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Putnam Permutations

E 3092 [1985, 360]. *Proposed by Rick Luttmann, Sonoma State University.*

Problem A-6 on the 1982 Putnam Competition was:

"Let σ be a bijection of the positive integers, that is, a one-to-one function from $\{1, 2, 3, \dots\}$ onto itself. Let $x_1, x_2, x_3, \dots$ be a sequence of real numbers with the following three properties:

(i) $|x_n|$ is a strictly decreasing function of n;

(ii) $|\sigma(n) - n| \cdot |x_n| \to 0$ as $n \to \infty$;

(iii) $\lim_{n\to\infty}\sum_{k=1}^{n} x_k = 1$.

Prove or disprove that these conditions imply that

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = 1."$$

Solve the problem with (ii) replaced by:

(ii)′ $|\sigma(n) - n| \cdot |x_{\sigma(n)}| \to 0$ as $n \to \infty$.

Solution by Victor Pambuccian, Bucharest, Romania. Unlike the Putnam problem, the answer to this problem is positive, i.e.,

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = 1. \qquad (*)$$

Proof. We begin by proving the convergence $(*)$ along a special sequence. Let $f(n) = \max\{\sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(n)\}$, so there is a $k \in \{1, 2, \dots, n\}$ such that $\sigma(f(n)) = k$. Then $f(n) \geqslant n$ and we have

$$\sum_{i=1}^{f(n)} x_{\sigma(i)} = \sum_{i=1}^{n} x_i + \sum_{j \in S_n} x_j, \qquad (1)$$

where S_n is a set with

$$S_n \subset \{n+1, n+2, \dots\} \tag{2}$$

and

$$|S_n| = f(n) - n. \tag{3}$$

Therefore,

$$\begin{aligned}\sum_{j \in S_n} |x_j| &\leqslant |S_n|\,|x_n| \qquad \text{by (i) and (2)}\\ &= (f(n)-n)|x_n| \leqslant (f(n)-k)\cdot|x_n|\\ &\leqslant (f(n)-k)|x_k| \qquad \text{by (i)}\\ &= |\sigma(f(n)) - f(n)|\,|x_{\sigma(f(n))}|.\end{aligned}$$

Since $f(n) \geqslant n$, this last sequence $\to 0$ as $n \to \infty$ by (ii)$'$. Therefore, $\lim_{n\to\infty} \sum_{j \in S_n} |x_j| = 0$ and, together with (1) and (iii), this gives

$$\lim_{n\to\infty} \sum_{i=1}^{f(n)} x_{\sigma(i)} = 1.$$

Finally, if N is a large positive integer, there exists a positive integer n such that $f(n) < N \leqslant f(n+1)$. Now

$$\begin{aligned}\left|\sum_{i=1}^{N} x_{\sigma(i)} - \sum_{i=1}^{f(n)} x_{\sigma(i)}\right| &= \left|\sum_{i=f(n)+1}^{N} x_{\sigma(i)}\right|\\ &\leqslant \sum_{i=f(n)+1}^{f(n+1)} |x_{\sigma(i)}|\\ &\leqslant |x_{n+1}| + \sum_{j \in S_{n+1}} |x_j|.\end{aligned}$$

The right-hand side of this inequality goes to zero as $N \to \infty$ and so $(*)$ is proved.

Also solved by O. P. Lossers (The Netherlands) and the proposer.

A Solution by Fractional Iterates

E 3113 [1985, 666]. *Proposed by Frank Schwellinger (student), Karlsruhe, West Germany.*

Prove that there is a continuous, strictly decreasing function $g(x)$ of the real line R onto R such that $g(g(x)) = 2x + 2$, but that there is no such function satisfying $g(g(x)) = x + 1$.

Solution and generalization by B. E. Rhoades, Indiana University. (i) There exists a continuous strictly decreasing function $g(x)$ of the real line R onto R such that

$g(g(x)) = ax + b$, $a > 0$ and $a \neq 1$, but (ii) there is no such function satisfying $g(g(x)) = x + c$, $c \neq 0$.

Proof. For (i) set $g(x) = -\sqrt{a}\,x + b/(1 - \sqrt{a})$. For (ii), if such a g existed, then, since it must be continuous, strictly decreasing with domain $\mathbb{R}$, it must cross the line $y = x$ and hence must have a fixed point p. Then $g(p) = p$ and $g(g(p)) = p \neq p + c$. ∎

Roger B. Eggleton and Danny R. O'Keefe, University of Newcastle, Australia, considered the following further generalization. Let $G(a, b)$ be the set of all functions $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $g_2(x) = ax + b$, where $g_2(x) := g(g(x))$. Let B, C, D, I be the sets of all functions $g\colon \mathbb{R} \to \mathbb{R}$ which are (respectively) bijective, continuous, strictly decreasing, and strictly increasing. They proved the following assertion.

THEOREM. (1) $G(a, b)$ *is uncountable*;
(2) $G(0, b) \cap (B \cup D \cup I) = \varnothing$;
(3) $G(a, b) \subseteq B$ *if and only if* $a \neq 0$;
(4) *if* $a \neq 1$, *then* $b/(1 - a)$ *is a fixed point of every* $g \in G(a, b)$;
(5) $G(a, b) \cap C = \varnothing$ *if and only if* $a < 0$;
(6) $G(a, b) \cap C \cap I \neq \varnothing$ *if and only if* $a > 0$;
(7) *when* $a \neq 1$, *then* $G(a, b) \cap C \cap D \neq \varnothing$ *if and only if* $a > 0$;
(8) $G(1, 0) \cap C \cap D \neq \varnothing$;
(9) $G(1, b) \cap D = \varnothing$ *if and only if* $b \neq 0$.

They note that results of this character are contained in Chapter 15 of a text by Marek Kuczma, *Functional Equations in a Single Variable*, PWN-Polish Scientific Publishers, Warsaw (1968).

Also solved by 90 other readers and the proposer.

The Variance of the Zeros of a Polynomial

E 3115 [1985, 666]. *Proposed by I. J. Schoenberg, University of Wisconsin-Madison.*

(a) Let z_j $(j = 1, \ldots, n)$ be the zeros of the polynomial

$$P_n(z) = z^n + a_2 z^{n-2} + \cdots + a_n \tag{1}$$

with complex coefficients. (Note $\Sigma z_j = 0$.) Show that

$$\sum_{j=1}^{n} |z_j|^2 \geqslant 2|a_2|,$$

with the equality sign if and only if the z_j all lie on a straight line passing through 0 in the complex plane.

$g(g(x)) = ax + b$, $a > 0$ and $a \neq 1$, but (ii) there is no such function satisfying $g(g(x)) = x + c$, $c \neq 0$.

Proof. For (i) set $g(x) = -\sqrt{a}x + b/(1 - \sqrt{a})$. For (ii), if such a g existed, then, since it must be continuous, strictly decreasing with domain $\mathbb{R}$, it must cross the line $y = x$ and hence must have a fixed point p. Then $g(p) = p$ and $g(g(p)) = p \neq p + c$. ∎

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with complex coefficients. (Note $\Sigma z_j = 0$.) Show that

$$\sum_{j=1}^{n} |z_j|^2 \geqslant 2|a_2|,$$

with the equality sign if and only if the z_j all lie on a straight line passing through 0 in the complex plane.

Distributions of Random Variables

E 3118 [1985, 736]. *Proposed by Gérard Letac, Université Paul-Sabatier, France.*

Consider 3 positive independent random variables U, X, Y such that U is uniform on $(0,1)$ and $\max(UX, UY)$ is distributed like X.

(1) If the distribution of Y is $\mu(dy) = (1+y)^{-2}\mathbf{1}_{(0,+\infty)}(y)\,dy$, show that X has distribution μ.

(2) If X and Y have the same distribution, show that there exists $c > 0$ such that cX has distribution μ.

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. Denote the distribution functions of X and Y by F and G, respectively. Conditioning on U immediately gives

$$F(z) = \int_0^1 F(z/u)G(z/u)\,du = z\int_z^\infty F(v)G(v)v^{-2}\,dv. \qquad (*)$$

(1) If $G(v) = \mu(0,v] = v/(1+v)$, then $(*)$ can be written as

$$h(z) := F(z)/z = \int_z^\infty h(v)(1+v)^{-1}\,dv.$$

It follows that h has a continuous derivative satisfying

$$h'(z) = -h(z)(1+z)^{-1},$$

and hence $h(z) = c(1+z)^{-1}$, or

$$F(z) = z(1+z)^{-1} = \mu(0,z]$$

as required.

(2) If $F = G$, then $(*)$ reduces to

$$h(z) = \int_z^\infty h^2(v)\,dv,$$

and, hence, $h'(z) = -h^2(z)$, or $F(z) = z/(c+z) = \mu(0, z/c)$ as required.

Also solved by R. A. Groeneveld, I. I. Kotlarski, D. Neuenschwander (Switzerland), M. F. Neuts, W. A. Newcomb, G. S. Rogers, Western Maryland College Problem Group, T. Zgonc, and the proposer.

Even Odder Than We Thought

E 3117 [1985, 735]. *Proposed by William C. Waterhouse, Pennsylvania State University.*

The 1983 Putnam Competition asked entrants to show (in essence) that there exists a positive real number u such that $[u^n]$ is odd for (positive) odd n and even for even n. The published proof [this MONTHLY, 91 (1984) 493] does not yield an explicit example. Find an explicit example of such a positive real number u.

Composite solution. Let a and b be positive integers with a odd, b even, and $b < a$. Let u and v be the positive and negative roots of the quadratic $x^2 - ax - b = 0$ and let $s_n = u^n + v^n$. Then $s_1 = a$ and $s_2 = a^2 + 2b$ are odd and the Newton identity $s_n = a \cdot s_{n-1} + b \cdot s_{n-2}$ implies that s_n is odd for all $n > 0$. Since $-1 < v < 0$ from the conditions on a and b, $[u^n] = s_n$ for odd n, and $[u^n] = s_n - 1$ for even n. Thus $[u^n] \equiv n \pmod 2$ as desired. (This solution is due to David Borwein and Jonathan Borwein (Canada), Stephen M. Gagola, Jr., and Walther Janous (Austria).)

Editorial comment. One slight variation gives $u = (a + \sqrt{d})/2$ and replaces conditions on a and b by conditions on a and d; namely, $d \equiv 1 \pmod 8$ is not a square and a is the largest odd integer $\leq \sqrt{d}$ (by Benne de Weger (Netherlands)). Or, $a = 2k + 1$ and $d = (2k+1)^2 + 8j$, $0 \leq j \leq k$ (by C. Georghiou (Greece), William G. Spohn, Paul K. Stockmeyer, and, with $j = 1$, Klaus Zacharias (West Germany)).

The problem asked for only *one* solution u. Particular instances of the foregoing, usually $(3 + \sqrt{17})/2$, were given by David G. Cantor, Lawrence J. Dickson, Zachary Franco, Keith Kearnes, O. P. Lossers (The Netherlands), J.G. Mauldon, Roger B. Nelsen, and Bjorn Poonen. The proposer's solution was the positive root of the cubic $x^3 - 11x^2 - 11x - 1 = 0$. The algorithm of the published solution to the 1983 Putnam problem was carried out by Delfin Hernandez and Michael Vowe (Switzerland) to obtain $u_1 = 3.16385673731437$ and $u_2 = 3.211247$ respectively valid for $n \leq 22$ and $n \leq 9$. De Weger remarks that for $u = a + \sqrt{(a^2 + j)}$, $1 \leq j \leq 2a$, $[u^n] \equiv n + 1 \pmod 2$, and several others gave special cases of this; the case $a = 1$, $j = 2$ goes back at least to Sylvester, *Nouvelles Ann. Math.*, 16(1857) pp. 125–126 and Quart. J. Pure and Appl. Math., 1(1857) p. 185.

By an iterative analysis, equivalent to dividing the half-closed interval $[1, 2)$ into 41 half-closed subintervals, Spohn proved that there is no u in $[1, 2)$ for which $[u^n] \equiv n \pmod 2$ for all $n \geq 1$. The only solutions valid for $n \leq 15$ are u in the interval $[548^{1/14}, 860^{1/15}) \approx [1.569015, 1.569037)$, but for any such u, $[u^{16}] = 1349$.

The 41 intervals come from 40 intermediate nodes at: $2^{1/2}$, $3^{1/3}$, $7^{1/5}$, $50^{1/10}$, $74^{1/11}$, $34^{1/9}$, $23^{1/8}$, $16^{1/7}$, $11^{1/6}$, $5^{1/4}$, $6^{1/4}$, $57^{1/9}$, $90^{1/10}$, $222^{1/12}$, $349^{1/13}$, $548^{1/14}$, $860^{1/15}$, $142^{1/11}$, $91^{1/10}$, $58^{1/9}$, $15^{1/6}$, $10^{1/5}$, $4^{1/3}$, $5^{1/3}$, $15^{1/5}$, $26^{1/6}$, $45^{1/7}$, $78^{1/8}$, $135^{1/9}$, $1195^{1/13}$, $693^{1/12}$, $233^{1/10}$, $234^{1/10}$, $697^{1/12}$, $404^{1/11}$, $405^{1/11}$, $136^{1/9}$, $79^{1/8}$, $46^{1/7}$, and $3^{1/2}$.

An Inequality Involving Curvature

E 3120 [1985, 736]. *Proposed by Kurt C. Foster and Lee A. Rubel, University of Illinois.*

Let f and g be two twice-continuously-differentiable functions on the interval $[0, 1]$. Define

$$K_f(x)\left(= f''(x)\left[1 + (f'(x))^2\right]^{-3/2}\right)$$

to be the curvature of the graph of $y = f(x)$ at the point $(x, f(x))$, with $K_g(x)$

similarly defined. Suppose $f(0) = g(0) = 0$ and $f'(0) = g'(0) = 0$, and that $K_g(x) \geq K_f(x)$ for all $x \in [0,1]$. Must $g(x) \geq f(x)$ for all $x \in [0,1]$?

Solution by Bjorn Poonen (student), Harvard College. Yes! We have

$$K_f(x) = \frac{d}{dx}\sin\left(\tan^{-1}f'(x)\right),$$

so for all $u \in [0,1]$

$$\sin\left(\tan^{-1}g'(u)\right) = \int_0^u K_g(t)\,dt \geq \int_0^u K_f(t)\,dt = \sin\left(\tan^{-1}f'(u)\right).$$

But $\sin(\tan^{-1}x)$ is an increasing function, so $g'(u) \geq f'(u)$. Hence for all $x \in [0,1]$,

$$f(x) = \int_0^x f'(u)\,du \leq \int_0^x g'(u)\,du = g(x).$$

Also solved by I. C. Bivens, R. -Fr. Bloden (Italy), R. X. Brennan, D. Cass, Z. Franco, P. L. Hon (Hong Kong), L. Kuntz and R. Martin (students, West Germany), K.-W. Lau (Hong Kong), O. P. Lossers (The Netherlands), J. B. M. Melissen (The Netherlands), J.-M. Monier (France), W. A. Newcomb, D. Spellman, Y.-L. Wong (student), and the proposers.

Encore

E 3125 [1986, 60]. *Proposed by I. J. Schoenberg, Madison, Wisconsin.*

Find two positive non-increasing sequences $\{a_n\}, \{b_n\}$ $(n = 1, 2, \ldots)$ such that $\Sigma_1^\infty a_n$ and $\Sigma_1^\infty b_n$ both diverge, while $\Sigma_1^\infty \min\{a_n, b_n\}$ converges.

Editorial Comment. This problem has occurred at least twice earlier as a MONTHLY problem, namely, as 4278 [1948, 34; 1949, 423] and as E 2437 [1973, 943; 1974, 1029]. It has also appeared on page 44 of the Otto Dunkel *Memorial Problem Book* (a supplement to the August–September, 1957 issue of the MONTHLY) and as Problem 9.11.1 in the appendix of Polya's book *Mathematical Discovery*, Vol. 2, 1967 or 1981 (Wiley, New York).

The essential point of the problem is that $\Sigma \min(a_n, b_n)$ can be taken as an arbitrary convergent series Σc_n of decreasing positive terms. Then it suffices to alternate blocks of terms where a_n remains constant and $b_n = c_n$ with blocks of terms where $a_n = c_n$ and b_n remains constant. The lengths of the blocks can be chosen so that both Σa_n and Σb_n diverge.

Solved by seventy-five readers.

A Theorem of Lagrange

E 3128 [1986, 60]. *Proposed by Florentin Smarandache, Lycée Sidi El Hassan Lyoussi, Sefrou, Morocco.*

If p is an odd prime, show that for all fixed $k \in \{1, 2, \ldots, p-2\}$, $\Sigma i_1 i_2 \cdots i_k$ is a multiple of p, where the summation is over all sequences $i_1, i_2, \ldots, i_k$ such that $1 \leqslant i_1 < i_2 < \cdots \leqslant i_k \leqslant p-1$.

Editorial comment. The result of the problem was obtained by Lagrange in the 1771 paper in which he gave the first published proof of Wilson's Theorem. His proof, which uses only Euclid's lemma and the binomial theorem, may be found in

Lagrange, *Oeuvres*, Vol. 3, pp. 425–438;

Dickson, *History of the Theory of Numbers*, Vol. 1, p. 62;

Hardy and Wright, *An Introduction to the Theory of Numbers*, §7.6.

Most solvers gave a proof based on Fermat's theorem, which implies the factorization

$$x^{p-1} - 1 = (x-1)(x-2)\cdots(x-p+1)$$

over the field of p elements.

Daniel Neuenschwander gave the following solution based on the existence of a primitive root b modulo p. Since multiplication by b permutes the coprime residue classes modulo p, we have

$$b^k \sum i_1 i_2 \cdots i_k \equiv \sum (bi_1)(bi_2)\cdots(bi_k) \equiv \sum i_1 i_2 \cdots i_k \pmod{p}.$$

But $b^k - 1$ is not a multiple of p and so the result follows.

Also solved by twenty-six other readers.

ADVANCED PROBLEMS

6551. *Proposed by A. Tissier, Montfermeil, France.*

Prove that the differential equation

$$y' = x - \frac{1}{y}$$

has a unique solution on $[0, +\infty)$ which is positive throughout and tends to zero at $+\infty$.

6552. *Proposed by Moshe Laub, Jerusalem, Israel.*

Let S be the set of positive integers n for which there exists a square equal to the sum of the squares of n consecutive positive integers. For example, $11 \in S$, since

$$77^2 = 18^2 + 19^2 + \cdots + 27^2 + 28^2.$$

Prove that

(a) S has density zero,

(b) S is infinite,

(c) if $n \in S$ and n is not itself a square, then there are infinitely many squares each of which is equal to the sum of the squares of n consecutive positive integers.

6553. *Proposed by Ian D. Macdonald, Lafayette College.*

Suppose that G is a group with generators a, b, c, d, where $abcd = cdab$, $ab = ba$, $ac = ca$, $bd = db$, $cd = dc$. Prove that the commutator subgroup of G is cyclic and contained in the center of G, but show that G is not necessarily abelian.

SOLUTIONS OF ADVANCED PROBLEMS

6374* [1982, 65]. *Proposed by Lee Whitt, Texas A & M University.*

Suppose $f(x)$, $-\infty < x < \infty$, is a real-valued function such that both $f(x)^2$ and $f(x)^3$ are C^∞. Must f be C^∞?

Editorial remark. An affirmative answer to the question raised in the problem was established by Henri Joris in his paper, "Une C^∞-application non-immersive qui possède la propriété universelle des immersions," *Arch. Math.*, (Basel) 39 (1982) 269–277. The question is discussed further by John Duncan, Steven G. Krantz, and Harold R. Parks in their paper "Nonlinear conditions for differentiability of functions," *Journal d'Analyse Mathématique*, 45 (1985) 46–68; they assert that the question was raised informally by M. R. Taylor in 1976.

A Set is Somewhere Thicker or Thinner Than Its Complement

6435. [1983, 403]. *Proposed by Thomas Q. Sibley, Cuttington University College, Liberia.*

Let m be any finitely additive extension of Lebesgue measure to all subsets of $[0,1]$. Are there two subsets A, B of $[0,1]$ such that $A \cap B = \varnothing$, $A \cup B = [0,1]$ and for any $0 \leqslant a < b \leqslant 1$

$$m(A \cap [a, b] = m(B \cap [a, b]) = \tfrac{1}{2}(b - a)$$

Solution by G. A. Edgar, Ohio State University, Columbus. The answer is "no." Let S be the set of all nonnegative finitely additive extensions of Lebesgue measure to all subsets of $[0,1]$. Then S is convex and compact (in the weak-star topology induced by the space of all bounded functions on $[0,1]$), so S has an extreme point by the Krein-Milman Theorem. Let m be an extreme point of S. If sets such as A and B exist, then

$$m = (m_1 + m_2)/2,$$

where m_1 and m_2 are defined by

$$m_1(E) = 2m(A \cap E), \qquad m_2(E) = 2m(B \cap E)$$

for all $E \subseteq [0,1]$. Now m_1 and m_2 agree with Lebesgue measure on all intervals $[a, b]$, so they are actually extensions of Lebesgue measure. But $m_1 \neq m_2$, contradicting extremality. This shows that no sets A, B exist.

The idea to use the Krein-Milman theorem in this way is due to J. Lindenstrauss; see A Remark on Extreme Doubly Stochastic Measures, *Amer. Math. Monthly*, 72 (1965) 379–382.

The editor remarks that some questions of a similar nature are examined in Andrew Simoson, An "Archimedean" Paradox, *Amer. Math. Monthly*, 89 (1982) 114–125, and Walter Rudin, Well-Distributed Measurable Sets, *Amer. Math. Monthly*, 90(1983) 41–42.

Embedding of Finite Groups

6426. [1983, 289]. *Proposed by J. L. Brenner, Palo Alto, California.*

(a) Show that every finite group can be embedded in one or another of the finite groups PSL(n, q). (b) The same for $O^+(n, q)$. $(c)^*$ For fixed q, find the smallest $n = n(k)$ such that the symmetric group S_k can be embedded in PSL(n, q). (Take representative values of q, k.)

Solution of (a) and (b) by E. C. Dade, Department of Mathematics, University of Illinois, Urbana. Since any finite group G can be embedded in the symmetric group $S_{|G|}$ on $|G|$ letters, we only need embed an arbitrary symmetric group S_k in PSL(n, g) for (a) and in $O^+(n, q)$ for (b). We may assume that $k > 2$, so that S_k has a trivial center. Then any embedding β of S_k in SL(n, q) leads to one in the factor group

$$\mathrm{PSL}(n, q) = \mathrm{SL}(n, q)/Z(\mathrm{SL}(n, q)),$$

since

$$Z(\mathrm{SL}(n, q)) \cap \beta(S_k) \subseteq Z(\beta(S_k)) = 1.$$

We can embed S_k in GL(k, q) via the usual permutation representation γ, in which a permutation

$$\pi\colon i \mapsto i\pi$$

of the set $\{1, 2, \dots, k\}$ corresponds to the permutation matrix

$$\gamma(\pi) = (\delta_{i\pi, j}),$$

where δ_{ij} is the kronecker delta. Unfortunately, the determinant $\det(\delta(\pi))$ is exactly the sign of the permutation π, and hence may be -1. To avoid this problem, take the direct sum $\gamma \oplus \gamma$ of two copies of the representation γ. This gives a faithful representation of S_k in SL$(2k, q)$, which then leads to an embedding of S_k in PSL$(2k, q)$.

To embed S_k in the orthogonal group $O^+(2k, q)$, simply notice that the representation $\gamma \oplus \gamma$ preserves the quadratic form sending any vector $(a_1, \dots, a_{2k})$, with $a_i \in GF(q)$ into $a_1^2 + a_2^2 + \cdots + a_{2k}^2$.

The solution of (c) may be found in L. E. Dickson, Representations of the General Symmetric Group as Linear Groups in Finite and Infinite Fields, *Trans. Amer. Math. Soc.*, 9(1908) 121–148. Since much of Dickson's terminology is obsolete, the reader might prefer the more modern treatment given by Ascher Wagner, The Faithful Linear Representation of Least Degree of S_n and A_n over a Field of Characteristic 2, *Math. Zeit.*, 151 (1976) 127–137 and The Faithful Linear

Representations of Least Degree of S_n and A_n Over a Field of Odd Characteristic, *Math. Zeit.*, 154(1977) 103–114.

The editors thank Gary Seitz and Bhama Srinivasan for providing us with clarifying information about (c), including the above references.

Nullstellensatz Always a Satz

6508 [1986, 65]. *Proposed by D. J. Newman, Temple University.*

The famous "Nullstellensatz" can be stated as follows: Let $P_1, P_2, \ldots, P_k$ be any polynomials in variables $x_1, x_2, \ldots, x_n$. If the P_i have no common zero, then there are polynomials Q_i such that $\sum_{i=1}^k Q_i P_i$ has no zero at all.

This is usually proved when the underlying field is algebraically closed. Prove that it holds over any field.

Solution by Paul R. Smith, Student, Universität Würzburg, West Germany. Suppose the field F is not algebraically closed. Let $f \in F[x]$ be an irreducible polynomial of degree $d > 1$. Define

$$\phi(x_1, x_2) = x_2^d f(x_1/x_2),$$

set

$$f_1(x_1) = f(x_1)$$
$$f_2(x_1, x_2) = \phi(x_1, x_2)$$

and for $n \geqslant 2$ define f_n inductively by

$$f_{n+1}(x_1, \ldots, x_{n+1}) = \phi(f_n(x_1, \ldots, x_n), x_{n+1}).$$

Clearly $x_1 = \cdots = x_k = 0$ is the only zero of f_k for $k \geqslant 2$. But for $P_1, \ldots, P_k \in F[x_1, \ldots, x_k]$ the expression

$$g = f_k(P_1, \ldots, P_k) = \sum_{i=1}^{k} Q_i P_i$$

belongs to the ideal generated by the P_i. Since these have no common zero, g has no zero at all.

Enzo R. Gentile (Argentina) notes that this problem is Aufgabe 7, Kapitel I, p. 23 of Ernst Kunz's Einführung in die kommutative Algebra und algebraische Geometrie. For some related material, D. D. Anderson refers us to R. Silhol, Géométrie algébrique sur un corps non algébriquement clos, *Comm. Algebra*, 6(11) (1978) 1131–1155.

Also solved by D. D. Anderson, Enzo R. Gentile (Argentina), Dennis Hamlin, S. V. Kanetkar, Budh Nashier and Warren Nichols (jointly), and the proposer.

Representations of Least Degree of S_n and A_n Over a Field of Odd Characteristic, *Math. Zeit.*, 154(1977) 103–114.

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Reviews

EDITED BY JOSEPH KONHAUSER
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The Banach-Tarski Paradox. By Stan Wagon. Cambridge University Press, 1985, vii + 251 pp.

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The Banach-Tarski paradox is not an inconsistency of mathematics, although it shows that 2 equals 1 in a certain sense. It is proved within the usual system of set theory (using the axiom of choice), and we have no reason to doubt the consistency of that system. But it destroys certain naive intuitions about point sets in three-dimensional space $\mathbb{R}^3$. So perhaps the reader of this book, which gives a very lush exposition of this striking result, or even the reader of this dry review, will be as intrigued as I was when I learned it for the first time. The main steps of this discovery were the following.

In 1905 G. Vitali proved that the circumference of a circle can be split into countably many sets that are congruent to each other by rotations. Hence there exists no *countably* additive probability measure over all subsets of the circle which would be invariant under all rotations. In 1914 F. Hausdorff proved that there exists no *finitely* additive probability measure over all subsets of the spherical surface which would be invariant under all rotations of the sphere. In 1923 S. Banach showed that *there exists* a finitely additive measure over all bounded subsets of the plane such that the unit square has measure one and the measure is invariant under all distance-preserving transformations of the plane. In 1924 Banach and Tarski, improving upon the work of Hausdorff, showed that if A and B are any subsets of the spherical surface $S^2 = \{(x, y, z)\colon\ x^2 + y^2 + z^2 = 1\}$, both having nonempty interiors, then there exists two finite partitions into disjoint pieces

$$A = A_1 \cup \cdots \cup A_n \quad \text{and} \quad B = B_1 \cup \cdots \cup B_n,$$

such that A_i is congruent to B_i by a rotation ρ_i of the sphere for $i = 1, \dots, n$. And the same result holds for any bounded subsets of $\mathbb{R}^3$ with nonempty interiors, with the corresponding pieces congruent by orientation- and distance-preserving transformations of $\mathbb{R}^3$.

For example, we can split the solid sphere in $\mathbb{R}^3$ into a finite number of pieces and rearrange them, like the pieces of a jigsaw puzzle, to produce two disjoint solid spheres each of the same size as the original one. This is called a *paradoxical decomposition*. Of course the pieces of such a decomposition cannot be measurable. This theorem is a striking demonstration that the unrestricted concept of a set of points has little to do with the idea of a physical body, and also that, to develop a reasonable theory of areas, volumes, etc., one must limit oneself to more special sets (e.g., Borel sets or Lebesgue measurable sets).

Drinfeld, Rosenblatt, Tits and others concerning the existence, properties and action of various groups in S^n, $\mathbb{R}^n$ and H^n are also discussed (without proofs). The book has an appendix about the open problems of the subject. Some of them are discussed also in the main text, for example, the open and apparently very difficult problem of Marczewski—whether paradoxical decompositions with parts having the property of Baire are possible. (Problem 10 on p. 231 has been solved in the affirmative for Lie groups by C. Bandt (to appear).)

The book is not strictly organized according to theorems or methods but according to topics of interest more or less defined by the author. It has an encyclopedic abundance and a notable emphasis on the intuitive motivations and the open character of the subject. This style does not allow the author to go into the proofs of the deepest recent achievements, but it gives the book the spontaneity of expositions in an older period of mathematics, a way of writing that is almost forgotten today in favor of a more compact and economic style that cannot be easily read outside of the school where the underlying concepts and problems have been discussed, and that betrays the slightly pathological intensity of our time.

A second edition of the book is to appear soon. It will contain a number of corrections and an addendum with a few updates and improvements.

I Want to Be a Mathematician: An Automathography. By Paul R. Halmos. Springer-Verlag, New York, 1985, xv + 403 pp.

GIAN-CARLO ROTA

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Every mathematician will rank other mathematicians in linear order according to their past accomplishments, while he rates himself on the promise of his future publications. Unlike most mathematicians, Halmos has taken the unusual step of printing the results of his lifelong ratings. By and large, he is fair to everyone he includes in his lists (from first-rate (Hilbert) to fifth-rate (almost everybody else)), except towards himself, to whom he is merciless (even in the choice of a title to the book: "I *want* to be a mathematician," as if there were any question in anybody's mind as to his professional qualifications).

Those scientists (even the very best) who take the unusual step of writing their autobiography are often driven by a desperate attempt to suppress an overwhelming "drop-in-the-bucket" feeling about their work. This disease strikes mostly in old age, an undeserved punishment at the end of many a productive career. In the majority of cases, autobiography has not proved to be good therapy. What is worse, the attempted cure will result in a pathetic "see-how-good-I-am" yarn of achievements and honors, a motley assortment of ephemeral episodes that will be soon forgotten if it is read (a questionable assumption).

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Halmos has discovered another therapy, one that may work. He admits from the start his own limitations (he overadmits them in fact). Once he has taken this courageous first step, the task of telling the truth, which for the average autobiographer is a formidable moral imperative to be faced up to, becomes for him an easy exercise in dispassionate description. And describe he does, down to the minutest detail: the colors of the cups in the tearoom at the University of Chicago, the number of pages in the bluebooks at the University of Hawaii, the bad quality of the chalk at Moscow S.U., etc. The leading thread of his exposition, what makes his narration entertaining (rather than just interesting), is mathematical gossip, which is freely allowed to unfold in accordance to its mysterious logic. The reader will be thankful for being spared the nauseating personal details that make most autobiographies into painful reading experiences ("My family was very musical." "The winter of 1932 was exceptionally severe," etc., etc.). Whatever does not relate to the world of mathematics is ruthlessly and justly left out (we hardly even learn whether he has a wife and kids).

At last we have a thorough account (one that stands the test of re-reading, and the only such, in fact) of the period that runs approximately from the forties to the present-day, a period that may go down in history as one of the golden ages of mathematics. However, the theme that emerges from this collection of amusing anecdotes is not the welcome lesson we would expect as the bequest of a golden age. Halmos's tales of incompetent department heads, of Neanderthal deans, of obnoxious graduate students unwittingly reveal, in the glaring light of gossip, the constant bungling, the lack of common sense, the absence of *savoir faire* that is endemic in mathematics departments everywhere. Take, for example, the turning point of the author's career, the incident of his leaving the University of Chicago. Even granting Halmos's contention that his papers may have lacked depth (at least, in someone's opinion) in comparison with those of certain colleagues of his (a debatable thesis, then and now), it still seems clear that the university made a mistake by dispensing with Halmos's services. Whatever his other merits, Halmos is now regarded as the best expositor of mathematics of his time. His textbooks have had an immense influence on the development of mathematics since the fifties, especially by their influence on mathematicians in their formative years. Halmos's glamor would have been a far sounder asset to the University of Chicago than the deep but dull results of an array of skillful artisans. What triumphed at the time is an idea that still holds sway in mathematics departments today, namely, the simplistic view of mathematics as a linear progression of problems solved and theorems proved, in which any other function that may contribute to the well-being of the field (most significantly, that of exposition) is to be valued roughly on a par with that of a janitor. It is as if in the filming of a movie all credits were to be granted to the scriptwriter, at the expense of other contributors (actors, directors, costume designers, musicians, etc.) whose roles are equally essential for the movie's success.

This well-worn tirade would hardly be worth repeating (and the author manages to keep it *sub rosa*), were it not for the fact that the mathematicians' share of the scientific pie is now shrinking. A strong case can be made that mathematics is today

the healthiest (and (what now begins to matter) the most honest) of the sciences. But just knowing the truth is not enough: the outside world must be made to believe it (it takes two to tango). Unfortunately, whereas physicists, chemists, and biologists have learned properly to appreciate everyone and justly to apportion rewards to all (too much like credits in a movie), even today (and even after reading Halmos), far *too* many mathematicians, when confronted with the painful chore of protecting (defending, asserting, popularizing, selling) mathematics, would rather be like the captain of a sinking ship, bravely (but pathetically) saluting while going down, like Clifton Webb in an old movie about the sinking of the Titanic.

Answer to "The Mathematician's Dictionary", page 645:

The soup norm.

(Contributions to this department are invited; acknowledgement of all of them will not be possible.)

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatibilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, S(13-18), L. *The Mind and the Machine: Philosophical Aspects of Artificial Intelligence.* Ed: S.B. Torrance. Ser. in Artif. Intell. Halsted Pr, 1984, 213 pp. [ISBN: 0-470-20104-5] Papers by a collection of philosophers, computer scientists, linguists, and psychologists on the influence of artificial intelligence on our philosophical and psychological understanding of the human mind. JAS

General, P*, L.** *Undergraduate Programs in the Mathematical and Computer Sciences: The 1985-1986 Survey.* Donald J. Albers, Richard D. Anderson, Don O. Loftsgaarden. MAA, 1987, xx + 180 pp, (P). [ISBN: 0-88385-057-5] Report on course enrollments and faculty characteristics in departments of mathematics, statistics, and computer science in two-year colleges, four-year colleges, and universities. Documents upturn in numbers of majors (40%), and in upper-division enrollments (52%); levelling off of remedial enrollments and of two-year college enrollments; and increase in student load per FTE faculty (30%) in mathematics. Fifth survey sponsored every five years since 1960 by the Conference Board of the Mathematical Sciences (CBMS). Provides a valuable benchmark for institutions to use in assessing their own situation. LAS

Precalculus, T*(13). *College Algebra and Trigonometry.* Steven Roman. Harcourt Brace Jovanovich, 1987, xiii + 802 pp, $31.95. [ISBN: 0-15-507911-5] Attractive, well-written presentation of the traditional topics; special features include marginal study suggestions with answers at the back of the book, and end-of-section "philosophical" remarks about the contents of each section. JNC

Precalculus, T(13: 1). *Intermediate Algebra, Second Edition.* Linda and Jimmie Gilbert. Prentice-Hall, 1987, xiv + 514 pp. [ISBN: 0-13-469487-2] Changes in this edition include revision of Chapter 2 (polynomial topics are now earlier, graphing later); new sections on synthetic division and solution of linear systems by matrix methods; applications now integrated throughout. Also available: Instructor's Manual, Test Banks, and Student's Study Guide. (*First Edition,* TR, January 1984.) JNC

Education, S, P*, L*. *School Mathematics in the 1990s.* Geoffrey Howson, Bryan Wilson. ICMI Study Ser. Cambridge U Pr, 1986, viii + 104 pp, (P). [ISBN: 0-521-33614-7] Based on a 1986 seminar in Kuwait, this brief volume outlines major issues facing all nations concerning school mathematics: Should the same mathematics be taught to all children? Should school mathematics be compulsory? Should mathematics teaching remain neutral on the social role of mathematics? Should content continue to dominate process in the development of curriculum? Explicit alternatives are emphasized to encourage different countries to develop strategies best suited to their needs and resources. Second volume in a new series sponsored by ICMI, the International Commission on Mathematical Instruction. LAS

History, S*, P*, L*.** *The History of Statistics: The Measurement of Uncertainty Before 1900.* Stephen M. Stigler. Harvard U Pr, 1986, xvi + 410 pp. [ISBN: 0-674-40340-1] A lucid, compelling exposition of how modern statistical methods evolved as solutions to intellectual puzzles posed by scientific data: how to distribute errors of measurement equitably; how to explain the "regression" of genetic variability; how to justify least square estimates. Eighteenth century contributions came from astronomy and geodesy, via Legendre, Laplace, and Gauss;

in the nineteenth century these ideas were adapted to social and biological sciences by Quetelet, Galton, Edgeworth, Pearson, and Yule. Superb background reading for every student and teacher of statistics. LAS

Logic, T(15-18: 1, 2), S, P, L. *Introduction to Combinators and λ-Calculus.* J. Roger Hindley, Jonathan P. Seldin. London Math. Soc. Stud. Texts, V. 1. Cambridge U Pr, 1986, 360 pp, $16.95 (P). [ISBN: 0-521-31839-4] Both combinatory logic and the λ-calculus serve to describe properties of operators and combinations of operators. This text introduces basic techniques and results in both fields. Main applications are in programming language theory, yet aren't treated in the text. Some exercises. Answers to selected exercises. Appendices; bibliography; index. RJA

Logic, T(15-17: 1, 2), S, P, L. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof.* Peter B. Andrews. Comput. Sci. & Appl. Math. Academic Pr, 1986, xv + 304 pp, $29.95 (P); $55. [ISBN: 0-12-058536-7; 0-12-058535-9] Text for first course in mathematical logic, and sequel course in type theory or higher order logic (specifically, typed λ-calculus), as taught at Carnegie Mellon. Type theory is widely applied in computerized automated deduction, and is presented as a more natural model of what mathematicians actually do than axiomatic set theory. Exercises. RB

Combinatorics, P. *Lecture Notes in Mathematics-1234: Combinatoire énumérative.* Ed: G. Labelle, P. Leroux. Springer-Verlag, 1986, xiv + 387 pp, $36.10 (P). [ISBN: 0-387-17207-6] Proceedings of a conference in enumerative combinatorics at the University of Quebec at Montreal, May 28-June 1, 1985, and a special session in combinatorics at the annual meeting of the Canadian Mathematics Society at the University of Laval. Twenty-three papers and open problems. LC

Discrete Mathematics. *Discrete Mathematics.* John A. Dossey, *et al.* Scott Foresman, 1986, 482 pp, $21.56. [ISBN: 0-673-18191-X] Differs in approach and emphasis from most of the new books on discrete mathematics. After two introductory chapters there are four chapters on graph theory including coloring, paths, algorithms, trees, matching, and network flows. Concludes with chapters on counting, recurrence, and finite state machines. Attractively produced; includes exercises, solutions, and bibliography. JS

Linear Algebra, T(14), S, L*. *Eclectic Linear Algebra.* Bobby Fincher. Spadra Pr, 1986, vii + 164 pp, $15 (P). [ISBN: 0-937161-34-9] A compact introduction to ideas and methods of abstract mathematics via elementary linear algebra. Covers the standard topics, together with the language of maps, rings, etc.; exercises emphasize proofs and investigation of hypotheses, minimizing drill. Terse presentation. Lacks discussion of proof techniques. A prelude to upper-level courses, for bright student with calculus, perhaps matrix background. RB

Linear Algebra, T(13-14). *Elementary Linear Algebra, Third Edition.* Stanley I. Grossman. Wadsworth, 1987, xvi + 446 pp, $36. [ISBN: 0-534-07422-7] *Second Edition* revised to bridge gap between the largely computational material in first half of book and theoretical material that followed. Changes include incorporating systems of equations and matrices into one chapter, introducing elementary matrices in Chapter 1, and streamlining the chapter on vector geometry of the Euclidean spaces. (*First Edition,* TR, April 1980; *Second Edition,* TR, January 1985.) LC

Linear Algebra, T(14), S, L. *Matrices and Society.* Ian Bradley, Ronald L. Meek. Princeton U Pr, 1986, 237 pp, $7.95 (P). [ISBN: 0-691-02404-9] An inexpensive source of social science applications of matrices and matrix operations. JNC

Linear Algebra, T*(14: 1). *Elementary Linear Algebra, Fifth Edition.* Howard Anton. Wiley, 1987, xv + 529 pp, $39.25. [ISBN: 0-471-84819-0] Changes in this edition include added examples and supplementary exercises, revision of the material on length and angle in inner product spaces, and revision of the chapter on numerical methods to include *LU*-decompositions and comparisons of number of operations required by various methods for solving linear systems. Supplements include a linear algebra software package and companion problem books. (*First Edition,* TR, March 1973; Extended Review, March 1974; *Second Edition,* TR, May 1977; *Third Edition,* TR, October 1981; *Fourth Edition,* TR, June-July 1985.) JNC

Group Theory, T(17-18: 1), S, P, L. *Finite Group Theory.* Michael Aschbacher. Stud. in Adv. Math., V. 10. Cambridge U Pr, 1986, ix + 274 pp, $32.50. [ISBN: 0-521-30341-9] A text and reference on finite groups, written in the light of the recent classification of finite simple groups, to which the author contributed powerfully. Approach through group representations—a central interest in finite simple groups and their irreducible representations—dictated choice of material. Presumes first course in algebra; proofs relatively terse; numerous exercises. A foundation for the proof of the classification theorem. RB

Group Theory, S(18), P. *Partially Ordered Abelian Groups with Interpolation.* K.R. Goodearl. Math. Surv. & Mono., No. 20. AMS, 1986, xxii + 336 pp, $68. [ISBN: 0-8218-1520-2] A solid foundation in the theory of partially ordered Abelian groups which satisfy the Riesz interpolation property. Detailed developments for that part of the subject on which most current applications rest. Includes a

list of unsolved problems and an extensive bibliography. CEC

Algebra, P. *Geometric Methods in Operator Algebras.* Ed: H. Araki, E.G. Effros. Res. Notes in Math. Ser., V. 123. Longman Scientific & Technical (US Distr: Wiley), 1986, 439 pp, $67.95 (P). [ISBN: 0-470-20376-5] The proceedings of the U.S.-Japan joint seminar on geometric methods in operator algebras which was held at Kyoto in July 1983. A collection of 24 papers from this first international conference on this newly emerging subject. CEC

Algebra, P. *Measures and Hilbert Lattices.* G. Kalmbach. World Scientific, 1986, xi + 247 pp, $23. [ISBN: 9971-50-009-4] First half extends classical measure theory to the theory of measures and states on orthomodular lattices. Second half investigates special properties of complete orthomodular lattices and gives characterizations of factors and Hilbert lattices. Extensive bibliography. BH

Calculus, T(13: 2). *Mathematics with Calculus and Its Applications to Management, Life, and Social Sciences.* Margaret B. Cozzens, Richard D. Porter. DC Heath, 1987, xiii + 992 pp, $27.95. [ISBN: 0-669-09366-1] Chapters on graphs, probability, mathematics of finance, difference equations, linear programming, matrices and game theory precede the traditional core topics of calculus. Prerequisite: high school algebra. JNC

Calculus, T*(13: 3). *Calculus.* James F. Hurley. Wadsworth, 1987, xxiii + 1136 pp, $48.75 [ISBN: 0-534-05592-3]; *Gateway to Hurley's Calculus and Its Supplementary Materials,* 63 pp, (P). Attractive, well written but huge; careful explanations of the standard topics with early coverage of trigonometric functions; includes occasional appearances of suggested algorithms for solving problems, illustrations of computer output, and optional subsections on numerically-oriented topics; supplements include student and instructor solutions manuals, a study guide, instructor's manual, a microcomputer diskette, and a forthcoming computer manual. JNC

Real Analysis, T(16-17). *Measure and Integration for Use.* H.R. Pitt. IMA Mono. Ser., V. 1. Oxford U Pr, 1985, xii + 143 pp, $22.95. [ISBN: 0-19-853608-9] The first part of the book presents in a concise form a general theory of integration and measure over a σ-ring including Radon-Nikodym theorem, Lebesgue convergence theorm, and product measures. The Lebesgue-Stieltjes integral in Euclidean space and some of its special properties are treated. PH

Real Analysis, S*(17), P*. *Introduction to the Theory of Fourier Integrals, Third Edition.* E.C. Titchmarsh. Chelsea, 1986, x + 394 pp, $23.95. [ISBN: 0-8284-0324-4] "Textually the same as the *Second Edition* (published in 1948), except for a few minor improvements." A classic text on the theory of Fourier integrals. BH

Partial Differential Equations, P. *Introduction to the Theory of Nonlinear Elliptic Equations.* Jindřich Nečas. Wiley, 1986, 164 pp, $29.95. [ISBN: 0-471-90894-0] Studies boundary value problems for nonlinear, second order, and elliptic partial differential equations. Includes a short introduction to Sobolev and Morrey-Campanato spaces and to methods of approximation. Studies regularity questions in detail. (1983 BG Teubner edition, TR, November 1984.) AM

Partial Differential Equations, P. *Lecture Notes in Mathematics-1224: Nonlinear Diffusion Problems.* Ed: A. Fasano, M. Primicerio. Springer-Verlag, 1986, viii + 188 pp, $15.80 (P). [ISBN: 0-387-17192-4] Contains the texts of a series of lectures given at the C.I.M.E. session on some problems in nonlinear diffusion held at La Querceta, Montecatini, Italy, from June 10-18, 1985. The lecturers were Donald G. Aronson, Ivar Stakgold, Jesus Hernandez, and Giorgio Talenti. The general theme of the session was the study of the effects of nonlinearity in diffusion problems. AM

Partial Differential Equations, P. *The Legacy of Sonya Kovalevskaya.* Ed: Linda Keen. Contemp. Math., V. 64. AMS, 1986, xiii + 297 pp, $29 (P). [ISBN: 0-8218-5067-9] Proceedings of a symposium sponsored by the Association for Women in Mathematics and The Mary Ingraham Bunting Institute, October 1985. Nine papers, three on Kovalevskaya's life and work, the rest on new mathematics involving differential equations in one form or another. LC

Partial Differential Equations, P. *Hydrodynamic Stability Theory.* Adelina Georgescu. Mechanics: Analysis, V. 9. Martinus Nijhoff (US Distr: Kluwer Boston), 1985, 307 pp, $58. [ISBN: 90-247-3120-8] In fluid mechanics, the Navier-Stokes equations admit solutions for any value of the Reynold's number. However, these solutions correspond to observed motion only for Reynold's numbers smaller than some critical value. Hydrodynamic stability theory deals with the determination of this critical value and with the description of unstable disturbances to fluid flow. The book begins with a general account of the subject. The latter part of the book presents applications of bifurcation and stability to the study of turbulence. AM

Numerical Analysis, P. *Vistas in Applied Mathematics: Numerical Analysis, Atmospheric Sciences, Immunology.* Ed: A.V. Balakrishnan, A.A. Dorodnitsyn, J.L. Lions. Transl. Ser. in Math. & Engin. Optimization Software, 1986, xii + 384 pp, $78. [ISBN: 0-911575-38-3] A volume commemorating the 60th birthday of Soviet applied mathematician G.I. Marchuk, containing 21 papers by international authors in numerical analysis, atmospheric sciences,

and immunology. Splitting methods for flow computations, finite element techniques, fictitious components and other iterative methods; modelling of weather/climate formation, oceanic circulation; disease models, related observational data. RB

Functional Analysis, P. *Lecture Notes in Mathematics-1221: Probability and Banach Spaces.* Ed: J. Bastero, M. San Miguel. Springer-Verlag, 1986, xi + 222 pp, $19.40 (P). [ISBN: 0-387-17186-X] Proceedings of a conference held in Zaragoza, Spain, June 17-21, 1985. JAS

Functional Analysis, P*. *Multivariate Approximation Theory: Selected Topics.* E. Ward Cheney. CBMS-NSF Reg. Conf. Ser. in Appl. Math. SIAM, 1986, v + 68 pp, $13.50 (P). [ISBN: 0-89871-207-6] Based on lectures given at a regional conference on approximation theory and numerical analysis at the University of Alaska in Fairbanks. Describes current status of multivariate approximation theory (mat) with a focus on best approximation, algorithms, and projection operators. Nicely motivated with excellent introductory chapters on mat and tensor products. Extensive bibliography. BH

Functional Analysis, P. *Probability in Banach Spaces—Stable and Infinitely Divisible Distributions.* Werner Linde. Wiley, 1986, 195 pp, $29.95. [ISBN: 0-471-90893-2] Studies stable measures on Banach spaces, first using a classical approach via infinitely divisible measures, and then a functional analytic approach using results about operator ideals and theorems from classical functional analysis. Presented in a dry theorem-proof-corollary fashion. Extensive bibliography. BH

Functional Analysis, P. *Lecture Notes in Mathematics-1208: Interpolation Functors and Duality.* Sten Kaijser, Joan Wick Pelletier. Springer-Verlag, 1986, iv + 167 pp, $15.70 (P). [ISBN: 0-387-16790-0] An exposition of a theory of interpolation which contains the classical theory and is suitable for duality. Intrinsic obstacles prevent a good formulation of duality in the category of Banach couples (usual setting for interpolation theory); doolittle diagrams of Banach spaces proposed instead. Notable as an application of category theory to analysis. RB

Analysis, S*(18), P*. *Noncommutative Harmonic Analysis.* Michael E. Taylor. Math. Surv. & Mono., No. 22. AMS, 1986, xvi + 328 pp, $68. [ISBN: 0-8218-1523-7] Survey of applications of Lie groups to analysis with the relationship between harmonic analysis and linear partial differential equations as unifying theme. Assumes basic understanding of Fourier analysis, functional analysis, and Lie groups. Very clearly written and well motivated. Extensive bibliography. BH

Analysis, T(16: 1). *Fourier Series and Boundary Value Problems, Fourth Edition.* Ruel V. Churchill, James Ward Brown. McGraw-Hill, 1987, x + 292 pp, $40.95. [ISBN: 0-07-010881-1] Revisions of the 1978 edition (TR, June-July 1978) include reorganization of topics (for example, boundary value problems involving Fourier series reached earlier), more figures, labeling of examples, and improvement of exposition. LC

Analysis, S(17-18), P, L. *The Geometry of Fractal Sets.* K.J. Falconer. Tracts in Math., V. 85. Cambridge U Pr, 1986, xiv + 162 pp, $16.95 (P). [ISBN: 0-521-33705-4] The first rigorous self-contained account of the mathematics of sets of fractional and integral Hausdorff dimension. Brings together material formerly only in original, technical papers; focuses on theory, not applications. Definition of Hausdorff dimension; local properties; net measures; projections onto lower dimensional subspaces; the Kakeya problem; examples. Exercises, extensive bibliography. (1985 hardcover text, TR, January 1986.) RB

Analysis, P. *Topics on Real Analytic Spaces.* Francesco Guaraldo, Patrizia Macrì, Alessandro Tancredi. Adv. Lect. in Math. Friedr Vieweg & Sohn, 1986, x + 163 pp, (P). [ISBN: 3-528-08963-6] "Comprehensive treatment of classical and recent results on global properties of real analytic spaces." Covers complexification, normalization, desingularization, pathology of noncoherence, embedding theorems, theory of relative approximation of differentiable functions by analytic functions, classification of analytic vector bundles. Presumes basic knowledge of sheaves, ringed spaces, and cohomology. BH

Analysis, T*(17-18), P. *Real-Variable Methods in Harmonic Analysis.* Alberto Torchinsky. Pure & Appl. Math., V. 123. Academic Pr, 1986, xii + 462 pp, $89. [ISBN: 0-12-695461-5] "Exploration of the unity of several areas in harmonic analysis, emphasizing real-variable methods, and leading to the study of the Calderón-Zygmund theory of singular integral operators, the Muckenhoupt theory of A_p weights, the Fefferman-Stein theory of H^p spaces, the Burkholder-Gundy theory of good λ inequalities, and the Calderón theory of commutators." Assumes elementary knowledge of Lebesgue integral. Historical notes and a large selection of problems at the end of each chapter. BH

Analysis, T(14-15: 2). *Advanced Engineering Mathematics, Second Edition.* Peter V. O'Neil. Wadsworth, 1986, xvi + 1224 pp, $50. [ISBN: 0-534-06792-1] Revisions include rewriting of some portions of the text, examples and problems, addition and expansion of various tables, plus new material such as a treatment of matrix methods for solving systems of differential equations with complex eigenvalues. (*First Edition*, TR, August-September 1983.) LC

Algebraic Geometry, P. *Lecture Notes in Mathematics-1226: Differential Function Fields and Moduli of Algebraic Varieties.* Alexandru Buium. Springer-Verlag, 1986, ix + 146 pp, $15.80 (P). [ISBN: 0-387-

17194-0] Deals with the connection between algebraic differential equations (ADE's!) with no movable singularity, and Galois theory of ADE's. Ends with applications to classical analytic settings: special values of automorphic functions (as in class field theory), foliations, and the Euler equations for rigid-body motion. BC

Differential Geometry, P. *Geometry of CR-Submanifolds.* Aurel Bejancu. Math. & Its Applic. D Reidel, 1986, xii + 169 pp, $49.50. [ISBN: 90-277-2194-7] An introduction to the problems of differential geometry of Cauchy-Riemann (CR-) submanifolds of Kählerian manifolds. Quick trip through background material; CR-submanifolds of almost Hermitian manifolds; various classes of CR-submanifolds; method of Riemannian fiber bundles; extensions of CR-structures to other geometric structures; pseudo-conformal mappings between CR-manifolds; application to relativity. RB

Differential Geometry, P. *Integral Geometry.* Ed: Robert L. Bryant, *et al.* Contemp. Math., V. 63. AMS, 1987, ix + 350 pp, $33 (P). [ISBN: 0-8218-5071-7] A collection of papers given as part of the AMS-SIAM-IMS summer research conference series during a one-week conference on integral geometry held at Bowdoin College in summer 1984. Contains papers ranging from the expository to the technical, and covers problems concerning computing geometric invariants, the Radon transform, and problems concerning integral geometric transforms. AM

Differential Geometry, T(18), P*. *Bieberbach Groups and Flat Manifolds.* Leonard S. Charlap. Universitext. Springer-Verlag, 1986, xiii + 242 pp, $34 (P). [ISBN: 0-387-96395-2] Designed as textbook and basic reference on flat Riemannian manifolds. Topics include classification of Bieberbach groups and holonomy groups and automorphisms of Bieberbach groups as well as helpful background information on differential topology, algebraic number theory, Riemannian geometry, cohomology of groups, and integral representations. Exercises interspersed throughout exposition of text persuade reader to work along. Extremely well motivated in a nice informal style. BH

Geometry, T(14: 1), S*, P*, L*. *Tilings and Patterns.* Branko Grünbaum, G.C. Shephard. WH Freeman, 1986, ix + 700 pp, $59.95. [ISBN: 0-7167-1193-1] Long awaited, this is the first comprehensive and systematic treatment of the subject. Focuses on classification and enumeration of tilings, including aperiodic tilings. Certain to appeal to artists, engineers, designers, and crystallographers as well as mathematicians. JNC

Algebraic Topology, S(18), P. *Lecture Notes in Mathematics-1213: Equivariant Stable Homotopy Theory.* L.G. Lewis, Jr., J.P. May, M. Steinberger. Springer-Verlag, 1986, ix + 538 pp, $44 (P). [ISBN: 0-387-16820-6] Equivariant homotopy theory, i.e., "the algebraic topology of spaces with group action" (here, compact Lie groups), and applications to classical (nonequivariant) homotopy theory. Fundamentals of equivariant theory, including duality theory, transfer maps associated to equivariant bundles; the Burnside ring in equivariant theory; applications to H_∞ ring spectra, generalized Thom spectra. RB

Algebraic Topology, T(18: 1), S, P. *Homotopic Topology.* A.T. Fomenko, D.B. Fuchs, V.L. Gutenmacher. Akademiai Kiado, 1986, 310 pp, $33. [ISBN: 963-05-3544-0] Translation of a Russian textbook offering homotopy concepts, singular homology, obstruction theory, spectral sequences of fiber bundles, Steenrod squares, Adams spectral sequence. Prerequisites: course in algebraic topology or considerable mathematical maturity. Vivid diagrams and imaginative, intriguing illustrations by mathematician/artist Fomenko are an organic part of the book. RB

Algebraic Topology, T(18: 1), P. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Douglas C. Ravenel. Pure & Appl. Math., V. 121. Academic Pr, 1986, xvii + 413 pp, $45 (P); $90. [ISBN: 0-12-583431-4; 0-12-583430-6] An exposition, reference, and exhibition of tools used for computing stable homotopy groups of spheres. Presumes working knowledge of algebraic topology, basic concepts of homotopy theory. Nontechnical introduction; foundational material worked out in generality; Adams spectral sequence; BP-theory and Adams-Novikov spectral sequence; chromatic spectral sequence; Moravara stabilizer algebras; computations and tables. RB

Differential Topology, P. *Lecture Notes in Mathematics-1217: Transformation Groups, Poznań 1985.* Ed: S. Jackowski, K. Pawalowski. Springer-Verlag, 1986, xiv + 396 pp, $35.80 (P). [ISBN: 0-387-16824-9] Proceedings of a symposium held in Poznań, Poland at the Adam Mickiewicz University, July 5-9, 1985. JAS

Topology, P. *Homogeneous Zero-dimensional Absolute Borel Sets.* A.J.M. van Engelen. CWI Tract, V. 27. Math Centrum, 1986, iii + 133 pp, Dfl. 20.10 (P). [ISBN: 90-6196-303-6] "Investigation into the internal topological structure of zero-dimensional absolute Borel sets." Characterizes various classes of these sets using the concept of the Wadge class of a space from game theory. BH

Dynamical Systems. *Symbolic Dynamics of Trapezoidal Maps.* J.D. Louck, N. Metropolis. Math. & Its Applic. D Reidel, 1986, viii + 312 pp, $59. [ISBN: 90-277-2197-1] The book is directed toward a detailed study, both theoretical and experimental, of the behavior of iterates of trapezoidal maps of the unit interval into itself. It presents a number of universality properties such as the ordering of

stable limit cycles and a partial proof that they can apply to a much broader class of functions. PH

Dynamical Systems, T(17-18: 1), P. *Global Stability of Dynamical Systems.* Michael Shub. Transl: Joseph Christy. Springer-Verlag, 1987, xii + 150 pp, $32. [ISBN: 0-387-96295-6] "...[A]n advanced text on the global theory of dynamical systems, including the stable and center manifold theorems, the stability of uniformly hyperbolic sets, Smale's Axiom A, the construction of Markov partitions, spectral decomposition, and Ω-stability theorems. For some of these topics, this is the first treatment in book form." RB

Dynamical Systems, P. *Chaotic Dynamics and Fractals.* Ed: Michael F. Barnsley, Stephen G. Demko. Notes & Rep. in Math. in Sci. & Eng., V. 2. Academic Pr, 1986, xi + 292 pp, $29.95. [ISBN: 0-12-079060-2] Proceedings of a conference on chaotic dynamics, Georgia Institute of Technology, March 1985. Fifteen research papers and a "loving ode to algorithmic complexity theory," concerning chaos and fractals, Julia sets, applications. Intended for mathematicians, physicists and other scientists working in or introducing themselves to the subject. RB

Systems Theory, T(17: 1), S, P, L. *Stabilization of Control Systems.* O. Hijab. Applic. of Math., V. 20. Springer-Verlag, 1987, xii + 129 pp, $32. [ISBN: 0-387-96384-7] A self-contained course on linear systems theory and the solution of the linear quadratic regulator control problem, and an exposition of the stochastic calculus of Brownian motion and its application to filtering theory. These concepts are then combined in an analysis of a stochastic control problem that is a generalization of the *LQ* regulator. Includes problems, solutions, and references. CEC

Probability, P. *Systems in Stochastic Equilibrium.* Peter Whittle. Prob. & Math. Stat. Wiley, 1986, ix + 460 pp, $64.95. [ISBN: 0-471-90887-8] Provides a study of statistical equilibrium in systems of interacting components. Central focus is a type of interaction called weak coupling. A system is weakly coupled if interaction between the various components is carried out by a mediator (e.g., in physics, the exchange of energy between two electrons would be mediated by exchange of a photon). These ideas have applications in areas such as statistical mechanics, chemical kinetics, ecological competition, and communication. AM

Probability, P. *Lecture Notes in Mathematics-1155: Stability Problems for Stochastic Models.* Ed: V.V. Kalashnikov, V.M. Zolotarev. Springer-Verlag, 1985, vi + 447 pp, $29.20 (P). [ISBN: 0-387-15985-1] The volume consists of 23 papers on stability problems for stochastic models, expanded and revised versions of presentations at the eighth seminar on stability problems for stochastic models at Uzhgorod in September 1984. Topics include probability metrics, queueing theory, limit theorems for sums of random variables, and mathematical statistics. KK

Probability, S, P*. *Ergodic Theorems.* Ulrich Krengel. Stud. in Math., V. 6. Walter de Gruyter, 1985, viii + 357 pp, $49.95. [ISBN: 3-11-008478-3] The main topics of the book are classical ergodic theorems, subadditive processes, maximal inequalities and dominated estimates, finite invariant measures, differentiation of integrals, entropy, ergodic theorems in von Neumann algebras. Accessible to nonspecialist. Most topics can be read independently of each other. A supplement by A. Brunel on Harris processes and the Zero-Two-Law. PH

Probability, P. *One-dimensional Stable Distributions.* V.M. Zolotarev. Transl. of Math. Mono., V. 65. AMS, 1986, ix + 284 pp, $92. [ISBN: 0-8218-4519-5] The class of stable distributions, which includes normal distributions and Cauchy distributions, is characterized by the fact that it is closed under convolution. These distributions appear naturally in problems from engineering, physics, astronomy, and economics. This book presents the fundamental notions of stable distributions and studies their analytical properties. AM

Probability, S(18), P. *Stopping Time Techniques for Analysts and Probabilists.* L. Egghe. London Math. Soc. Lect. Note Ser., V. 100. Cambridge U Pr, 1984, xvi + 351 pp, $29.95 (P). [ISBN: 0-521-31715-0] "Studies convergence theory of generalized martingales in Banach spaces, and of extensions of generalized sub- or super-martingales in Banach lattices." Intended for use by probabilists as well as analysts wishing to learn important stopping time techniques. Assumes some familiarity with measure theory and functional analysis. Extensive bibliography. BH

Probability, P. *Characterization Problems Associated with the Exponential Distribution.* T.A. Azlarov, N.A. Volodin. Transl: Margaret Stein. Springer-Verlag, 1986, 137 pp, $24. [ISBN: 0-387-96316-2] Theoretical monograph dealing with ways of characterizing the exponential distribution. Also includes some characterizations of the geometric and multivariate exponential distributions. RSK

Probability, P. *Probability Theory and Harmonic Analysis.* Ed: J.-A. Chao, Wojbor A. Woyczyński. Pure & Appl. Math., V. 98. Dekker, 1986, viii + 291 pp, $59.75. [ISBN: 0-8247-7473-6] Collection of fifteen papers from the mini-conference on probability and harmonic analysis in Cleveland, Ohio, May 1983, and from seminars of the probability consortium of the Western Reserve. Contains expository and survey papers as well as original research. Topics include martingales, stochastic integrals, diffusion processes on manifolds, random walks and harmonic functions on graphs, random Fourier series, invariant differential and degenerate elliptic operators, and

singular integral transforms. BH

Probability, P. *Inequalities for Stopped Brownian Motion.* D.P. van der Vecht. CWI Tract, No. 21. Math Centrum, 1986, iii + 88 pp, Dfl. 13.90 (P). [ISBN: 90-6196-296-X] Theoretical monograph deriving upper bounds and related results for standardly stopped Brownian motion. RSK

Probability, P. *Lecture Notes in Mathematics-1212: Stochastic Spatial Processes.* Ed: P. Tautu. Springer-Verlag, 1986, viii + 311 pp, $27.50 (P). [ISBN: 0-387-16803-6] Proceedings of a conference held in Heidelberg, West Germany, September 10-14, 1984, covering both mathematical theory and biological applications. JAS

Statistics, P. *Lecture Notes in Statistics-36: Spatial Variation, Second Edition.* Bertil Matérn. Springer-Verlag, 1986, 151 pp, $18.30 (P). [ISBN: 0-387-96365-0] Corrected version, with some additions but no change in content, of the original 1960 edition. Concerned with the distribution of objects in a plane or in space, modeled by stationary stochastic processes. Includes some applications to forestry. RSK

Statistics, P. *Lecture Notes in Statistics-33: An Asymptotic Theory for Empirical Reliability and Concentration Processes.* Miklós Csörgő, Sándor Csörgő, Lajos Horváth. Springer-Verlag, 1986, 171 pp, $17.40 (P). [ISBN: 0-387-96359-6] Theoretical monograph presenting "a *unified* asymptotic theory for empirical total time on test, Lorenz, and concentration processes." Of relevance to reliability theory and life testing. RSK

Statistics, T(17-18: 1, 2). *Time Series: Theory and Methods.* Peter J. Brockwell, Richard A. Davis. Ser. in Stat. Springer-Verlag, 1987, xiv + 519 pp, $48. [ISBN: 0-387-96406-1] Systematic treatment of linear time series models and their applications. Covers both time and frequency domain methods. RSK

Statistics, T(18: 1), P*. *Asymptotic Theory of Statistical Inference.* B.L.S. Prakasa Rao. Wiley, xiv + 438 pp, $49.95. [ISBN: 0-471-84335-0] In the Wiley Series in Probability and Mathematical Statistics. Modern approach, exploring the relations between recent advances in probability theory and stochastic processes and the asymptotic theory of (mostly parametric) statistical inference. Good set of references. No exercises. RSK

Statistics, P. *Lecture Notes in Statistics-40: Boundary Crossing of Brownian Motion.* Hans Rudolf Lerche. Springer-Verlag, 1986, v + 142 pp, $15.70 (P). [ISBN: 0-387-96433-9] "Discusses the tangent approximation for Brownian motion as a global approximation device which connects results of fluctuation theory of Brownian motion with classical methods of sequential statistics. Then makes use of these connections to derive optimal properties of tests of power one and repeated significance tests for the simplest model of sequential statistics, the Brownian motion with unknown drift." BH

Statistics, T(16-18), P, L. *Mathematical Theory of Statistics: Statistical Experiments and Asymptotic Decision Theory.* Helmut Strasser. Stud. in Math., V. 7. Walter de Gruyter, 1985, xii + 492 pp, DM 158. [ISBN: 3-11-010258-7] "The aim of this book is to connect the classical parts of mathematical statistics with new developments of asymptotic decision theory. For this it makes use of the theory of statistical experiments which goes back to LeCam. Beginning with an exposition of important results of testing and estimation, the book gives an introduction to statistical decision theory and the theory of experiments." KK

Statistics, P. *The Structure of Asymptotic Deficiency of Estimators.* Masafumi Akahira. Papers in Pure & Appl. Math., No. 75. Queen's U, 1986, iv + 202 pp, (P). Technical monograph dealing with asymptotic efficiency and deficiency, where the latter is the limit of the additional number of observations needed for one statistical procedure to perform as effectively as another. RSK

Statistics, P. *Lecture Notes in Statistics-31: Asymptotic Expansions for General Statistical Models.* J. Pfanzagl. Springer-Verlag, 1985, vii + 505 pp, $36 (P). [ISBN: 0-387-96221-2] General investigation of approximation of the local structure of probability measures by Edgeworth expansions. Treatment is restricted to statistical procedures based on independent, identically distributed observations. KK

Statistics, T(15-16), L. *Principal Component Analysis.* I.T. Jolliffe. Ser. in Stat. Springer-Verlag, 1986, xiii + 271 pp, $39. [ISBN: 0-387-96269-7] A book on principal component analysis intended for readers with some background in probability, statistics, and matrix algebra. Divided into three main sections: theory of principal components, various applications of principal component analysis, and various special situations and some generalizations. KK

Statistics, T*(14-16: 1, 2), S, L. *Probability and Statistics for Engineering and the Sciences, Second Edition.* Jay L. Devore. Brooks/Cole, 1987, xiii + 672 pp, $44. [ISBN: 0-534-06828-6] Revision of the author's 1982 text (TR, December 1982). Presentation is more coherent and some advanced material has been dropped, making it more readable. Excellent exercises, many based on real data. RSK

Computer Literacy, T(13: 1), S, L. *An Introduction to Information Processing.* Harvey M. and Barbara Deitel. Academic Pr, 1986, xxi + 455 pp, $18 (P). [ISBN: 0-12-209005-5] An up-to-date computer literacy text. Chapter topics include the information revolution, the evolution of computers, the processor, input, output, secondary storage, data communications, structured programming, programming languages, structured system analysis, database man-

agement systems, and operating systems. Features include high-quality graphics, problems and exercises, and appendices on Basic programming, number systems, and the computing profession. CEC

Computer Literacy. *dBASE III: Tips & Traps.* Dick Andersen, Cynthia Cooper, Bill Dempsey. Osborne McGraw-Hill, 1986, xiii + 271 pp, $17.95 (P). [ISBN: 0-07-881195-3] A nice supplement to any standard user's or reference manual. Good points: lots of useful information, better-than-usual explanations, points out bugs (traps) carefully. Drawbacks: some out-of-date information, PROLOCK is no longer used; few if any programming examples; no comparison with *dBASE II.* JAS

Computer Programming, P. *Lecture Notes in Computer Science-239: Mathematical Foundations of Programming Semantics.* Ed: Austin Melton. Springer-Verlag, 1986, vi + 395 pp, $27.50 (P). [ISBN: 0-387-16816-8] Twenty-one papers on programming semantics and mathematics immediately or potentially applicable to semantics research; proceedings of Manhattan, Kansas conference (April 1985). Category theory, topology, partial ordering; semantic domains, database queries, semantics for variants of ALGOL. RB

Computer Programming, S(16-17), P. *The Prototyping Methodology.* Kenneth E. Lantz. Prentice-Hall, xi + 191 pp. [ISBN: 0-8359-5897-3] Guide to software prototyping, including a systematic methodology for prototyping within the context of system development, from an information systems viewpoint. Presented primarily for programmer/analysts, project leaders, executives, consultants involved with system development, this "how-to" manual stresses practical considerations. Not a text, but interesting for MIS programs. RB

Computer Programming, T(14-18: 1), S*. *Advanced Turbo Prolog.* Herbert Schildt. Osborne McGraw-Hill, 1987, xi + 299 pp, $19.95 (P). [ISBN: 0-07-881007-8] Assumes knowledge of programming and of Prolog. Advances to high level of Prolog programming through study of a series of artificial intelligence basic topics and techniques. Appendix; index. Could be valuable as a supplement in a programming language course. RJA

Computer Programming. *Advanced dBASE III: Programming & Techniques.* Miriam Liskin. Osborne McGraw-Hill, 1986, xiii + 656 pp, $19.95 (P). [ISBN: 0-07-881196-1] A substantial, but not really advanced, introduction to dBASE III programming with some applied software engineering thrown in. The author presents, via the example of an accounts receivable system, the standard ways of developing a nice, useable application program with dBASE III. The reviewer was unable to find any material on converting or improving older dBASE II programs. JAS

Computer Programming, T(14-18: 1, 2), S, P, L.** *The Art of Prolog: Advanced Programming Techniques.* Leon Sterling, Ehud Shapiro. Ser. in Logic Prog. MIT Pr, 1986, xx + 427 pp, $29.95. [ISBN: 0-262-19250-0] Gives a clear, well-organized, in-depth introduction to both logic programming and Prolog while maintaining the connection between the two. Contains much of the folklore of both Prolog and logic programming. As well, one finds logic programming versions of many classic algorithms from computer science. Each chapter contains a section with pointers to appropriate background information. Chapter exercises; appendices; references; index. RJA

Computer Programming, T*(13-18: 1), S, L. *A Programmer's Guide to Common LISP.* Deborah G. Tatar. Digital Pr, 1987, x + 327 pp, $16.10 (P). [ISBN: 0-932376-87-8] A pedagogically attractive introduction to the *de facto* LISP standard for programmers. Many good examples and exercises. Finishes with a toy expert system, including code. Annotated bibliography. Solutions to exercises. Index of defined procedures and macros. RJA

Computer Programming, S(15-18), P. *Executing Temporal Logic Programs.* B.C. Moszkowski. Cambridge U Pr, 1986, xiii + 125 pp, $14.95 (P). [ISBN: 0-521-31099-7] Begins with syntax and semantics of temporal logic. Uses Interval Temporal Logic (ITL) as its underlying formalism. Proceeds to presenting Tempura, an imperative programming language based on subsets of temporal logic. Includes examples of programming in Tempura, and details of implementing an interpreter. Bibliography. RJA

Computer Programming, T(13-18: 1), S. *The Art of C Programming.* Robin Jones, Ian Stewart. Springer-Verlag, 1987, xiii + 186 pp, $18.50 (P). [ISBN: 0-387-96392-8] The text divides into two parts. First, the language itself is developed using examples, exercises, and some projects. Secondly, genuine applications are presented: rational arithmetic, turtle graphics, random number generation. An attempt is made to present the C-style of programming. Quick reference guide; index. RJA

Software Systems, S(16-18), P. *Advanced Database Techniques.* Daniel Martin. Ser. in Inform. Sys. MIT Pr, 1986, xxi + 377 pp, $35. [ISBN: 0-262-13215-X] An overview of databases and their use for researchers, practitioners and advanced students, suitable for quick reading or long study. Precise definitions, unifying concepts combined with examples, specific timing information, etc. Overview of database management; relational techniques in detail; data representation, packing, protection; selection of a database; database techniques and architectures, optimization; product review of ORACLE. RB

Software Systems. *Guide argent du système Unix.* Peter P. Silvester. Springer-Verlag, 1986, xi + 199

pp, $22 (P). [ISBN: 0-387-15869-3] A translation of the original English edition of a general user's guide to UNIX including the standard version 7 utilities and compilers. Of course the UNIX commands are still given in English, but translations of these key words, as well as the English text, are provided. JAS

Software Systems, S(17-18), P. *The Misconstrued Semicolon: Reconciling Imperative Languages and Dataflow Machines.* A.H. Veen. CWI Tract V. 26. Math Centrum, 1986, ix + 180 pp, Dfl. 27.50 (P). [ISBN: 90-6196-302-8] Main subject of the text is a compiler to translate imperative programs (i.e., written in imperative languages) into dataflow programs (that run on dataflow machines). Appendices; references; index. RJA

Software Systems, P. *Lecture Notes in Computer Science-215: Mathematical Methods of Specification and Synthesis of Software Systems '85.* Ed: W. Bibel, K.P. Jantke. Springer-Verlag, 1986, 245 pp, $16.40 (P). [ISBN: 0-387-16444-8] First workshop on the topic, Berlin, East Germany, April 1985. Eight invited lectures and fifteen contributed papers focusing on abstract data type theory and inductive inference, emphasizing basics from mathematical logic and universal algebra. RB

Software Systems, P. *Lecture Notes in Computer Science-234: Concepts in User Interfaces: A Reference Model for Command and Response Languages.* Ed: David Beech. Springer-Verlag, 1986, x + 116 pp, $14.30 (P). [ISBN: 0-387-16791-9] A document developed by International Federation for Information Processing working group to clarify and facilitate conceptual issues concerning general purpose user interfaces to computers. Less a development structure for standards than an organizational scheme for concepts. Employs "objects," encourages separation of physical medium (e.g., keyboard, mouse) from internal computer system commands. RB

Software Systems, P. *Software System Design Methods: The Challenge of Advanced Computing Technology.* Ed: Jozef K. Skwirzynski. NATO ASI Ser. F, V. 22. Springer-Verlag, 1986, xiii + 747 pp, $115. [ISBN: 0-387-16765-X] Proceedings of NATO Advanced Study Institute, Durham, United Kingdom (1985). Fault-tolerant software; human factors in software development and use; empirical/statistical reliability models for software; cost of software projects; security in computer communications and data storage. 26 papers, six transcripts or summaries of panel discussions. RB

Software Systems, P. *Data Base Organization for Data Management, Second Edition.* Sakti P. Ghosh. Comput. Sci. & Appl. Math. Academic Pr, 1986, xiii + 487 pp, $39.50 (P); $65. [ISBN: 0-12-281852-0; 0-12-281851-2] Fundamentals of data management and organization. Formal and mathematical treatment of query languages, searching, address transformation, retrievals, filing schemes, access paths, metadata, etc. (*First Edition,* TR, January 1979.) RM

Software Systems, P. *Designing Computer-Based Learning Materials.* Ed: Harold Weinstock, Alfred Bork. NATO ASI Ser. F, V. 23. Springer-Verlag, 1986, ix + 285 pp, $65.50. [ISBN: 0-387-16080-9] Proceedings of a NATO Advanced Study Institute involving participants from eighteen countries. Nine papers, including psychology of learning; Socratic dialogs; demands on pedagogical developers of good materials; industrial educational software; computer testing; cost effectiveness; laboratory use. Includes one of the preliminary example dialogs produced in daily workshops. RB

Software Systems, T(13). *Mastering CAD with the ROBO Systems CAD-2.* Harry M. Hawkins, Kolan K. Bisbee. Computer Science Pr, 1986, xi + 163 pp, $19.95 (P). [ISBN: 0-88175-107-3] Introduces beginning ideas in CAD and mechanical drafting using the Robo Systems CAD-2 (a CAD package written for the Apple IIe). Intended as a student workbook or a self-paced introduction to the ROBO CAD system, the book assumes some understanding of drafting fundamentals. Exercises are generally based on those found in an introductory drawing course. AM

Computer Science, P. *Cellular Logic Image Processing.* Ed: M.J.B. Duff, T.J. Fountain. Academic Pr, 1986, x + 277 pp, $37.50. [ISBN: 0-12-223330-1] Nine articles and extracts from Ph.D. theses concerning the award-winning CLIP systems, developed by the Image Processing Group in the Department of Physics and Astronomy, University College London. Motivated by tracking of charged nuclear particles, the group has developed the CLIP4 system involving an array of 9216 parallel processors, now available commercially. RB

Computer Science, P. *Current Advances in Distributed Computing and Communications.* Ed: Yechiam Yemini. Elec. Eng. Commun. & Signal Processing. Computer Science Pr, 1987, vi + 377 pp, $49.95. [ISBN: 0-88175-128-6] Fifteen papers presented at the Distributed Computing and Communications Seminars at Columbia University, concerning performance analysis of distributed systems and various communications issues for distributed processes. The papers emphasize new seminal concepts for approaching current research problems, as opposed to fully mature techniques. RB

Computer Science, P. *Categorical Combinators, Sequential Algorithms and Functional Programming.* P-L Curien. Pitman, 1986, 300 pp, $24.95 (P). [ISBN: 0-273-08722-3] Aim is to illuminate the semantics of sequential programming languages. Chapters on categorical combinators, sequential algorithms on concrete data structures, the functional

language CDSO, and the full abstraction problem. References; index of definitions; index of symbols. Mathematical prerequisites outlined. RJA

Computer Science, P. *Advances in Computers, Volume 25.* Ed: Marshall C. Yovits. Academic Pr, 1986, viii + 417 pp, $59.50. [ISBN: 0-12-012125-5] Latest in a longstanding annual series which documents a wide range of computing subjects by means of extended expository position papers. Natural language access of knowledge databases; design and performance evaluation of database computers; programs for parallel processing; computing and high energy physics; social issues in office automation. RB

Computer Science, P. *Control Flow and Data Flow: Concepts of Distributed Programming.* Ed: Manfred Broy. Springer-Verlag, 1986, viii + 525 pp, $39.50 (P). [ISBN: 0-387-17082-0] Second printing of proceedings of NATO Advanced Study Institute "summer school" directed by Bauer, Dijkstra, and Hoare in Marktoberdorf, West Germany (1985 hardcover text, TR, May 1986). Fourteen papers on distributed systems; operational models; abstract modelling; hardware; design and verification. Includes Dijkstra's remarkable and enjoyable dinner speech "On the nature of computing science." RB

Computer Science, P. *Products of Automata.* Ferenc Gécseg. EATCS Mono. on Theoret. Comput. Sci., V. 7. Springer-Verlag, 1986, viii + 107 pp, $35. [ISBN: 0-387-13719-X] Treats special products of automata, called α_i-products. First chapter presents necessary results from universal algebra, automata, and sequential machines. Subsequent chapters cover homomorphic representations, isomorphic representations, generalized products and simulations, infinite products and representations of automaton mappings in finite lengths. References; subject index. RJA

Computer Science, T(15-16: 1), S*, L.** *Operating Systems: Design and Implementation.* Andrew S. Tanenbaum. Prentice-Hall, 1987, xvi + 719 pp, $24.95. [ISBN: 0-13-637406-9] Text for an undergraduate course in operating systems: easily accessible survey of standard design principles; detailed case study of the MINIX operating system (Unix rewritten). MINIX sources, executables (IBM PC) generously available through publisher; cross-referenced source listing in text facilitates specific discussions, e.g., interrupt handling, floppy disk driver, file server. Well written. Offers balanced mixture of principles and practical considerations; built-in programming project opportunities. RB

Computer Science, T(14-16: 1, 2). *Principles of Programming Languages: Design, Evaluation, and Implementation, Second Edition.* Bruce J. MacLennan. Holt, Rinehart & Winston, 1986, xiv + 568 pp. [ISBN: 0-03-005163-0] This *Second Edition* has added more straightforward exercises (as opposed to open-ended "thought questions"), and has otherwise expanded the treatments of functional programming, Smalltalk, and Prolog. The basic structure of presenting programming language principles via case studies remains founded on the examples of FORTRAN, ALGOL-60, Pascal, Ada, LISP, Smalltalk, and Prolog. (*First Edition,* TR, October 1983.) JAS

Computer Science, T(13-15: 1, 2), L. *Programming 16-Bit Machines: The PDP-11, 8086, and M68000.* William H. Jermann. Prentice-Hall, 1986, xi + 436 pp, $32.95. [ISBN: 0-13-729161-2] Designed to provide an electrical engineering student with software experience at the machine and assembly level. An introductory text which is far more conceptually oriented than most programming books while still being relatively concrete in its presentation of specific processors. Presents only the RT-11 operating system and a smattering of PC-DOS—only four terminal I/0 functions. Chapter-by-chapter references and a reasonable but incomplete index. JAS

Computer Science, T(15-16: 1), L. *Computer Organization and Architecture: Principles of Structure and Function.* William Stallings. Macmillan, 1987, xiv + 511 pp. [ISBN: 0-02-415480-6] Fundamental concepts of the structure, function, organization, and architecture of computers. Approach is top down hierarchical, from computer as a CPU, memory, I/O modules, system interconnections, operating system, down through CPU and control unit level. Discussion of historical issues, different systems (mainframe to micro), parallel and RISC architectures. RM

Computer Science, P. *Lecture Notes in Computer Science-233: Mathematical Foundations of Computer Science 1986.* Ed: J. Gruska, B. Rovan, J. Wiedermann. Springer-Verlag, 1986, ix + 650 pp, $50.50 (P). [ISBN: 0-387-16783-8] Proceedings of the Twelfth Symposium on Mathematical Foundations of Computer Science, Bratislava, Czechoslovakia, August 1986. 62 papers on complexity, automata, simulation, parallelism, graph coloring, algebraic systems, etc., by an international array of authors. RB

Computer Science, S, P, L. *Artificial Intelligence: Concepts, Techniques and Applications.* Yoshiaki Shirai, Jun-ichi Tsujii. Ser. in Comput. Transl: F.R.D. Apps. Wiley, 1985, viii + 177 pp, $19.95 (P). [ISBN: 0-471-90581-X] A translation of a 1982 Japanese original text introducing the techniques, languages, and future of artificial intelligence. JAS

Computer Science, P. *Lecture Notes in Computer Science-240: Category Theory and Computer Programming.* Ed: David Pitt, *et al.* Springer-Verlag, 1986, vii + 519 pp, $36.60 (P). [ISBN: 0-387-17162-2] Proceedings of a tutorial and workshop held at Guildford, United Kingdom, September 16-20, 1985. The tutorial appears as introductory essays aimed at

academia and industry. The rest is organized into four parts: semantics of programming languages; program specification; categorical logic; and categorical programming. JAS

Computer Science, T*(16-18: 1, 2), S, L. *Prolog Programming for Artificial Intelligence.* Ivan Bratko. Intern. Comput. Sci. Ser. Addison-Wesley, 1986, xvii + 423 pp, $25.95 (P). [ISBN: 0-201-14224-4] Work is in two parts. Part one treats the Prolog language through a presentation of examples of problems and their solutions in Prolog. The presentation brings out the contrasts and connections between procedural and declarative approaches to problem solving. Part two investigates certain central topics in artificial intelligence from the Prolog/logic programmer's point of view. This approach provides the reader with a concrete, practical form for expressing abstract constructions. Exercises and solutions to selected exercises. Index. RJA

Computer Science, P. *Mathematics of Information Processing.* Ed: Michael Anshel, William Gewirtz. Proc. of Symp. in Appl. Math., V. 34. AMS, 1986, xi + 233 pp, $32. [ISBN: 0-8218-0086-8] Lecture notes from the AMS short course held in Louisville, Kentucky, January 23-24, 1984. JAS

Computer Science, P. *Application Development Systems: The Inside Story of Multinational Product Development.* Ed: Tosiyasu L. Kunii. Springer-Verlag, 1986, viii + 382 pp, $55. [ISBN: 0-387-70017-X] Revised papers on software engineering from 1983, 1984, and 1985 IBM computer science symposia. JAS

Applications, P. *Lecture Notes in Computer Science-219: Advances in Cryptology: EUROCRYPT '85.* Ed: Franz Pichler. Springer-Verlag, 1986, ix + 281 pp, $18.40 (P). [ISBN: 0-387-16468-5] The proceedings of a workshop on the theory and application of cryptographic techniques, held in Linz, Austria, April 1985. CEC

Applications, T(17-18: 1), S, P. *Process Modeling.* Morton M. Denn. Pitman, 1986, xi + 324 pp, $34.95. [ISBN: 0-273-08704-5] "...[A] systematic treatment of the methods that are useful in the development and application of mathematical models." Not an applied mathematics text; focuses on the use of physical principles to arrive at the proper mathematical formulation. Valuable as a chemical engineer's viewpoint toward modelling. RB

Applications, P. *Computer-Aided Design and Manufacturing: Methods and Tools, Second, Revised and Enlarged Edition.* Ed: U. Rembold, R. Dillmann. Symbolic Comput. Springer-Verlag, 1986, xiv + 458 pp, $88.50. [ISBN: 0-387-16321-2] A discussion of advanced and future technologies applying computers to industrial design and manufacture: nine revised sets of lecture notes from Advanced Course on Computer Integrated Manufacturing, Karlsruhe, FRG (1983). Computer control for manufacturing equipment; interfacing CAD and CAM systems; technology planning, economic analysis for manufacturing systems; quality assurance, robotics. RB

Applications, P. *Pictorial Information Systems in Medicine.* Ed: Karl Heinz Höhne. NATO ASI Ser. F, V. 19. Springer-Verlag, 1986, xii + 525 pp, $142.50. [ISBN: 0-387-13921-4] Proceedings of NATO Advanced Study Institute in Braunlage/Harz, Germany (1984). The design of picture archiving and communication systems (PACS) involves medicine (especially radiology), database technology, computer graphics, human-machine interaction issues, hardware technology, expert systems. De-emphasis on specific storage, communication technologies and standards: predominantly long papers of interdisciplinary interest. RB

Applications (Artificial Intelligence), P. *Talking with Computers in Natural Language.* Eduard V. Popov. Springer-Verlag, 1986, xii + 305 pp, $69. [ISBN: 0-387-16320-4] Translation of a Russian monograph (1982) which asserts that natural language interfaces for computers should be designed around a model of a participant in the conversation (MPC), not merely a model of the natural language. An MPC includes models of language, surrounding world, and the system itself. Includes description of author's POET system (1977). RB

Applications (Artificial Intelligence), P. *Catalogue of Artificial Intelligence Tools, Second, Revised Edition.* Ed: Alan Bundy. Symbolic Comput. Springer-Verlag, 1986, 168 pp, $27.50 (P). [ISBN: 0-387-16893-1] A reference work listing artificial intelligence (AI) techniques and portable software, with abstract-length descriptions and references/availability information. Online version available in England; paperback version periodically updated and reprinted. 284 items categorized into 19 fields within AI, from automatic programming and computer architecture to theorem proving and vision. (*First Edition,* TR, October 1985.) RB

Applications (Artificial Intelligence), P. *Lecture Notes in Computer Science-232: Fundamentals of Artificial Intelligence: An Advanced Course.* Ed: W. Bibel, Ph. Jorrand. Springer-Verlag, 1986, v + 313 pp, $24.80 (P). [ISBN: 0-387-16782-X] Seven articles summarizing lectures delivered at the first Advanced Course in Artificial Intelligence, Vignieu, France, July 1985. Knowledge representation; knowledge processing, including machine learning and automated reasoning; parallel programming in languages FP2, concurrent Prolog. Inattention to other subfields of artificial intelligence, e.g., computer vision, natural language, automatic programming. RB

Applications (Communication Theory), T(16-17: 1), P. *Adaptive Signal Processing: Theory and*

Applications. S. Thomas Alexander. Texts & Mono. in Comput. Sci. Springer-Verlag, 1986, ix + 179 pp, $36. [ISBN: 0-387-96380-4] Echo in long distance phone lines, data transmission from a mobile unit are adaptive signal processing problems: something unknown, unpredictable must be learned or tracked. Text suitable for students (electrical engineering, computer science, related fields), professional engineers as bridge between classroom, professional literature. Physical, geometric interpretation accompanies mathematical principles. Includes foundations of recent fast adaptive filtering methods. RB

Applications (Economics), P. *Lecture Notes in Economics and Mathematical Systems-268: Information Evaluation in Capital Markets.* Volker Firchau. Springer-Verlag, 1986, vii + 103 pp, $14.60 (P). [ISBN: 0-387-16462-6] An investor assembling a securities portfolio in a capital market can make better decisions with added information, so information has value to him. Such value is determined explicitly in this monograph, based on Bayesian decision theory and the hybrid model of capital markets (constant absolute risk aversion, normally distributed rates of return). RB

Applications (Physics), S(15-17), L. *A Collection of Problems on the Equations of Mathematical Physics.* Ed: V.S. Vladimirov. Transl: Eugene Yankovsky. Springer-Verlag, 1986, 288 pp, $19.95. [ISBN: 0-387-16647-5] Several hundred exercises (with answers to computational problems) on analytic techniques in physics: boundary-value problems, integral equations, the Cauchy problem, etc. Includes some measure theory and Lebesgue integration to get at generalized functions and Fourier and Laplace transforms. BC

Applications (Physics), P. *Clifford Algebras and Their Applications in Mathematical Physics.* Ed: J.S.R. Chisholm, A.K. Common. NATO ASI Ser. C, V. 183. D Reidel, 1986, xix + 592 pp, $114. [ISBN: 90-277-2308-7] Clifford algebras, defined in 1878, generalize Hamilton's quaternions to n-dimensional spaces. The definition was neglected, and specific cases rediscovered in mathematics, mathematical and theoretical physics, electrical engineering. This 1985 NATO workshop (Canterbury, United Kingdom), gathered speakers in the various fields to provide background fundamentals of their disciplines and share recent research in Clifford algebras. 51 papers. RB

Applications (Physics), P. *Mathematical Problems of Statistical Mechanics and Dynamics: A Collection of Surveys.* Ed: R.L. Dobrushin. Math. & Its Applic. D Reidel, 1986, xiv + 261 pp, $89. [ISBN: 90-277-2183-1] Five papers on probabilistic methods in mathematical physics: Phase diagrams for continuous-spin models: an extension of the Pirogov-Sinai theory; Spectrum analysis and scattering theory for a three-particle cluster operator; Stochastic attractors and their small perturbations; Statistical properties of smooth Smale horseshoes; Space-time entropy of infinite classical systems. BH

Applications (Physics), T(16-17).** *Principles of Thermodynamics and Statistical Mechanics.* D.F. Lawden. Wiley, 1987, xiii + 154 pp, $34.95. [ISBN: 0-471-91172-0] Based on a course of lectures given to third-year mathematics students at the University of Aston in Birmingham. Covers the standard syllabus of a course in statistical mechanics and thermodynamics, but tries to emphasize the general principles of thermodynamics and their mathematical development. Contains exercises and a number of worked sample problems. AM

Applications (Physics), P. *Metastability and Incompletely Posed Problems.* Ed: Stuart S. Antman, *et al.* Inst. for Math. & Its Applic., V. 3. Springer-Verlag, 1986, xii + 372 pp, $31.50. [ISBN: 0-387-96462-2] Proceedings of a workshop (part of 1984/85 program in continuum physics and partial differential equations, Institute for Mathematics and Its Applications, Minnesota); experts from engineering, mathematics, physics discuss equilibrium events in nature which do not realize minimum energy (metastable), for which available data is insufficient to determine a unique configuration (incompletely posed). 22 papers. RB

Applications (Physics). *Annual Review of Fluid Mechanics, Volume 19, 1987.* Ed: John L. Lumley, Milton Van Dyke, Helen L. Reed. Annual Reviews, 1987, 626 pp, $32. [ISBN: 0-8243-0719-4] 20 expositions of current frontiers in fluid mechanics ranging from tsunamis to magnetic fluids, from cavitation bubbles to tornadic thunderstorms. Cumulative author and title indices for all 19 volumes in the series. LAS

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THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 8 October 1987

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(ISSN 0002-9890)

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Reprint Permission: A. B. WILLCOX, Executive Director, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Advertising Correspondence: MS. ELAINE PEDREIRA, Advertising Manager, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Subscription correspondence, changes of address, and inquiries about nondelivered or defective issues: Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Back Issues: P. and H. BLISS CO., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

Ringing the Cosets

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The author received his A.B. from Oberlin College in 1961 and his M.S. and Ph.D. from Michigan State University in 1966 and 1969. In between he served as actuarial trainee for the Home Life Insurance Company and as Communications-Electronics Officer for the United States Air Force. Since 1969 he has been on the faculty at Western Michigan University, and he has been Professor of Mathematics (with occasional excursions into the Department of English) since 1979. He has enjoyed two sabbaticals in England (Royal Holloway College, University of London; Mathematical Institute, Oxford University), has been Managing Editor for the *Journal of Graph Theory*, Visiting Fellow to Wolfson College (Oxford), and Faculty Teaching Fellow, Western Michigan University. His book, *Graphs, Groups and Surfaces* (North-Holland), is now in its second edition.

1. Introduction

Throughout history, man has announced important events—some joyful, some sorrowful—by making appropriate sounds. A natural means of doing this is by percussion, and this has led to the instruments called drums and bells. The former are used primarily for military purposes, and the latter primarily for religious purposes. Thus we associate bells with churches, and their sound with coronations, weddings, funerals, calls to service. (In fact, for centuries church towers were almost the only structures substantial enough to accommodate sizable bells.)

Bells are chimed—swung through an arc, with clapper and bell meeting to produce the sound—using ropes or levers. Prior to the fourteenth century, church bells in Europe were usually hung on a spindle and chimed by pulling a rope attached to the spindle. The next two centuries saw the development, in England, of a more sophisticated method of hanging a bell, to improve the control that a ringer had over it. The bell was mounted first on a quarter-wheel, then on a half-wheel, and finally on a full wheel—so that it would swing through a full 360-degree arc each time it rang. The further refinement of the slider and the stay made possible the setting of the bell (in mouth-up position), allowing the ringer to temporarily halt his bell and restart it precisely, thus leading to the development of change ringing (sometimes called campanology) in England. For example, a 'ring' of eight bells, tuned to the notes of a diatonic scale, could (and can) be rung in prescribed sequence, instead of being jangled haphazardly. However, as each bell (weighing perhaps a ton or more) can be advanced or retarded only slightly in its period of ringing (about once every two seconds), one expects to hear not melody, but instead an orderly sequence of permutations on the bells.

This practice crossed the channel only into Belgium—but without the slider and stay, so that purely mechanical methods of ringing eventually led to the carillon there. In Great Britain, nonconformist chapels usually had but one bell, and for

centuries Roman Catholic churches were allowed no bells at all. Thus change ringing became a peculiarly English art, formalized by Fabian Stedman in 1668, with the publication of his *Tintinnalogia—or the Art of Change Ringing*.

Now there are over 5,000 towers in England where bells are rung in changes, exactly sixteen in the United States, six in Canada, and perhaps two dozen in Australia. There are two bell foundaries still actively producing bells for changes: the Loughborough Bell Foundary (Loughborough, England) and the Whitechapel Bell Foundary in London. (The latter manufactured Big Ben and the Liberty Bell.)

Let us denote the n bells in a tower by the natural numbers $1, 2, \ldots, n$—arranged in descending order of pitch, from bell 1 (the *treble*) to bell n (the *tenor*). A change is a ringing of the n bells, once each, in some order. (The terminology is not standard. Many ringers call this a *row* and use 'change' to mean the transition from one row to the next.) To avoid confusing the number of a given bell with the number of the position it occupies in a particular change, we regard a change as a permutation $f\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$, where the domain numbers represent the positions, and the range numbers represent the bells themselves. Thus a change f, recorded as $f(1), f(2), \ldots, f(n)$, would ring bell $f(1)$ first, bell $f(2)$ second, and so on. The very special change given by the identity permutation is called *rounds* and is denoted by r. The central problem in change ringing is to ring an *extent* on n bells; this is a sequence of $n! + 1$ changes satisfying:

(i) The first and last change are both rounds.
(ii) No other change is repeated (so that each other change is rung exactly once.)
(iii) From one change to the next, no bell changes its order of ringing by more than one position.

It should be recorded that ringers regard an extent as consisting of $n!$—not $n! + 1$—changes, but I believe that the way an extent begins and ends is best described mathematically as I have done above.

Certain additional conditions that an extent might meet are often regarded as desirable (or even necessary, in some cases); but it is only the three conditions given above that are *always* required. Among the additional conditions, here regarded as being optional, the three following are the most noteworthy:

(iv) No bell occupies the same position in its order of ringing for more than two successive changes.
(v) The working bells each do the same work.
(vi) Each division (or lead) of the extent, and thus the plain course and perhaps even the extent itself, is palindromic in the transitions employed to pass from one change to the next; that is, the extent is *symmetric*. (In the case of the entire extent, the condition applies to the first $n! - 1$ transitions, exclusive of the last transition, returning to rounds.)

We need additional terminology to understand the last two conditions. First, we remark that there are two basic types of construction for composition in change

ringing: methods and principles. A *method* is *treble-dominated*; that is, the treble *plain hunts*—occupying successively positions $1, 2, \ldots, n$; $n, \ldots, 2, 1$; $1, 2, \ldots, n$; $n, \ldots, 2, 1$; and so on. It is not considered to be working. (There are also methods—Surprise, Delight, Treble Bob—with the treble dodging as it hunts: $1, 2, 1, 2, 3, 4, 3, 4, \ldots, n-1, n, n-1, n$; $n, n-1, n, n-1, n-2, n-3, n-2, n-3, \ldots, 2, 1, 2, 1$; and so on, as well as methods—Treble Place, Alliance—with the hunt bell making internal places. The methods we consider here, however, all have the treble plain hunting. In a *principle*, all the bells are working—that is, they are performing more intricate tasks, such as dodging around other bells, making internal places, and so forth. (A plain hunt bell makes only the external places 1 and n.) Finally, each composition is composed of basic units called, for methods, *leads* (the changes progressing from one treble lead to the next, consisting of $2n$ changes —if the treble is plain hunting among n bells) or, for principles, *divisions*.

It is often useful to know that rounds is rung at *backstroke* at both the start and conclusion of an extent, and that all changes alternate between backstroke and *handstroke*. (These terms distinguish the two directions of revolution of the bell upon its wheel and the two corresponding techniques the ringer uses to pull, release, and catch the rope (via the sally, for handstroke).

It appears that rule (i) is for musicality, rule (ii) for thoroughness, and rule (iii) for mechanical considerations, due to the manner in which the bells are hung on their wheels (recall that each can be advanced or retarded only slightly in its order of ringing). Rules (iv) and (v) help keep the performance interesting for the ringers, and rule (vi) is to ease their memory burden. In fact, no ringer is allowed any visual aid to memory within the ringing changer. One ringer—the conductor—will occasionally make a *call* (an oral instruction, either a *bob* or *single*) to instruct the ringers to modify their pattern of ringing slightly at an appropriate time. What each ringer must bring into the tower is the clear memory of the path (called the *blue line*; see, for example, [**15**]) that his or her bell follows through the other bells in the sequence of changes to be rung. This path is then, normally, followed by *ropesight*: pulling your rope about one quarter of a second after the rope of the bell yours is to follow is pulled by your fellow ringer. Waiting to *hear* the sound of that bell will leave you lagging!

Typically the number n of bells in the ring is between three and twelve, with eight being common. There is a nomenclature for extents, and the last part of this nomenclature specifies the number of bells—as in Table 1.1. The odd-bell names reflect the maximum number of pairs of bells that could be exchanged, in their order of ringing.

As an extent of Major takes about eighteen to twenty-two hours to ring—surely one of the 'major' physical and intellectual feats of mankind—extents on more than eight bells clearly surpass the limits of human endurance. Even on eight bells they are extremely rare: on 27 and 28 July, 1963, Plain Bob Major was rung on tower bells at the Loughborough Bell Foundary; on 27 and 28 December, 1977, the same extent was rung on handbells in a private residence in Farnham, Surrey. (Each handbell is readily controlled by a flick of the wrist; but then, each ringer—having

TABLE 1.1. Part of the nomenclature.

n	Name	$n!$
3	Singles	6
4	Minimus	24
5	Doubles	120
6	Minor	720
7	Triples	5,040
8	Major	40,320
9	Caters	362,880
10	Royal	3,628,800
11	Cliques	39,916,800
12	Maximus	479,001,600

two hands—has *two* blue lines to memorize. In a tower, each ringer has both hands full—of rope and sally—controlling one bell. In fact, it takes many months of practice to learn this control, let alone learning to strike uniformly as part of rounds, to say nothing of ringing constantly changing changes.)

An extent of triples, usually requiring just under three hours of concentrated ringing, is much more readily attainable, and many *peals* are attempted, completed, and reported in the weekly publication *The Ringing World* (published in Guildford, Surrey, England.) Technically, a peal consists of at least 5,000 and, for $5 \leqslant n \leqslant 7$, exactly 5,041 successive changes satisfying the rules above (except that (ii) is waived for $n < 7$, where a peal consists of several extents strung together.) Thus for $n = 7$, a peal *is* an extent; and, for $n > 7$, a peal is a partial extent, called a *touch* (rule (i) still holds). A peal could be performed in celebration of some religious or secular event, or for the purely intellectual and aesthetic satisfaction of it.

As generally the number of bells in a tower is even, for odd-bell extents (or peals or touches) a *covering bell*—always the tenor—rings last in every change, in tolerated violation of rules (iv) and (v). In fact, some listeners find the stability and regularity this provides to be pleasing musically.

For more information regarding the history and practice of change ringing, the reader should seek out Wilson [**22**], Camp [**3**], and *The Ringing World*. For many helpful diagrams, see [**15**]. As might be readily imagined, composers and ringers are intimately involved with mathematics in their art, as they are ringing permutations in a very structured way (we expand on this below); however it is only recently that mathematicians have turned their attention to change ringing. The interested reader should consult works by Rankin [**12**], Fletcher [**7**], Dickinson [**6**], Price [**10**], Budden [**2**], and White ([**18**], [**19**], [**20**], or chapter 15 of [**21**]). Change ringing has also been popularized, to an extent, by Dorothy Sayers in her novel *The Nine Tailors* [**14**]; the title refers to the nine strokes, on the tenor, signifying the death of a man (hence the expression 'Nine Tailors make a man'; 'tailors' is a corruption of 'tellers').

In the present paper, we continue the approach of [**18**], [**19**], [**20**] and regard an extent on n bells as a hamiltonian cycle in a Cayley color graph for the symmetric group S_n; it is often helpful, in finding such a cycle, to imbed the graph in an

appropriate surface. From this point of view it is the individual changes which are being studied. But composers of extents (and the ringers themselves) have been doing coset decomposition in symmetric groups since considerably before mathematicians invented the concept; in this point of view the study focuses on blocks of the composition consisting of several changes each. Again Cayley color graphs—and related structures—are relevant, and again surface imbeddings—and even covering projections for surfaces—facilitate the study. In this paper the two viewpoints are contrasted, with emphasis on the second. Particular attention will be paid to the methods of Plain Bob and Grandsire, to the principles of Stedman and Erin, and to the author's own principle on five bells. Both left and right coset decompositions will be found to be useful.

2. Ringing the changes

In [**19**] and [**20**] extents are studied in terms of their individual changes. This allows an algebraic interpretation of the six conditions of Section 1 and, in turn, a graph-theoretical characterization of extents. In particular, by condition (iii) each transition δ from one change to the next can be regarded as a product of disjoint transpositions of adjacent positions in S_n, the symmetric group of degree n. Of course, in this context positions 1 and n are *not* adjacent, for $n \geqslant 3$. (As always, n denotes the number of bells.) Conversely, each such element in S_n is a candidate for a transition rule for a composition. (We recall, from Theorem 2.1 of [**19**], that the number of such candidates is one less than the nth Fibonacci number $F(n)$—where $F(0) = F(1) = 1$.) Thus, if $\delta_1, \delta_2, \ldots, \delta_k$ represent the first k transitions, then the $(k+1)$st change is given by $r\delta_1\delta_2 \cdots \delta_k$. (In this notation we compose permutations from right to left; the δ_i permute positions and r—rounds—maps from positions to bells.) Now form the set Δ of all transitions employed for a given extent; by condition (ii) the set Δ of these involutions generates S_n. Construct the *Cayley graph* $G_\Delta(S_n)$: $V(G_\Delta(S_n)) = S_n$, $E(G_\Delta(S_n)) = \{\{g, g\delta\} \mid g \in S_n, \delta \in \Delta\}$; then by conditions (i) and (ii) an extent corresponds to a hamiltonian cycle in this graph. In summary, we have the basic result:

THEOREM 2.1. *An extent on n bells, using transition rules from* Δ, *can be composed if and only if* $G_\Delta(S_n)$ *is hamiltonian.*

Consider now condition (v), that all working bells work alike. The *plain course* is the collection of changes beginning with rounds and continuing without *calls* (special generating involutions, either *bobs* or *singles*) until rounds occurs again—perhaps prematurely. The condition ringers use for (v) is that the plain course consist of *the same number of divisions* (*or leads*) *as there are working bells.* Designate this number by m, and let w be the word in the generators from Δ in S_n describing the first division (or lead), so that the plain course is given by $w^m = e$ (the identity element of S_n); w is an m-cycle. Then the work done by bell i $(n - m + 1 \leqslant i \leqslant n)$ in division (lead) j is precisely duplicated by bell $w(i)$ in division (lead) $j + 1$ (mod m)—so that, indeed, all working bells work alike. (The non-working bells are stabilized by w.)

Now let w have length l, as a word in the generators from Δ: $w = \delta_1\delta_2 \cdots \delta_l$. Then condition (iv) requires that if δ_i fixes j, $1 \leqslant j \leqslant n$, then δ_{i+1} does *not* fix j (subscripts on w are modulo l.)

Finally, condition (vi) requires the word $w^* = \delta_1\delta_2 \cdots \delta_{l-1}$ be a palindrome.

In the event that the plain course is the full extent (we call this a *no-call* extent), then $ml = n!$. Condition (i) is that $w^m = e$ and condition (ii) is that w^m has no proper identity subwords. This latter may be difficult to verify, so we typically refer to Theorem 2.1 instead and search for an appropriate hamiltonian cycle in $G_\Delta(S_n)$.

In [**19**] such a search was fruitful, for the case $n = 5$, $\Delta = \{(12)(34), (23)(45), (45)\}$. For ease of notation, we set $a = (12)(34)$, $b = (23)(45)$, $c = (45)$. The Cayley graph $G_\Delta(S_5)$ was imbedded into the surface N_{10} (the closed nonorientable 2-manifold of genus 10) with a 5-fold rotational symmetry that allowed the hamiltonian cycle (necessarily of length 120)—and hence the extent—to be generated by a path of length 24, which exactly corresponded to one division of the principle. In particular, $w = (ac)^3(ab)^3ac(ab)^2(ac)^2ab = (14253)$ ($l = 24$) and $m = 5$ in the above notation; thus $w^5 = e$ completely describes the resulting composition, which was prematurely titled "No-call Doubles" in [**19**]. Conditions (i), (ii), and (iii) of Section 1 are guaranteed to hold, by Theorem 2.1. Condition (iv) can be immediately verified by observing that transition a—moving bells in all but the last position—alternates throughout the extent, and transitions b and c both move the bell in this last position. Condition (v) holds by the remarks given above, since there are five divisions ($m = 5$) and 5 working bells. However, condition (vi) fails—as w^* is not quite palindromic.

On 9 December 1984, this extent was rung to quarter peal length (eleven replications, 1320 changes in all) on the tower bells at the church of St. Thomas the Martyr in Oxford. The band consisted of J. D. Alford 1, M. E. Ovenden 2, J. G. Pusey 3, R. L. Wilden 4, I. M. Gardiner 5, Rachel Pusey (C) 6. (Recall that for an odd number of bells, the tradition is to ring an additional "covering" bell—the tenor—last in every change, in allowed violation of conditions (iv) and (v).) Following the performance, the band—as is the custom—named the composition (over the not-too-strenuous objections of the composer): "White's 'No Call' Doubles." This was duly reported, by M. E. Ovenden, in *The Ringing World* [**8**].

It was noted above that White's "No Call" Doubles does not satisfy condition (vi). Surprisingly, this turns out to be an asset, for—as called to my attention by J. G. Pusey and A. P. Smith (personal communications)—an asymmetric principle automatically generates three companion extents. It can be rung forwards or backwards and/or reversed: the *reverse* of an extent on n bells is obtained by replacing each transposition (ij) in each generator in Δ with $(n - i + 1, n - j + 1)$. This gives a vertical reflection of sorts, whereas the "backwards" gives a horizontal reflection.

On 17 February 1985, "Reverse White's 'No Call' Doubles" was rung to quarter peal length at Carfax Tower, Oxford (M. E. Ovenden 1, R. L. Wilden 2, J. C. Machell 3, N. G. Robinson 4, J. G. Pusey 5, J. J. Strange (C) 6). This was reported, by J. G. Pusey, in *The Ringing World* [**11**]. Letting $d = (12)$ denote the reverse of

$c = (45)$ and noting that $a = (12)(34)$ and $b = (23)(45)$ are the reverse of each other, we record this extent as $[(bd)^3(ba)^3bd(ba)^2(bd)^2ba]^5$. The "backwards" and "reverse backwards" (or "backwards reverse") variants are respectively $[ba(ca)^2(ba)^2ca(ba)^3(ca)^3]^5$ and $[ab(db)^2(ab)^2db(ab)^3(db)^3]^5$; apparently these have yet to be rung.

All this arises from one serendipitous application of topological graph theory.

3. Ringing the cosets

We now switch our attention from individual changes to blocks of changes corresponding to cosets of S_n in certain natural ways. These blocks may or may not agree with the decomposition of an extent into leads or divisions. In this section we illustrate, with the extent Plain Bob Minimus, that both left and right coset decompositions may apply. In the final two sections of the paper, we specialize to left and then right cosets.

Plain Bob Minimus is one of eleven extents on four bells with the treble plain hunting (see **[15]**, for example), and one of only three of the eleven satisfying conditions (iv), (v), and (vi) as well. It is generated by $\Delta = \{a, b, c\}$ for S_4, where $a = (12)(34)$, $b = (23)$, $c = (34)$. Figure 2 of **[19]** shows $G_\Delta(S_4)$ imbedded into N_1, the projective plane. This imbedding decomposes N_1 into three octagons, four hexagons, and six squares. The three octagons correspond to the three leads of this method, and also to the three left cosets of the dihedral subgroup D_4 of degree four and order eight in S_4. Table 3.1 shows this decomposition, by columns. The first row of column two, for example, is the change $r(ab)^3ac$, where $w = (ab)^3ac = (234)$. Thus in position one we ring bell 1; in position two, bell 3; in position three, bell 4; and in position four, bell 2: 1, 3, 4, 2 or 1342. The first column is determined by the palindromic $w^* = (ab)^3a$, and we see from Figure 3.1 that a and b do indeed generate D_4, there regarded as the symmetry group of a square. In fact, the first column (lead) gives, in order, $\{e, a, ab, aba, \ldots, w^*\} = D_4$. The second column gives, in order, $\{w, wa, wab, waba, \ldots, ww^*\} = wD_4$, and the third column gives, in order, $\{w^2, w^2a, w^2ab, w^2aba, \ldots, w^2w^*\} = w^2D_4$. Finally, applying generator c a third time returns us to rounds, since $(w^2w^*)c = w^2(w^*c) = w^3 = e$. Thus the plain course is the full extent: $[(ab)^3ac]^3$.

TABLE 3.1. Plain Bob Minimus.

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

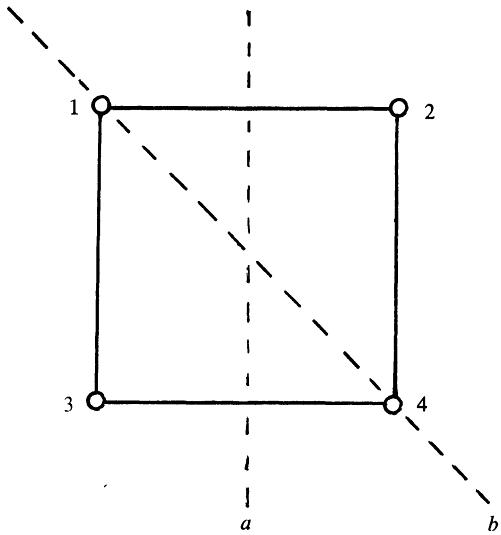

FIG. 3.1.

Now if we focus on the rows of the matrix of Table 3.1 rather than the columns, we get a decomposition of S_4 into the right cosets of row one: $H = \{e, w, w^2\}$. We see that row two consists of $\{a, wa, w^2a\} = Ha$, row three of $\{ab, wab, w^2ab\} = Hab$, etc., down to row eight: $\{w^*, ww^*, w^2w^*\} = Hw^*$.

These observations will be exploited in the following sections.

There is a third coset decomposition important to ringers, given in the example above by the alternating group A_4 and its complement in S_4. A change is *in-course* or *out-of-course* according as the permutation it represents is even or odd respectively; ringers use the terms "even" and "odd" just as mathematicians do, depending upon the parity of the number of transpositions the change is away from rounds. In Table 3.1, the changes in rows 1, 2, 5 and 6 are even; all others are odd. This situation generalizes to *every* extent: A_n contains the in-course changes, and $\bar{A}_n$ (the other coset) contains the out-of-course changes. Of course, since $[S_n : A_n] = 2$, the out-of-course changes constitute both a left and right coset.

Two additional decompositions into right cosets will be discussed in Section 5.

4. Left cosets

Recall that the plain course of an extent is given by $w^m = e$, where w is a word in the generators from Δ for S_n. The word w is called a *plain lead*, since typically (in a method) it takes you from one treble lead (bell 1 in position one) at backstroke (in Plain Bob, for example) to the next treble lead at the same stroke and—of course—it is the basis for the plain course. Two basic types of modification to w—usually at or near the end of the word—give rise to a *bob lead* and a *single lead*, respectively, depending upon whether the *coursing order* (the order of the working

bells as they come to each piece of work) of three (sometimes: an odd number of) or two (sometimes: an even number of) bells is changed by the call. For clarity, we will denote plain, bob, and single leads, respectively, by P, B, and S.

A successful composition typically results by combining the words P, B, and S in an appropriate manner. Thus it is reasonable, and often considerably more efficient, to consider a graph in which the vertices represent not elements of S_n, but treble leads, and where the edges represent not involutions from Δ, but the words P, B, and S in a stabilizer of 1 in S_n—usually either S_{n-1} or A_{n-1} (up to isomorphism). In the nicest case, each treble lead can be identified with a left coset of D_n (called the *hunting group*, as the treble hunts once up and once down in each coset) in S_n. This is the situation we want to highlight here.

There is significant historical precedent for this approach. Here is an example of the sort of question that greatly interests ringers and, especially, composers of extents: Can a full extent of Grandsire Triples (all 5040 changes, on seven bells) be composed using only plain and bob leads (that is, with no single leads whatsoever)? The negative answer to this question was first given by W. H. A. Thompson in 1886 [**17**], by means of a parity argument based on a complicated graphical diagram which resembles—but is not exactly—a Cayley color graph for the group A_6. The 360 vertices (representing half of all the treble leads) are partitioned into 72 blocks of 5 leads each. Thompson called each of these blocks a "Q-set," and they are in fact cosets of the five-element subgroup generated by $P^{-1}B$. For a more detailed outline of Thompson's analysis, consult [**7**].

The same conclusion (singles necessary, for Grandsire Triples) arises as a special case of a special case of a considerably more general 1948 result due to the mathematician R. A. Rankin [**12**]. I state this result in the context that we are studying, where, for $g \in \Gamma$ (a group), $\langle g \rangle$ is the subgroup generated by g and $|\langle g \rangle|$ is the order of that subgroup (and of g).

THEOREM 4.1. *Let a group* Γ *be generated by* $\Delta = \{x, y\}$, *with* $k = |\Gamma|/|\langle x \rangle|$, $l = |\Gamma|/|\langle y \rangle|$, *and* $m = |\langle x^{-1}y \rangle|$. *For m odd, if* $G_\Delta(\Gamma)$ *is hamiltonian, then k and l are both odd.*

It is instructive to see how the Thompson result is contained in Theorem 4.1. We take $x = P = (34675)$ and $y = B = (165)(347)$, so that $x^{-1}y = P^{-1}B = (14675)$. In part B of this section, we will see that P and B *are* as given for Grandsire Triples and that they *do* generate $\Gamma = A_6$. Then $m = 5$, $k = 360/5 = 72$, and $l = 360/3 = 120$. Thus $G_\Delta(A_6)$ cannot be hamiltonian, and plain and bob leads alone do not suffice for Grandsire Triples.

We will need the following propositions.

PROPOSITION 4.2. *The permutations* $(12z)$, $3 \leqslant z \leqslant n$, *generate* A_n.

Proof. This is well known (see, for example, p. 66 of [**4**]). Or, see Project 5.8.4 of [**1**].

PROPOSITION 4.3. *The permutations* (123) *and* $(34 \cdots n)$ *generate* A_n, *for n odd.*

Proof. See p. 67 of [**4**]. Or, conjugate (123) by powers of $(34 \cdots n)$ and apply Proposition 4.2.

A. *Plain Bob.* The diagram in Figure 4.1 is a slight modification of one due to David W. Struckett [**16**]. The graph is the 1-skeleton of a truncated octahedron. The 24 vertices are labelled with the 24 treble leads on five bells. (Bell 1 is first in every change, and is suppressed.) Let $a = (12)(34)$, $b = (23)(45)$, $c = (34)$, and $d = (23)$; then $P = (ab)^4ac = (2354)$ and $B = (ab)^4ad = (45)$ are the plain and bob leads respectively. The directed edges in the figure represent right multiplication by P, and the undirected edges represent right multiplication by B. Thus the figure is actually a spherical imbedding of the Cayley color graph $C_\Delta(\Gamma)$, where $\Delta = \{P, B\}$ for $\Gamma = (S_5)_1 \cong S_4$ (the stabilizer in S_5 of object 1, acting on $\{2,3,4,5\}$; this is precisely what is needed to study the treble leads on 5 bells!).

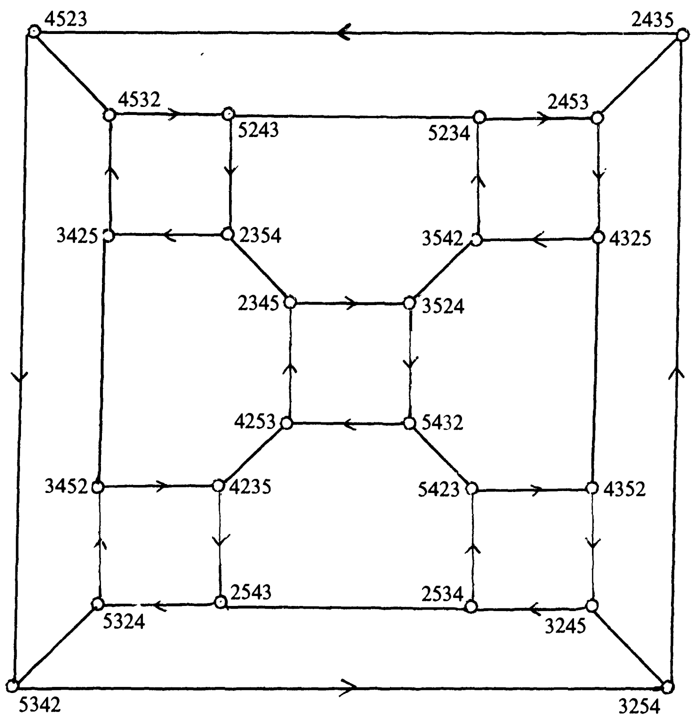

FIG. 4.1.

(A Cayley *color* graph is a Cayley graph with each edge "colored" by the color associated with the generator whose action that edge models, and directed in accordance with that action (from g to $g\delta$, $g \in \Gamma$, $\delta \in \Delta$)—unless the generator has order two, in which case no ambiguity results from omitting the arrows. In "ringing the changes," all generators are involutions and hence no edge needs an arrow. Such

is not the case in "ringing the cosets." Thus we employ Cayley color graphs here. It is clear that a hamiltonian cycle in $C_\Delta(\Gamma)$ (which must respect the arrows) always induces one in $G_\Delta(\Gamma)$, but not conversely. Thus the contrapositive of Theorem 4.1—the sense in which we always use it—applies to $C_\Delta(\Gamma)$ as well.)

An example calculation may help at this stage. If we start at vertex 5423 in Figure 2, we must realize that this change is identified with permutation (2534)—in position two, ring bell 5; in position five, ring bell 3; etc.—so that when we apply P, we are taking the product $(2534)(2354) = (245)$, identified with change (1)4352. Thus there is a directed edge from vertex 5423 to vertex 4352, colored with the color associated with P, in the diagram.

But the figure has another interpretation, and this is the one given in [**16**]. Each vertex represents a block of *ten* changes, commencing with the treble lead given and continuing as determined by applying $w^* = (ab)^4a = (23)(45)$ one letter at a time. We make two remarks: 1) $w^*c = P$ and $w^*d = B$, so that w^* is the maximal subword in common to both leads; 2) since $w^* \in (S_5)_1$, the tenth change in each block is also a treble lead. Thus each of the 24 treble leads appears in each of two blocks, once as the first change (the one given in the figure) and once as the last. As Mr. Struckett indicates, these two blocks are diametrically opposite on the spherical imbedding. (This is verified by noting that diametrically opposite vertices are separated by the following Petrie path: $P^2BP^{-2}B = (23)(45) = w^*$.) Furthermore, in each such pair of blocks, one is the backwards version of the other; this is immediate, since w^* is a palindrome.

But more! The involutions a and b generate D_5 in S_5, as a diagram similar to Figure 1 readily verifies. Thus each vertex of Figure 2 represents a left coset of D_5 in S_5, with each coset appearing twice—represented by each of the two treble leads it contains. (This illustrates an advantage accruing to ringers when changes can be decomposed into cosets. Since two cosets are either disjoint or identical, comparing treble leads is sufficient to make the distinction. This vastly simplifies the verification of the "truth" of an extent: ensuring that no change is repeated.)

Diagrams such as Figure 4.1 are used to combine the leads P and B so as to get either a full or a partial extent. (Imbedding the diagram on an appropriate surface can be helpful in at least three different ways: 1) cycles are easier to trace out and verify when edges only intersect where they are supposed to—which does not happen when a nonplanar graph is represented on the plane, for example; 2) significant portions of the cycle may often be taken from certain region boundaries—as was the case for Plain Bob Minimus in Section 3; 3) symmetries of the imbedding may suggest how to find the desired cycle more easily, as was the case with White's "No Call" Doubles in Section 2.) But care must be taken to avoid two vertices representing the same coset. The situation at hand is ideally suited to circumvent this difficulty very nicely. It is a standard result in surface topology that antipodal identification on the sphere produces a 2-fold covering projection down to the projective plane N_1. If we perform this process on Figure 2, then the result is Figure 3—where each pair of identical cosets has been identified to one vertex. (*Warning*: this is *not* a Schreier coset graph, as we are using *right* multiplication on

left cosets. Thus the choice of coset representative is *not* arbitrary; we select the treble lead at backstroke.)

Now each hamiltonian cycle (respecting the arrows—except that a bob lead passing through the crosscap on N_1 (these are labelled a, b, c) requires traversing against the arrows on the plain leads until another bob lead passes through the crosscap—this is required, since passing through the crosscap corresponds to changing hemispheres above, where—for fixed vantage point—orientation is reversed) gives an extent of Plain Bob Doubles. These are *precisely* $(P^3B)^3$, $P^2B(P^3B)^2P$, $PB(P^3B)^2P^2$, and $B(P^3B)^2P^3$—all starting at 2345 (rounds)—as may be readily verified from the figure. Thus topological graph theory shows very nicely that there are precisely four extents of Plain Bob Doubles using the standard bob B. (For example, the one given by $(P^3B)^3$ is $[((ab)^4ac)^3(ab)^4ad]^3$.)

Figure 4.2 can be used, as observed in [**16**], to show that there is no "true 240" (each change repeated at the opposite stroke); this also follows from Theorem 4.1, with $\Gamma = S_4$, $k = 6$, $l = 12$, and $m = 3$: thus there is no hamiltonian cycle in Figure 4.2.

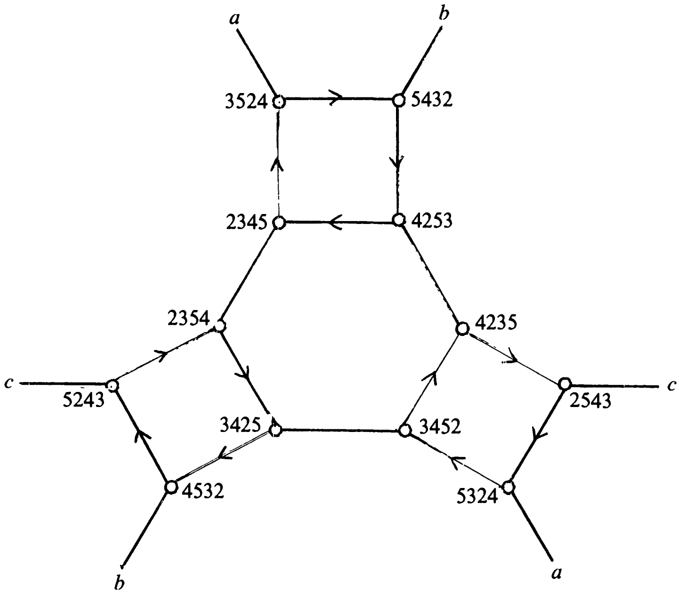

FIG. 4.2.

Let us try to extend the very nice situation for Plain Bob on five bells to the general case $n \geqslant 5$. As is traditional, we start with $a = (12)(34)(56)\cdots$, $b = (23)(45)(67)\cdots$, and $c = (34)(56)(78)\cdots$ to get the plain lead $P = (ab)^{n-1}ac = (3579\cdots 8642)$, an $(n-1)$-cycle (as it must be, to satisfy condition (v), since only the treble is a non-working bell and we therefore need P^{n-1} to give the plain course). We observe that $\langle a, b\rangle = D_n$ (by an easy generalization of Figure 3.1), so

we expect a decomposition into left cosets of D_n. If we use the standard bob $d = (23)(56)(78)\cdots$, we get the standard bob lead $B = (ab)^{n-1}ad = (579\cdots 864)$. We need the following lemma.

LEMMA 4.4. *The permutations* $P = (3579\cdots 8642)$ *and* $B = (579\cdots 864)$ *in* S_n *generate* S_{n-1} *for n odd and* A_{n-1} *for n even.*

Proof. We compute that $P^{-1}B = (243)$, and conjugation of (243) by B gives all 3-cycles of the form $(2z3)$, $4 \leqslant z \leqslant n$; by Proposition 4.2, these 3-cycles generate A_{n-1} (on the symbols $2,3,\ldots,n$). Now if n is odd, neither P nor B—both even cycles—is in A_{n-1}, so that A_{n-1} is a proper subgroup of $\langle P, B\rangle$, which must, therefore, be S_{n-1}. On the other hand, if n is even, then both P and B—odd cycles—are in A_{n-1}, and $A_{n-1} \leqslant \langle P, B\rangle \leqslant A_{n-1}$.

Thus for n odd we do generalize the case for $n = 5$ above. We obtain a Cayley color graph $C_\Delta(\Gamma)$, with $\Delta = \{P, B\}$ for $\Gamma = (S_n)_1 \cong S_{n-1}$. Each left coset of D_n is represented twice, once by each of the treble leads it contains. (Note that $|S_{n-1}| \cdot |D_n| = 2|S_n|$.) Since $P^{-1}B$ has order $m = 3$, Rankin's analysis (Theorem 4.1) applies to show that a single is required for a "true $2n!$" (each change rung twice, once at each stroke).

For n even, however, we obtain a Cayley color graph $C_\Delta(\Gamma)$, again with $\Delta = \{P, B\}$ but now for $\Gamma = A_{n-1} \leqslant S_{n-1} \cong (S_n)_1$. For $n \equiv 0 \pmod 4$, each left coset of D_n is represented once, just by the treble lead at backstroke ($|A_{n-1}| \cdot |D_n| = |S_n|$). The other treble leads, representing the "backwards" cosets, appear in an isomorphic copy of $C_\Delta(A_{n-1})$, with the vertices labelled by the elements of the other coset of A_{n-1} in S_{n-1} (the treble leads at handstroke); this is because $w^* = (ab)^{n-1}a$ is not in A_{n-1} for this case. For $n \equiv 2 \pmod 4$, w^* is in A_{n-1}, and each copy of $C_\Delta(A_{n-1})$ has exactly half of the cosets, each represented twice.

Theorem 4.1 applies in the first subcase, to show that an extent of Plain Bob on n bells ($n \equiv 0 \pmod 4$, $n \geqslant 8$) requires a single; it cannot be rung on plain and standard bob leads alone. For $n \equiv 2 \pmod 4$, $n \geqslant 6$, we draw the same conclusion without recourse to Theorem 4.1, since the cosets of $C_\Delta(A_{n-1})$ do not exhaust S_n. Theorem 4.1 does tell us, in this subcase, that we cannot have a touch of $n!$ changes containing each of the $n!/2$ changes accessible without singles once at handstroke and once at backstroke.

Thus in each case (n odd or even, $n \geqslant 5$), neither $C_\Delta(\Gamma)$ nor $G_\Delta(\Gamma)$ contains a hamiltonian cycle. This conclusion is false for $n = 4$ ($C_\Delta(A_3)$ is a directed 3-cycle), but here Theorem 4.1 does not rule this out: $\Gamma = A_3$, so k and l *are* both odd.

In summary, we have shown the following:

THEOREM 4.5. *For Plain Bob on n bells,* $n \geqslant 5$:

(1) *If n is odd, there is no true* $2n!$ *using plain and bob leads only*;

(2) *If n is even, there is no extent using plain and bob leads only.*

Now, let us be daring and replace the standard bob $d = (23)(56)(78)\cdots$ with the call $d' = (45)(67)(89)\cdots$; this is a bob for n odd, and a single for n even. Now the

bob lead $B = (ab)^{n-1}\, ad$ is replaced with $X = (ab)^{n-1}ad' = (23)$, and regardless of parity $\langle P, X\rangle = S_{n-1} \cong (S_n)_1$.

LEMMA 4.6. *The permutations* $P = (3579 \cdots 8642)$ *and* $X = (23)$ *in* S_n *generate* S_{n-1}, *for all* n.

Proof. This is well known (see, for example, p. 63 of [**4**]). Or, start with X and conjugate successively by P^{-1} to obtain transpositions (23), (35), (57), ..., (42); by Project 5.8.2 of [**1**], these generate S_{n-1}.

Thus, for each $n \geqslant 5$, we obtain a Cayley color graph $C_\Delta(\Gamma)$, with $\Delta = \{P, X\}$ for $\Gamma = S_{n-1} \cong (S_n)_1$; each vertex also represents a left coset of D_n in S_n, and each such coset is represented twice—once by each of the stabilizers of 1—the treble leads—it contains. The spherical voltage graph of Figure 4.3 determines (see Chapter 10 of [**21**], for instance) an imbedding of $C_\Delta(S_{n-1})$ into the closed orientable 2-manifold of genus $1 + (n-3)!(n^2 - 7n + 8)/4$, as a branched covering space over the sphere. (The case $n = 5$ gives genus 0 as expected—except we should note that the Struckett diagram (Figure 2) uses $B = (45)$ instead of $X = (23)$, its reverse in $(S_5)_1$.) There are $(n-2)! \cdot (n-1)$-gonal regions above, and $(n-1)$ $(n-3)!$ $(2n-4)$-gonal regions; each of the former covers the loop in Figure 4.3 and corresponds to the plain course.

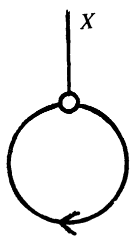

FIG. 4.3.

It is well known that each orientable surface double-covers a nonorientable surface of half the characteristic, and we illustrate that here by identifying each vertex g of $C_\Delta(S_{n-1})$ with $gw^* = g(ab)^{n-1}a$ (the other treble lead in the same coset). This readily extends to a double-covering projection (with no branching) to the closed nonorientable surface of genus $2 + (n-3)!(n^2 - 7n + 8)/4$, which has half of everything above: half as many vertices (each left coset is now represented exactly once), half as many $(n-1)$-gonal regions (each still modeling the plain course), etc. A hamiltonian cycle (respecting the arrows properly) in this configuration would represent a Plain Bob extent on plain and bob leads alone (n odd) or on plain and single leads alone (n even). Such is not precluded by Theorem 4.1, since the quotient graph is not a Cayley color graph.

On the other hand, Theorem 4.1 does apply to $G_\Delta(S_{n-1})$, before projection, precisely when n is odd—for then $P^{-1}X = (468 \cdots 753)$ has order $m = n - 2$,

odd. Since $k = (n-1)!/(n-1) = (n-2)!$ and $l = (n-1)!/2$ are both even for $n \geqslant 5$, we see that a single is required for a true $2n!$ in this situation also.

In summary, we have shown:

THEOREM 4.7. *For the modified Plain Bob on n bells, $n \geqslant 5$ (using $d' = (45)(67)(89)\cdots$):*

(1) *If n is odd, there is no true $2n!$ using plain and bob leads only. An extent using plain and bob leads only is not ruled out by Theorem* 4.1.

(2) *If n is even, an extent using plain and single leads only is not ruled out by Theorem* 4.1. *Nor is a true $2n!$, using plain and single leads only.*

Let us attempt to generalize the approach advocated in this section. This will serve two purposes: (1) to solidify our understanding of how a hamiltonian cycle for the leads can extend to one for all the changes, and (2) to apply Theorem 4.1 from a different perspective: instead of starting with a fixed set of transition rules and seeing what they might generate, we start with the hope of an extent and see what transition rules might allow us to attain this goal.

We say that an involution in S_n is of *type A* if it is a disjoint product of transpositions of adjacent numbers; that is, if it is a candidate for a transition rule for an extent.

THEOREM 4.8. *Let $\Delta = \{a, b, c, d\}$ be a set of generating involutions for S_n, each of type A. Let $b, c, d, \in \overline{A_{n-1}} \subseteq (S_n)_1$. Let $\langle a, b\rangle = D_n$, with ab of order n. Set $X = (ab)^{n-1}ac$, $Y = (ab)^{n-1}ad$, and suppose $\Delta' = \{X, Y\}$ generates A_{n-1}. Then there is an extent on n bells using transitions from Δ—and leads X and Y—if and only if $C_{\Delta'}(A_{n-1})$ is hamiltonian.*

Proof. Since $(ab)^n = e$, $X^* = Y^* = (ab)^{n-1}a = b \in (S_n)_1$. Since c and $d \in (S_n)_1$, so do X and Y. Thus we can regard $\langle X, Y\rangle = A_{n-1}$ as being imbedded in $S_{n-1} \cong (S_n)_1$. Now form the decomposition of S_n into left cosets of D_n. (D_n consists of the successive subwords $e, a, ab, aba, \ldots, (ab)^{n-1}a$ of $X^* = Y^*$.) Since $b \in \overline{A_{n-1}}$, each coset contains two elements of $(S_n)_1$—the treble leads—exactly one of which is in A_{n-1}. Take this element as the coset representative. Thus each vertex of $C_{\Delta'}(A_{n-1})$ represents not only a unique element of A_{n-1} (on the symbols $2, 3, \ldots, n$), but also an entire left coset of D_n in S_n—that is, an entire lead of an extent on n bells. Now, if $C_{\Delta'}(A_{n-1})$ is hamiltonian, then there is an identity word $f(X, Y) = e$ in the leads X and Y that visits each vertex of $C_{\Delta'}(A_{n-1})$ exactly once. Thus $f((ab)^{n-1}ac, (ab)^{n-1}ad) = e$ is an identity word in the generators from Δ. Since distinct cosets are disjoint, this represents a cycle in $C_\Delta(S_n)$. Since the word $f(X, Y)$ necessarily has length $(n-1)!/2$, the cycle in $C_\Delta(S_\Delta)$ has length $(((n-1)!/2)2n) = n!$; that is, it is a hamiltonian cycle. By Theorem 2.1, there is an extent on n bells, using transitions from Δ. By construction, it uses leads X and Y. We think of the word $X^* = Y^*$ as ringing each lead—passing from the treble lead at lead head (in A_{n-1}) to the treble lead at lead end (in $\overline{A_{n-1}}$). Then either c or d takes us to another treble lead at lead head (in A_{n-1} again); and so on.

The converse follows from similar considerations.

THEOREM 4.9. *Again let* $\Delta = \{a, b, c, d\}$ *be a set of generating involutions for* S_n, *each of type A. Let* $b, c, d, \in (S_n)_1$, *and let* $\langle a, b\rangle = D_n$, *with ab of order n. Again set* $X = (ab)^{n-1}ac$ *and* $Y = (ab)^{n-1}ad$, *but now suppose* $\Delta' = \{X, Y\}$ *generates* S_{n-1}. *Then there is a true* $2n!$ *on n bells using transitions from* Δ*—and leads X and Y —if and only if* $C_{\Delta'}(S_{n-1})$ *is hamiltonian.*

Proof. As before, $X, Y \in (S_n)_1$, and we regard $\langle X, Y\rangle$ as being imbedded in $S_{n-1} \cong (S_n)_1$. Again form the decomposition of S_n into left cosets of D_n. Now, each vertex of $C_{\Delta'}(S_{n-1})$ represents not only a unique element of S_{n-1} (on the symbols $2, 3, \ldots, n$), but also an entire left coset. Each coset is represented twice, however—once by each treble lead it contains. If $C_{\Delta'}(S_{n-1})$ is hamiltonian, then we find an identity word $f(X, Y) = e$ that visits each vertex of $C_{\Delta'}(S_{n-1})$ exactly once. Thus $f((ab)^{n-1}ac, (ab)^{n-1}ad) = e$ gives the double extent that we seek: each coset is rung exactly twice—once forwards and once backwards. Thus each of the $n!$ changes on the n bells is rung once at handstroke and once at backstroke, and we have a true $2n!$.

The converse follows from similar considerations.

Now let us see what type of extent might fit the description in Theorems 4.8 and 4.9. Since $\langle a, b\rangle = D_n$ with ab of order n and $b \in (S_n)_1$, we must have $b = (23)(45)(67)\cdots$ and $a = (12)(34)(56)\cdots$. Assume that we are describing a method with one hunt bell, the treble, on n bells. Then the plain course must be $P^{n-1} = e$; without loss of generality we take $P = X = (ab)^{n-1}ac = bc = (23)(45)(67)\cdots c$, and this must be an $(n-1)$-cycle fixing 1. This forces $c = (34)(56)(78)\cdots$, and we are ringing plain bob on n bells! Moreover, since $b \notin A_{n-1}$, $n \equiv 0$ or $3 \pmod 4$; furthermore, since $c \notin A_{n-1}$ also, $n \equiv 0 \pmod 4$. Then for $d \notin A_{n-1}$, both $X, Y \in A_{n-1}$. (Our choice of d will determine whether the other lead Y is a bob or a single.)

Consider now the implications of Theorem 4.1, as pertaining to Theorem 4.8. Since $k = (n-1)!/(2|\langle X\rangle|) = (n-1)!/(2(n-1)) = (n-2)!/2$ is even for $n \geqslant 6$, we must have $m = |\langle X^{-1}Y\rangle|$ even, to have any hope of finding $C_{\Delta'}(A_{n-1})$ hamiltonian. But

$$X^{-1}Y = ca(ba)^{n-1}(ab)^{n-1}ad = cd = (34)(56)(78)\cdots d;$$

this element in S_n *must* have even order (to find an extent of plain bob (n) by application of Theorem 4.8). The standard bob of $d = (23)(56)(78)\cdots$ gives $\langle X, Y\rangle = A_{n-1}$, as required, but $cd = (243)$, of odd order. The standard single $d = (56)(78)\cdots$, for n even, gives $cd = (34)$ of even order, so there might seem to be hope of finding an extent using plain and single leads only by this approach. We would first need to verify that $\langle X, Y\rangle = A_{n-1}$; however, see Lemma 4.10.

LEMMA 4.10. *The permutations* $X = (3579\cdots 8642)$ *and* $Y = (23)(579\cdots 864)$ (*arising from* $X = (ab)^{n-1}ac$, $Y = (ab)^{n-1}ad$ *as above*) *generate* S_{n-1}, *for n even.*

Proof. Since n is even, $n-3$ is odd, and hence $Y^{n-3} = (23)$—since $(579\cdots 864)$ is an $(n-3)$-cycle. But (23) and X generate S_{n-1}, by Lemma 4.6.

Now, what other candidates for d might be tried? For example, if $n = 8$, we can take $d = (78)$; then $cd = (34)(56)$, $X = (3578642)$, and $Y = (23)(45)(678)$. Then $\langle X,Y\rangle = A_7$ ($Y^2 = (876)$, and $XY^2 = (23574)$; now, apply Proposition 4.3). Thus it makes sense to look for a hamiltonian cycle in $G_{\{X,Y\}}(A_7)$. This search will be successful, as the next result guarantees.

THEOREM 4.11. *Let $\Delta = \{a, b, c, d\}$ be a set of type A generating involutions for S_n, with $b, c, d \in A_{n-1} \subseteq (S_n)_1$. Let $\langle a, b\rangle = D_n$, with ab of order n. Set $X = (ab)^{n-1}ac$, $Y = (ab)^{n-1}ad$. Suppose that $\Delta' = \{X, Y\}$ generates A_{n-1} and that $cd = dc$. Then there is an extent on n bells using transitions from Δ and leads X and Y.*

Proof. By Theorem 4.8, it will suffice to construct a hamiltonian cycle in $C'_\Delta(A_{n-1})$. We start by noting that generator X for A_{n-1} partitions the vertices of the graph into disjoint directed m-cycles corresponding to the relation $X^m = e$, where m is the order of X. Since $\langle x, y\rangle = A_{n-1}$, the graph is connected, and we find an edge colored Y joining points u and v in separate cycles. (Then $v = uY$.) We claim that the situation is as depicted in Figure 4.4; that is, there is an edge colored Y joining uYX^{-1} to uX. So, we need to verify that $YX^{-1} = XY^{-1}$. But

$$\begin{aligned} YX^{-1} &= \left((ab)^{n-1}ad\right)\left(ca(ba)^{n-1}\right) = (ab)^{n-1}a(dc)a(ba)^{n-1} \\ &= (ab)^{n-1}a(cd)a(ba)^{n-1} = \left((ab)^{n-1}ac\right)\left(da(ba)^{n-1}\right) = XY^{-1}. \end{aligned}$$

Thus two cycles of length m are combined into one cycle of length $2m$ (starting at uX, say): $(X^{m-1}Y)2_{m-1} = e$. This produces a *touch* of $4mn$ changes. We repeat this process until all of the m-cycles are incorporated into a hamiltonian cycle. (The full process involves $((n-1)!/2m) - 1$ "mergers" in all, and produces a touch of $n!$ changes—that is, an extent.)

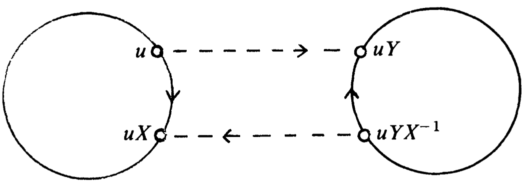

FIG. 4.4.

We remark that the construction of Theorem 4.11 is due to P. A. B. Saddleton (personal communication), and is reminiscent of one used by Rapaport [13] which, in conjunction with Theorem 2.1, shows that an extent on three transition rules exists, for *every n*.

Next, consider the implications of Theorem 4.1, as pertaining to Theorem 4.9. Now $k = (n-1)!/(n-1) = (n-2)!$ is even for $n \geqslant 4$, and so again we need cd to have even order. Again the standard single $d = (56)(78)\cdots$ (for n even) leaves hope, as does the more exotic call $d = (45)(67)(89)\cdots$ for n even only: $cd =$

$(468 \cdots 753)$, an $(n-2)$-cycle—of even order if and only if n is even. For each of these we need to verify that $\langle X, Y\rangle = S_{n-1}$; for this see Lemmas 4.10 and 4.6, respectively.

Then the construction given for the proof of Theorem 4.11, together with Theorem 4.9, establishes also:

THEOREM 4.12. *Let $\Delta = \{a, b, c, d\}$ be a set of type A generating involutions for S_n, with $b, c, d \in (S_n)_1$ and $\langle a, b\rangle = D_n$ with ab of order n. Set $X = (ab)^{n-1}ac$, $Y = (ab)^{n-1}ad$. Suppose that $\Delta' = \{X, Y)$ generates S_{n-1} and that $cd = dc$. Then there is a true $2n!$ on n bells using transitions from Δ and leads X and Y.*

The Theorem does not apply for $d = (45)(67)(89)\cdots$ (n even), as $c = (34)(56)(78)\cdots$ and d do *not* commute. However, we *do* obtain:

COROLLARY 4.13. *There is a true Plain Bob $2n!$ on n bells using singles only, for n even ($n \geqslant 6$).*

Proof. Take $a = (12)(34)(56)\cdots$, $b = (23)(45)(67)\cdots$, $c = (34)(56)(78)\cdots$, and $d = (56)(78)\cdots$. Let $X = (ab)^{n-1}ac$ and $Y = (ab)^{n-1}ad$. Then X and Y generate S_{n-1}, by Lemma 4.10. Note that $cd = dc$, and apply Theorem 4.12.

B. *Grandsire*. We attempt to analyze the method of Grandsire, on $n \geqslant 5$ bells (here, n is always odd; in practice, Grandsire is occasionally rung for n even as well). In Grandsire there are two hunt bells, 1 and 2. The other bells should all work alike, so the plain course should be described by $P^{n-2} = e$. Again we decompose into left cosets of D_n (each lead consists of $2n$ changes). The situation is complicated for Grandsire, however, because the first coset begins not with rounds, but with the first change after rounds. The coset D_n then occurs at the *end* of the extent rather than the beginning. This occurs because the first transition (generating involution) Grandsire uses is $f = (12)(45)(67)\cdots$; this is followed by the word $(ba)^{n-1}b$—same a, b as for Plain Bob, but now with $w^* = (ba)^{n-1}b$ instead of $(ab)^{n-1}a$. Nevertheless, a and b still generate D_n.

To be consistent with our analysis of Plain Bob, we take the first change of each coset as our coset representative (and vertex label): these are changes $2, 2n+2, 4n+2, \ldots$. This has the effect—due to the action of f—of stabilizing 1 in the *second* position.

Let us begin with the case $n = 5$: Grandsire Doubles. We have $a = (12)(34)$, $b = (23)(45)$, and $f = (12)(45)$; the plain lead is $P = (ba)^4bf = (345)$, and the bob lead is $B = (ba)^3(bf)^2 = (15)(34)$. Both P and B are in A_4 (on the symbols $1, 3, 4, 5$). We are regarding A_4 as a subgroup of $S_4 \cong (S_5)_2$: bell 1 is fixed in second placç. Conjugating P by B gives (431); this and P^{-1} generate A_4 (on $1, 3, 4, 5$), by Proposition 4.2. The Cayley color graph $C_\Delta(A_4)$, $\Delta = \{P, B\}$, imbeds on the sphere as a truncated tetrahedron. Its twelve vertices represent six "cosets" of D_5 in S_5, twice each. A second isomorphic Cayley color graph (and homeomorphic imbedding) represents the other six "cosets" of D_5 in S_5, again twice each. (We abuse terminology here, as the bob leads fail to be cosets in their final two changes.) The

two graphical components together form a disconnected Cayley color graph for S_4 on $\{1, 3, 4, 5\}$: $2C_\Delta(A_4) = C_\Delta(S_4)$. Each "coset" in the second component is contained in A_5 (its vertex is an element of $\overline{A_4}$); each "coset" in the first component is contained in $\overline{A_5}$ (its vertex is an element of A_4). Thus the 24 vertices in the two components taken together cover each element of S_5 twice. (For example, the vertex labelled with change 21345 is in the first component, and represents the identity in $A_4 \leqslant S_4 \cong (S_5)_2$ as well as the coset containing $(12) \in \overline{A_5}$; the same coset is also represented by the vertex labelled 41523—also in the first component.)

Clearly P and B do not suffice to ring Grandsire Doubles, as everything in sight —a, b, f, P, and B—is in A_5. So, a single $g = (45)$ is introduced, leading to the single lead $S = (ba)^3bfgf = (1435)$. Since $S \in \overline{A_4}$, $\langle P, B, S\rangle = S_4$. The Cayley color graph $C_{\Delta'}(S_4)$, $\Delta' = \{P, B, S\}$, is connected, and can be obtained by adding the proper 24 directed edges (colored with S) between the two components of $C_\Delta(S_4)$. It is then a simple matter to find a closed cycle—such as $((BP)^2SP)^2$—which rings Grandsire Doubles very nicely. (We need to check that the final two changes in each bob and single lead are true—but this is indeed the situation. All other changes are true, by the "coset" decomposition.) We note, however, that this is not a hamiltonian cycle in $C_\Delta(S_4)$, which still represents each "coset" twice. Unfortunately, identifying vertices representing the same coset does *not* give a covering projection (neither for surfaces nor for graphs), as it did for Plain Bob; so there does not seem to be a related configuration in which a *hamiltonian* cycle gives us what we want.

It is worth noting that the method of Plain Bob—the standard beginning point for all ringers, due to its beauty and simplicity—is also (by far!) the best behaved mathematically.

Nevertheless, we can generalize Grandsire Doubles to $n \geqslant 5$ and odd as follows: let $a = (12)(34)(56)\cdots$, $b = (23)(45)(67)\cdots$, and $f = (12)(45)(67)\cdots$. Form $P = (ba)^{n-1}(bf) = (468\cdots9753)$ and $B = (ba)^{n-2}(bf)^2 = (6, 10, 14, \ldots, 9, 5, 1)\cdot(4, 8, 12, \ldots, 11, 7, 3)$. Note that each bob lead fails to be a coset of D_n in S_n at its last two changes (since $B = (ba)^{n-2}(bf)^2$, instead of $(ba)^{n-1}bf$). Thus *if* we could find a hamiltonian cycle in $C_{\{P, B\}}(A_{n-1})$, we would still need to check the final two changes of the bob leads for trueness. However, as we shall see, there is no such cycle. Thus the deviation from a perfect coset decomposition does not affect our argument.

LEMMA 4.14. $\langle P, B\rangle = A_{n-1} \leqslant S_{n-1} \cong (S_n)_2$.

Proof. P and B are both in A_{n-1}, as they both stabilize 1 in the second position and are both even (a, b, and f all involve the same number—$(n-1)/2$—of transpositions, and both words P and B have length $2n$ in a, b, and f.) Thus $\langle P, B\rangle \leqslant A_{n-1}$. To demonstrate the reverse containment, we show that $(13x) \in \langle P, B\rangle$, $4 \leqslant x \leqslant n$, and apply Proposition 4.2. First, we calculate that $P^{-2}B = (135)$. Then we conjugate by powers of P to see that $(1, 3, k)$ and $(1, 3, k+1)$ are both in $\langle P, B\rangle$ if either is, $4 \leqslant k \leqslant n-1$, k even. Next, we conjugate P by (135); call the result y. Clearly $y \in \langle P, B\rangle$. Finally, we conjugate by powers of y to see that for

(135), (137) and for $(1,3,h)$, $(1,3,h+5)$ — $4 \leqslant h \leqslant n-5$, h even—both are in $\langle P, B\rangle$ if either is. But, (135) *is* in $\langle P, B\rangle$.

So, we form the Cayley color graph $C_\Delta(A_{n-1})$, $\Delta = \{P, B\}$. Each vertex represents a left "coset" of D_n in S_n, but how many times is each coset represented? We note that the two changes in each "coset" that have 1 in the second position (these are the two changes which label vertices) are connected by the word $z = (ba)^{n-2}b$ —as can be readily verified by following the plain hunt of the treble. Recall that a and b each involve $(n-1)/2$ transpositions. Thus $z \in A_n$ if and only if $n \equiv 1 \pmod 4$. In these cases, as for Grandsire Doubles, each component of $C_\Delta(S_{n-1})$ (isomorphic to $C_\Delta(A_{n-1})$) contains half the "cosets" of D_n in S_n, twice each. We see instantly that a single is required to ring the extent—as we must have access to the other component.

On the other hand, if $n \equiv 3 \pmod 4$, then $z \in \overline{A_n}$ and each component of $C_\Delta(S_{n-1})$ (isomorphic to $C_\Delta(A_{n-1})$) contains each left "coset" of D_n in S_n exactly once. Hence, even though $\langle P, B\rangle = A_{n-1}$ and not S_{n-1}, a hamiltonian cycle in $C_\Delta(A_{n-1})$ would ring Grandsire on n bells, with plain and bob leads only, in a manner similar to that of Theorem 4.8—providing, as stated earlier, that the final two changes of the bob leads proved true.

To apply Theorem 4.1 to these cases, we compute $P^{-1}B = (468 \cdots 9751)$, an $(n-2)$-cycle, so that m is odd. But $k = (n-1)!/(2(n-2)) = (n-1)(n-3)!/2$ and $l = ((n-1)!/2)/((n-1)/2) = (n-2)!$ are both even, for $n \geqslant 7$. Thus no hamiltonian cycle exists in $C_\Delta(A_{n-1})$, and again a single is required for the extent. (Note that this includes Thompson's case, $n = 7$.) We have proved:

THEOREM 4.15. *There is no extent of Grandsire on n (odd) bells using plain and bob leads only, for $n \geqslant 5$.*

Finally, we note that the spherical voltage graph of Figure 4.5 imbeds $C_\Delta(A_{n-1})$ on a closed orientable 2-manifold of genus $1 + (n-3)!(n^2 - 7n + 8)/4$. Curiously, this is the same genus as calculated for the Plain Bob Cayley color graph surface for S_{n-1} in part A of this section. We also note that $n = 7$ gives genus 49. Thus it is not surprising that Thompson's diagram was so complicated.

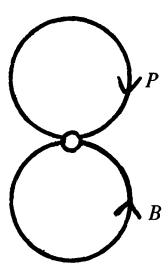

FIG. 4.5.

We have analyzed the two most popular methods—Plain Bob and Grandsire—as deeply as seems feasible, using the techniques of topological and algebraic graph theory. Now we turn our attention to the most popular principle—Stedman—and its close relative, Erin. For the sake of brevity, we omit most details in the following two subsections.

C. *Stedman.* Stedman on n bells (n odd) is a principle; that is, all n bells are working. The plain course is thus described by $P^n = e$. Each plain lead involves *twelve* changes, for all odd n. These twelve consist of *two* left cosets of D_3 in S_n.

We take $a = (12)(34)(56)\cdots$, $b = (23)(45)(67)\cdots$, and $f = (12)(45)(67)\cdots$ just as for Grandsire; also, let $d = (12)(34)(56)\cdots(n-4, n-3)$ and $g = (12)(34)(56)\cdots(n-4, n-3)(n-1, n)$—making $(n-2)$nd place. The plain lead is, and the bob and single leads could be, given by: $P = (fb)^2fa(bf)^2ba$, $B = (fb)^2fg(bf)^2ba$, and $S = (fb)^2fd(bf)^2ba$. It is clear that b and f generate a subgroup of S_n isomorphic to D_3. The involutions a, g, and d are responsible for linking the left cosets of this group together in pairs, in their respective leads.

It is routine to show that $\langle P, S\rangle = S_n$, $n \geqslant 5$. Note that there is no efficiency in this; we still have $n!$ vertices in any corresponding graphical configuration. The problem is that nothing is stabilized in $\langle P, S\rangle$. Moreover, $m = |\langle P^{-1}S\rangle| = 2$, so that Theorem 4.1 does not apply in any event.

I suspect that $\langle P, B\rangle = A_n$, $n \geqslant 7$, but have troubled to show this only for $n = 7, 9, 11$. Here there is some efficiency, but precious little. For $n = 9$, everything in sight (a, b, f, g, P, and B) is in A_n, so of course a single is required for the extent. For both $n = 7$ and 11, $m = |\langle P^{-1}B\rangle| = 3$ and Theorem 4.1 applies (to the isomorphic copy of $\underline{C_\Delta}(A_n)$ in $C_\Delta(S_n)$ whose vertices are labelled from $\overline{A_n}$, since the lead heads are in $\underline{A_n}$) to show that there is no "true $6n!$" (each change *six* times, with each element of $\overline{A_n}$ serving as lead head exactly once) possible, with plain and bob leads alone.

But even these analyses are inconclusive, as there are other options frequently employed for B: $(fb)^2fa(bf)^2bg$, or even $(fb)^2gf(bf)^2bg$; and for S: $(fb)^2fa(bf)^2bd$, or even $(fb)^2fd(bf)^2bd$; and even combinations such as $(fb)^2fg(bf)^2bd$.

D. *Erin.* Erin on n bells (n odd) is also a principle, so that $P^n = e$ once again. For each odd n, each plain lead involves *six* changes and is *one* left coset of D_3 in S_n.

Again, $a = (12)(34)(56)\cdots$, $b = (23)(45)(67)\cdots$, and $f = (12)(45)(67)\cdots$; now we set $d = (12)(34)(67)(89)\cdots$ (making 5th place only). Take $P = (fb)^2fa$ and $B = (fb)^2fd$. Again, $\langle f, b\rangle \cong D_3$. Since both words P and B have length six in the letters f, b, a, and d, each lead—unlike Stedman—consists of just one coset. It is routine to show that $\langle P, B\rangle = A_n$, for $n \geqslant 7$. We calculate $m = |\langle P^{-1}B\rangle| = n - 4$, which is odd. Thus Theorem 4.1 applies (k and l are easily seen to be even), for $n \equiv 3 \pmod{\underline{4}}$, to show that there is no "true $3n!$ (each change *three* times, with each element of $\overline{A_n}$ serving as lead head exactly once) possible, with plain and bob leads alone. Again we are examining the copy of $C_\Delta(A_n)$ in $C_\Delta(S_n)$ whose vertices

are labelled from $\overline{A_n}$, since—as for Stedman—the lead heads are all in $\overline{A_n}$. M. E. Ovenden (personal communication) observes that such a $3n!$ is equivalent to an extent of Original Triples; thus no such is possible, using bob d only.

For $n \equiv 1 \pmod 4$, everything is in A_n, so a single is required, even for one extent.

5. Right cosets

For Plain Bob Minimus (a method), we saw that each row of Table 1 corresponds to a right coset of Z_3 in S_4. Similarly, for White's "No Call" Doubles (a principle), each row of Table 4 of **[19]** corresponds to a right coset of Z_5 in S_5. These are instances of the following general situation. Let an extent of n bells (method or principle) have m working bells, so that $P^m = e$ describes the plain course. Arrange the changes into m columns, so that each column contains one lead (or division). Then each row of changes corresponds to a right coset of Z_m in S_n. If the extent is a "no call" extent (the plain course *is* the full extent), then the rows in fact give a full coset decomposition of S_n. (This requires the word P have length $l = n!/m$ in the generating involutions for the extent.) This is the case for both Plain Bob Minimus and White's "No Call" Doubles.

Plain Bob Minimus, and Table 1, illustrate another decomposition into right cosets which is vital to plain bob in general, as well as to other extents. Form the subgroup S_3 of S_4 consisting of all the permutations stabilizing 1—that is, the treble leads. These are precisely the lead heads and lead ends, and thus correspond to rows 1 and 8 of Table 1. The other right cosets i of S_3 in S_4 correspond to rows $\pm i$, $i = 2, 3, 4 \pmod 9$. Thus each half lead is a transversal of S_3 in S_4: it intersects each of the four cosets in a singleton.

Right transversals of PSL(2, 5) in S_7 are commonly used to compose extents of Stedman Triples and Erin Triples. P. A. B. Saddleton (personal communication) comments that William Hudson was the first to use this group (of order 60) in this way: As in Section 4C, take $a = (12)(34)(56)$, $b = (23)(45)(67)$, $f = (12)(45)(67)$, and $g = (12)(34)(67)$; add $h = (12)$. Set $P = fba(fb)^2fabfb$ (conjugate to the P of Section 4C), $B = fbg(fb)^2fgbfb$ (two bobs), and $D = fbh(fb)^2fabfb$. Then the courses $X = PB^2P^4 = (16235)$ and $Y = DB^2P^4 = (154)(236)$ generate PSL(2, 5) and are both transversals in S_7. Since $YX^{-1} = XY^{-1}$, the construction of Theorem 4.11 applies to guarantee an extent of Stedman Triples (such as $(XY^2X^2Y(XY^2X^3)^4)^2$).

We also outline an elegant construction of an extent of 'true' triples (just one place made at each transition) due to Philip Saddleton. Take $x = (23)(45)(67)$, $y = (12)(45)(67)$, and $z = (12)(34)(56)$—making places 1, 3, 7, respectively—to generate S_7. Form the Schreier coset graph of S_7 mod PSL(3, 2) (30 vertices, each representing a right coset of order 168). The two hamiltonian cycles $P = yxz(xyxyxz)^4xyz$ and $B = yzx(yxyxyz)^2(xyxyxz)^2xyz$ in this configuration give the plain and bob leads respectively. Moreover, $\Delta = \{P, B\}$ generates PSL(3, 2), and $BP^{-1} = PB^{-1}$. Thus the construction of Theorem 4.11 applies once again, to find a

hamiltonian cycle in $C_\Delta(PSL(3,2))$, such as:

$$\left(\left((B^2P^5)^2((PB)^2P^3)^2\right)^2(B^2P^5)^2(PB)^2P^6BP^3\right)^2.$$

This is the extent. It is interesting to contrast this right-coset construction with the left-coset constructions of the preceeding section. Set $\Delta = \{P, B\}$ in both contexts, for uniformity. In each case we find a hamiltonian cycle in $C_\Delta(\Gamma)$, where Γ is a useful subgroup of S_n. But in Section 4 we have one Cayley color graph $C_\Delta(\Gamma)$, each vertex of which represents a left coset of D_n, whereas in the Saddleton construction we have one Schreier (right) coset graph (for Γ in S_n), each vertex of which represents a $C_\Delta(\Gamma)$. Thus, although the Saddleton extent rings 168 leads of 30 changes each, it is found as $[S_7\colon \mathrm{PSL}(3,2)] = 30$ cosets of 168 changes each. In contrast, the Plain Bob Major extent found by Theorem 4.11 (for example) is rung as 2520 leads of 16 changes, each *and* is found as $|A_7| = [S_8 : D_8] = 2520$ cosets of $|D_8| = 16$ changes each.

Yet another right-coset decomposition of S_4 for Plain Bob Minimus is apparent from the 4-fold rotational symmetry of Figure 2 of [**19**]. If, for example, we label alternate vertices of the inner octagon (consistent with the edge-coloring) by $\{e, (1243), (14)(23), (1342)\} \cong Z_4$, then this determines a decomposition of S_4 into right cosets of Z_4 in an obvious manner. (Each vertex is grouped with its three images under rotation by 90, 180, and 270 degrees.) The graphical structure obtained by identifying vertices in each coset *is* a Schreier coset graph for Z_4 in S_4, as we are now using right multiplication on *right* cosets. This quotient structure does not appear to be of any practical benefit in a ringing context. But if we apply the same idea to White's "No Call" Doubles, we do get new information.

Figure 7 of [**19**] represents an imbedding of $C_\Delta(S_5)$ into N_{10}, and its 5-fold symmetry was exploited to find a hamiltonian cycle producing White's "No Call" Doubles. We now use this symmetry in a more systematic way. Group each vertex with its four images under rotation by 72, 144, 216, and 288 degrees. This gives a decomposition of S_5 into right cosets of Z_5. The quotient graph is a Schreier coset graph and its quotient imbedding into N_2 (the Klein bottle) is depicted in Figure 5.1 below. The covering projection is 5-fold, with branching (two of the twelve 10-gons above wrap five times each around a respective digon below).

Recall from Section 2 that White's "No Call" Doubles is given by $w^5 = e$, $w = (ag)^3(ab)^3ag(ab)^2(ag)^2ab$. (Here I use g instead of c, to be consistent with [**12**].) The word w corresponds to the hamiltonian circuit in Figure 5.1 commencing at the vertex designated "*". By construction, it is clear that this coset decomposition coincides with that given by the rows of Table 4 of [**19**].

We seek other doubles principles latent in Figure 5.1. We continue to desire condition (iv)—conditons (i), (ii), (iii) are guaranteed by the method of construction, as is condition (v); condition (vi) we not only do not insist on, but are actually happy to fail, for the reason given in Section 2. Thus generator *a must* alternate throughout—as it is the only one affecting the bell in position one. There are exactly twelve hamiltonian cycles in Figure 5.1 with solid edges alternating and,

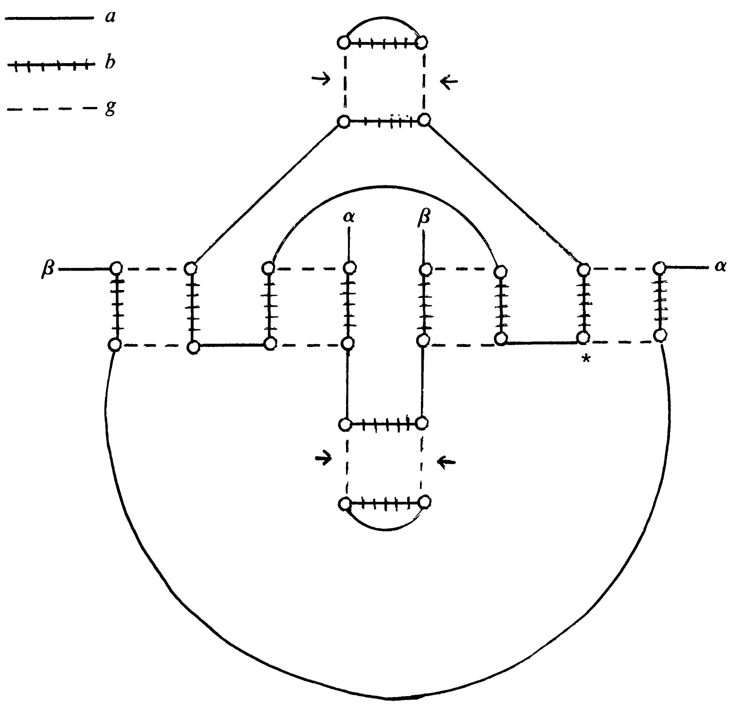

FIG. 5.1.

without loss of generality, starting at vertex * with the solid edge. (In verifying this, it is helpful to note that each of the four dashed edges designated by an arrow *must* also be employed. We should also comment that the Schreier coset graph of Figure 5.1 is *not* vertex transitive, so that a fixed word giving a hamiltonian cycle at one vertex may not do so at another.) These twelve fall into four equivalence classes under rotation (conjugation); for example, $(ab)^2ag(ab)^2(ag)^2ab(ag)^3ab$ and $(ag)^3(ab)^3ag(ab)^2(ag)^2ab$ are in the same class. The classes are represented by:

I. $(ag)^3(ab)^3ag(ab)^2(ag)^2ab = (14253)$

II. $(ab)^3(ag)^3ab(ag)^2(ab)^2ag = (12345)$

III. $(ag)^3(ab)^2agab(ag)^4ab = (12345)$

IV. $[(ag)^5ab]^2 = e$.

I is, of course, White's "No Call" Doubles. II is, after suitable conjugation, the backwards version of I, as given in Section 2. IV brings up rounds after a touch of 24 changes. (Thus a hamiltonian cycle in the quotient structure is *not* sufficient for one in $C_\Delta(S_5)$; it is necessary that the word it represents in S have order five.)

What is new from all this is III: this *does* lift to a hamiltonian cycle in $C_\Delta(S_5)$ and thus represents another asymmetric doubles principle. Thus it has three companion extents, given in reverse and/or backwards. We note that the reverse versions are not described explicitly by Figure 5.1, since the generators differ. The backwards

version *is* described within Figure 5.1, by reversing the orientation of any of the four cycles (of the twelve computed) giving III.

Both I and III are an example of an extent of Original Doubles. This means that the changes for five bells (doubles) can be regarded as calls for Plain Bob Minor (on six bells), and conversely. This is also the case $n = 5$ of Original on n bells: plain hunt on n bells is used to generate a principle on n bells. Specifically, recall that, for Plain Bob on n bells, $a = (12)(34)(56)\cdots$ and $b = (23)(34)(56)\cdots$ generate the *hunting group* D_n, and $c = (34)(56)(78)\cdots$ is used to give the plain lead $P = (ab)^{n-1}ac$. Recall from Section 4A that, for n even ($n \geqslant 6$), extents of Plain Bob using plain and single leads only are *not* ruled out by Theorem 4.1—and, by Theorem 4.5 (2), single leads are *required* for extents by Plain Bob on such n. Such an extent would be given by a calling $[f(P, S)]^{n-1}$, where $f(P, S)$ is a word in P and S of length $(n-2)!/2$—since P and S are each composed of $2n$ changes.

THEOREM 5.1. *For each calling* $[f(P, S)]^{n-1}$ *of Plain Bob on n bells (n even), there is a principle on* $n - 1$ *bells with* $b' = (12)(34)(56)\cdots$ *alternating with* $c' = (23)(45)(67)\cdots$ *and* $s' = (45)(67)(89)\cdots$; *and conversely.*

Proof. We note that $w^* = (ab)^{n-1}a = b$ (since $(ab)^n = e$: ab is a rotation in D_n); hence w^* takes us through each lead (whether plain or single), from one treble lead—at the lead head, to the next treble lead—at the lead end. Either c or s then takes us from the lead end of one lead to the lead head of the next. We suppress the treble, in position one, (since it has been stabilized at the two extremes of each lead) and allow b, c, and s to act on the $n - 1$ remaining positions. We renumber these positions, decreasing each number by one, and this induces $b' = (12)(34)(56)\cdots$, $c' = (23)(45)(67)\cdots$, and $s' = (45)(67)(89)\cdots$. Then $P = w^*c = bc$ becomes $b'c'$, and $S = w^*s = bs$ becomes $b's'$. Since $[f(P, S)]^{n-1}$ rings all the changes, and hence all the left cosets of D_n, for Plain Bob (n), $[f(b'c', b's')]^{n-1}$ rings all the changes for an extent on $n - 1$ bells (that is, all the elements of $(S_n)_1$—the two representatives of each coset). Since Plain Bob is a method with one hunt bell, the other $n - 1$ bells all work alike; thus the extent on $n - 1$ *is* a principle. (This is similar to the situation studied in Section 4A, except that b rings $2n$ changes in the method, but b' only rings one in the principle.) Recalling that $f(P, S)$ has length $(n-2)!/2$, we check that $[f(b'c', b's')]^{n-1}$ has length $((n-1)(n-2)!/2)2 \doteq (n-1)!$ It is clear that the argument reverses.

Thus, from White's "No Call" Doubles, setting a, b, and g of I equal to b', c', and s', respectively, we find $[S^3P^3SP^2S^2P]^5$ as a calling for Plain Bob Minor. (This also appears in [**5**]; see also [**9**].) Thus the hamiltonian cycle I (of length 24), when interpreted fully, rings all 720 changes of Plain Bob Minor!

Conversely, we find in [**5**] (and [**9**]) another calling $[S^3P^2SPS^4P]^5$ for Plain Bob Minor. This yields w^5 as another Original Doubles principle, where $w = (ag)^3(ab)^2agab(ag)^4ab = (12345)$. But this is just III above!

Finally, by commencing with the calling $[P^3B]^3$ for Plain Bob Doubles—as found in Section 4A from Figure 4.2—and replacing the role of single lead S for Plain Bob Minor above with the bob lead B for Plain Bob Doubles, we find an

extent of Original Minimus to be in the form $[(ab)^3ac]^3$; that is, this Original Minimus is just Plain Bob Minimus, as in Section 3. This rings us round to our starting point for coset ringing, and so we end.

Acknowledgements. I am indebted to the many ringers who enthusiastically shared their knowledge, their skill, and their artistry with me. In particular, I thank Mike Ovenden and John Pusey for their advice and their efforts in organizing performances of White's "No Call" Doubles and Reverse White's "No Call" Doubles, and for their patient and thorough correspondence. I thank Len Porter, Chris Robson, and Anthony P. Smith for their advice, support, and encouragement. I thank David Struckett for making reference [**17**] available to me. I thank Philip Saddleton for attracting my attention to right cosets, Theorem 4.11 and its proof, and the construction leading to the original doubles principle III of Section 5. I thank T. Jefferson Smith, who first suggested that I expand my study from changes to cosets. And, I thank Steve Anderson for introducing me to the mathematics of change ringing.

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The William Lowell Putnam Mathematical Competition

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The following results of the forty-seventh William Lowell Putnam Mathematical Competition, held on December 6, 1986, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Douglas S. Jungreis, Bjorn M. Poonen, and David J. Zuckerman; each was awarded a prize of two hundred fifty dollars.

The second prize, $2,500, was awarded to the Department of Mathematics of Washington University, St. Louis. The members of its team were: Daniel N. Ropp, Dougin A. Walker, and Japheth L. M. Wood; each was awarded a prize of $200.

The third prize, $1,500, was awarded to the Department of Mathematics of the University of California, Berkeley. The members of its team were: Michael J. McGrath, David P. Moulton, and Christopher S. Welty; each was awarded a prize of $150.

The fourth prize, $1,000, was awarded to the Department of Mathematics of Yale University. The members of its team were: Thomas O. Andrews, Kamal F. Khuri-Makdisi, and David R. Steinsaltz; each was awarded a prize of $100.

The fifth prize, $500, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology. The members of its team were: David Blackston, James P. Ferry, and Waldemar P. Horwat; each was awarded a prize of fifty dollars.

The six highest-ranking individual contestants, in alphabetical order, were David J. Grabiner, Princeton University; Waldemar P. Horwat, Massachusetts Institute of Technology; Douglas S. Jungreis, Harvard University; David J. Moews, Harvard University; Bjorn M. Poonen, Harvard University; and David J. Zuckerman, Harvard University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $500 by the Putnam Prize Fund.

The next four highest-ranking individuals, in alphabetical order, were Yong Yao Du, University of Waterloo; Gregory J. Kuperberg, Harvard University; John A. Overdeck, Stanford University; and Michael Reid, Harvard University.

The following teams, named in alphabetical order, received honorable mention: University of British Columbia, with team members Wayne J. Broughton, Marek J. Radzikowski, and Russil Wvong; California Institute of Technology, with team members Leland Brown, Stanley Chen, and Darien G. Lefkowitz; Princeton University, with team members Douglas Davidson, David J. Grabiner, and Randall G. Rose; Rice University, with team members Charles R. Ferenbaugh, Thomas Hyer, and John M. Steinke; and the University of Waterloo, with team members Yong Yao Du, Eric Veech, and Minh Tue Vo.

Honorable mention was achieved by the following thirty-seven individuals named in alphabetical order: Thomas O. Andrews, Yale University; John B. Boyland, University of California, Davis; William Chen, Auburn University; Constantine N. Costes, Harvard University; William P. Cross, California Institute of Technology; Henri R. Darmon, McGill University; Michael B. Davis, Case Western Reserve University; Samuel S. Dooley, Texas A. & M. University; Glenn D. Ellison, Harvard University; Bryan K. Feir, University of Waterloo; Charles R. Ferenbaugh, Rice University; James P. Ferry, Massachusetts Institute of Technology; Patrick T. Headley, Case Western Reserve University; Earl A. Hubbell, California Institute of Technology; William J. Jockusch, Carleton College; Daniel W. Johnson, Rose-Hulman Institute of Technology; Kamal F. Khuri-Makdisi, Yale University; Darien G. Lefkowitz, California Institute of Technology; Michael J. McGrath, University of California, Berkeley; David P. Moulton, University of California, Berkeley; Matthew D. Mullin, Princeton University; Du Nguyen, University of Ottawa; Ravi K. Ramakrishna, Cornell University; Daniel N. Ropp, Washington University, St. Louis; Randall G. Rose, Princeton University; David B. Secrest, University of Illinois, Champaign-Urbana; Kenneth W. Shirriff, University of Waterloo; Stephen A. Smith, University of Waterloo; Theron W. Stanford, California Institute of Technology; John M. Steinke, Rice University; David R. Steinsaltz, Yale University; Constantin T. Teleman, Indiana University, Bloomington; Dougin A. Walker, Washington University, St. Louis; Christopher S. Welty, University of California, Berkeley; Karl M. Westerberg, Carnegie-Mellon University; Japheth L. M. Wood, Washington University, St. Louis; and James C. Yeh, Princeton University.

The other individuals who achieved ranks among the top 96, in alphabetical order of their schools, were: Auburn University, Darren C. Abbott; University of British Columbia, Wayne J. Broughton, Marek J. Radzikowski, Russil Wvong; California Institute of Technology, Leland F. Brown, Stanley Chen, Thomas J. Lenosky; California Polytechnic State University, San Luis Obispo, Daniel L. Krejsa; California State University, Fresno, Chiu Liu; University of California, Berkeley, John Stanley Tillinghast; Carleton University, Serge Elnitsky, Lones A. Smith; Carnegie-Mellon University, Petros I. Hadjicostas, Joseph G. Keane; University of Chicago, Robert M. Beals; Columbia University, Martin J. Strauss, Ali F. Yegulalp; Concordia University, Chinh Mai; University of Connecticut, Anthony J.

Zajac; Harding University, Scott T. Burleson; Harvard University, Chenteh Kenneth Fan, Glen T. Whitney; Massachusetts Institute of Technology, David T. Blackston, Larry Buxbaum, Jordan A. Drachman, Mark Kantrowitz; University of Massachusetts, Amherst, James F. Riordan; University of Missouri, Rolla, David A. Betz; Université de Montréal, François Bedard; University of New Brunswick, Iain G. DeMille; Ohio State University, Thomas E. Barrett; Princeton University, Joseph J. Bohman, Rahul V. Pandharipande; Queen's University, Neale Ginsburg, Krishna Rajagopal; Rice University, Thomas M. Hyer; Stanford University, Thomas H. Chung, Joshua R. Zucker; Texas A. & M. University, Mark G. Yarbrough; University of Texas, El Paso, Luis Gerardo Valdes Sanchez; University of Toronto, Edward J. Doolittle, Jeffrey S. Rosenthal; University of Washington, Seattle, Dean M. Yasuda; University of Waterloo, Christopher K. Anand, Frank M. D'Ippolito, Peter J. Fowler, John S. Omielan, Minh Tue Vo; University of Wisconsin, Madison, Robert C. Mattson.

There were 2094 individual contestants from 358 colleges and universities in Canada and the United States in the competition of December 6, 1986. Teams were entered by 270 institutions.

The Questions Committee for the forty-seventh competition consisted of Richard P. Stanley (Chairman), Abraham P. Hillman, and Harold M. Stark; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leqslant 13x^2$.

Problem A-2. What is the units (i.e., rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leqslant x$.

Problem A-3. Evaluate $\sum_{n=0}^{\infty}\operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geqslant 0$ denotes the number θ in the interval $0 < \theta \leqslant \pi/2$ with $\cot\theta = t$.

Problem A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

Problem A-5. Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on R^n. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Prove that there is a function $g(x)$ on R^n such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leqslant i \leqslant n$. (A linear function is one of the form

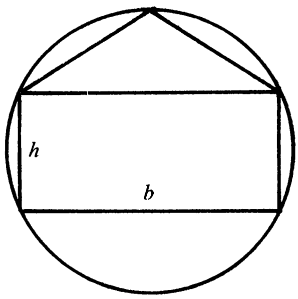
$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

Problem A-6. Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

Problem B-1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.
For what value of h do the rectangle and triangle have the same area?



Problem B-2. Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$ where x, y, and z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

Problem B-3. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r, and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.

Problem B-4. For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}\,|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r \to \infty} G(r)$ exists and equals 0.

Problem B-5. Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, $r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x$, $\pm y$, $\pm z$, where the number of minus signs is 0 or 2.

Problem B-6. Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqq i \geqq 0$ is the number of students among the top 201 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (152, 23, 10, 7, 0, 0, 0, 2, 2, 3, 1, 1)
The condition that $x^4 + 36 \leqslant 13x^2$ is equivalent to

$$(x - 3)(x - 2)(x + 2)(x + 3) \leqslant 0.$$

The latter is satisfied if and only if x is in the closed interval $[-3, -2]$ or the closed interval $[2, 3]$. The function f is increasing on these intervals because for such x, $f'(x) = 3(x^2 - 1) > 0$. It follows that the maximum value of f over this domain is $\max\{f(-2), f(3)\} = 18$.

A-2 (155, 0, 0, 0, 0, 0, 0, 0, 0, 0, 33, 13)
The greatest integer is

$$I = \frac{10^{20000} - 3^{200}}{10^{100} + 3}$$

since the remainder is

$$\frac{3^{200}}{10^{100} + 3} < 1.$$

We have

$$I \equiv \frac{-3^{200}}{3} \pmod{10} \equiv -3^{199} \pmod{10} \equiv -3^3(3^4)^{49} \pmod{10} \equiv -27 \pmod{10}$$
$$\equiv 3 \pmod{10}.$$

The last digit is 3.

A-3 (53, 6, 15, 1, 0, 0, 0, 1, 12, 1, 26, 86)
Using

$$\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha},$$

one sees that $\operatorname{Arccot}(1 + n + n^2) = \operatorname{Arccot} n - \operatorname{Arccot}(n+1)$. Then the series telescopes to $\lim_{n\to\infty}(\operatorname{Arccot} 0 - \operatorname{Arccot}(n+1)) = \pi/2$.

A-4 (21, 3, 4, 4, 5, 7, 0, 6, 3, 7, 23, 118)
We first prove:

LEMMA. *If an $n \times n$ matrix (α_{ij}) satisfies (b), then there are unique numbers $c_1 = 0, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n$ such that $\alpha_{ij} = c_i + d_j$. Conversely, any such choice of c_i's and d_j's yields a unique matrix (α_{ij}) satisfying (b).*

Proof. If $\alpha_{ij} = c_i + d_j$ then any transversal of A sums to

$$\sum_{i=1}^{n} c_i + \sum_{j=1}^{n} d_j,$$

so (b) is satisfied.

Conversely, suppose (α_{ij}) satisfies (b). Define $d_j = \alpha_{1j}$ and $c_i = \alpha_{i1} - d_1 = \alpha_{i1} - \alpha_{1j}$ (so $c_1 = 0$). Since (b) is satisfied, we have $\alpha_{ij} + \alpha_{11} = \alpha_{i1} + \alpha_{1j}$, so $\alpha_{ij} = \alpha_{i1} + \alpha_{1j} - \alpha_{11} = (c_i + d_1) + d_j - d_1 = c_i + d_j$, as desired.

The c_i's and d_j's are unique since $c_1 = 0$ and $\alpha_{1j} = c_1 + d_j$ forces $d_j = \alpha_{1j}$, and then $\alpha_{i1} = c_i + d_1$ forces $c_i = \alpha_{i1} - d_1$. This proves the lemma.

Thus $f(n)$ is equal to the number of $2n$-tuples $(c_1 = 0, c_2, \ldots, c_n, d_1, \ldots, d_n)$ for which $c_i + d_j = 0, \pm 1$. We break the possibilities into the following eight cases.

distinct c_i's	possible d_i's	number of c_i's	number of d_i's	product
0	$0, -1, 1$	1	3^n	3^n
$0, -2$	1	$2^{n-1} - 1$	1	$2^{n-1} - 1$
$0, -2, -1$	1	$3^{n-1} - 2^n + 1$	1	$3^{n-1} - 2^n + 1$
$0, 2$	-1	$2^{n-1} - 1$	1	$2^{n-1} - 1$
$0, 1, 2$	-1	$3^{n-1} - 2^n + 1$	1	$3^{n-1} - 2^n + 1$
$0, 1$	$0, -1$	$2^{n-1} - 1$	2^n	$\frac{1}{2}4^n - 2^n$
$0, -1$	$0, 1$	$2^{n-1} - 1$	2^n	$\frac{1}{2}4^n - 2^n$
$0, -1, 1$	0	$3^{n-1} - 2^n + 1$	1	$3^{n-1} - 2^n + 1$

Summing the last column gives

$$f(n) = 4^n + 2 \cdot 3^n - 4 \cdot 2^n + 1.$$

A-5 (13, 4, 0, 0, 0, 0, 0, 1, 0, 2, 39, 142)

Note that $c_{ji} = -c_{ij}$ for all i and j. Let $h_i = \frac{1}{2}\Sigma_j c_{ij} x_j$ so that $\partial h_i / \partial x_j = \frac{1}{2} c_{ij}$. Then

$$\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = \frac{1}{2} c_{ij} - \frac{1}{2} c_{ji} = c_{ij} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$$

and so

$$\frac{\partial (h_i - f_i)}{\partial x_j} = \frac{\partial (h_j - f_j)}{\partial x_i}$$

for all i and j. Hence $(h_1 - f_1, \ldots, h_n - f_n)$ is a gradient and so there is a function g such that $\partial g / \partial x_i = h_i - f_i$. In other words, $f_i + \partial g / \partial x_i = h_i$ is linear.

A-6 (1, 4, 1, 1, 0, 1, 0, 0, 6, 4, 64, 119)

Write $(b)_j = b(b-1)\ldots(b-j+1)$. Differentiating $(1-x)^n f(x) = 1 + \Sigma_{i=1}^n a_i x^{b_i}$ j times $(0 \leqslant j \leqslant n)$ and putting $x = 1$ yields

$$\begin{aligned} 0 &= 1 + \textstyle\sum a_i, \\ 0 &= \textstyle\sum a_i b_i, \\ 0 &= \textstyle\sum a_i (b_i)_2, \\ &\ \ \vdots \\ 0 &= \textstyle\sum a_i (b_i)_{n-1}, \\ n! f(1) &= \textstyle\sum a_i (b_i)_n. \end{aligned}$$

Solve the first n equations for $a_1, \ldots, a_n$ by Cramer's rule and substitute into the last equation. We get

$$n! f(1) = \frac{\displaystyle\sum_{i=1}^{n} (b_i)_n (-1)^i \begin{vmatrix} b_1 & \cdots & \hat{b}_i & \cdots & b_n \\ (b_1)_{n-1} & \cdots & (\hat{b}_i)_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & & b_n \\ & & \vdots & \\ (b_1)_{n-1} & (b_2)_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}$$

where indicates a missing entry. The denominator D is a polynomial of total degree $\binom{n}{2}$ in $b_1, \ldots, b_n$ and vanishes whenever $b_i = b_j$, $i \neq j$. Hence, $D =$

$C\prod_{i<j}(b_i - b_j)$. The constant C is seen to be 1 by considering, say, the coefficient of $b_2 b_3^2 \cdots b_n^{n-1}$, so $D = \prod_{i<j}(b_i - b_j)$. (This can also be proved by performing elementary row operations on the Vandermonde determinant.)

Put $b_i = b_j$ $(i \neq j)$ in the numerator. All terms vanish except two, which are of equal magnitude but opposite sign. Hence the numerator is divisible by $b_i - b_j$ and thus by the denominator. Therefore $n!f(1)$ is a polynomial in $b_1, \dots, b_n$. The degree of this polynomial is $\leqslant n$, since each term in the numerator has degree n more than the degree $\binom{n}{2}$ of the denominator.

Now put $b_i = 0$. The denominator doesn't vanish, but each term in the numerator does. Thus, $n!f(1)$ is divisible by b_i, so $n!f(1) = kb_1b_2 \cdots b_n$ for some constant k. By putting $f(x) = 1$ (so $b_i = i$), we see $k = 1$. Thus $f(1) = b_1 \cdots b_n/n!$.

B-1 (183, 3, 7, 0, 0, 0, 0, 0, 4, 2, 1, 1)

The altitude of the triangle is $\frac{1}{2}(2 - h)$. Equal area means

$$h = \frac{1}{2}(\text{altitude of triangle}) = \frac{1}{4}(2 - h),$$

so $h = 2/5$.

B-2 (123, 31, 16, 3, 0, 0, 0, 0, 16, 5, 2, 5)

The system is equivalent to $0 = (x - y)(x + y - 1 - 2z) = (y - z)(y + z - 1 - 2x) = (z - x)(z + x - 1 - 2y)$. If no two of x, y, z are equal, then $x + y - 1 - 2z = y + z - 1 - 2x = z + x - 1 - 2y = 0$. Adding these gives the contradiction $-3 = 0$. So at least two of x, y, z are equal. If $x = y$ and $y \neq z$, then $z = 2x + 1 - y = x + 1$. In this case we find that $x - y = 0$, $y - z = -1$, and $z - x = 1$. A similar result follows when $y = z$ and $z \neq x$, and when $z = x$ and $x \neq y$. Thus, the only possibilities for $(x - y, y - z, z - x)$ are $(0, 0, 0)$, $(0, -1, 1)$, $(1, 0, -1)$, and $(-1, 1, 0)$. One shows easily that each of these actually occurs.

B-3 (26, 5, 4, 1, 0, 1, 0, 4, 3, 5, 33, 119)

Suppose for $k \geqslant 1$ that we have polynomials F_k and G_k with integer coefficients such that $F_k \equiv f \pmod p$, $G_k \equiv g \pmod p$, and $F_k G_k \equiv h \pmod{p^k}$. For $k = 1$ this can be done with $F_1 = f$, $G_1 = g$. Let $h - F_k G_k = tp^k$ for some $t \in \Gamma$. Let $F_{k+1} = F_k + stp^k$ and $G_{k+1} = G_k + rtp^k$. Then $F_{k+1} \equiv F_k \equiv f \pmod p$ and $G_{k+1} \equiv G_k \equiv g \pmod p$, and

$$\begin{aligned} F_{k+1}G_{k+1} &= F_k G_k + tp^k(rF_k + sG_k) + rst^2p^{2k} \\ &\equiv F_k G_k + tp^k(rF_k + sG_k) \pmod{p^{k+1}}. \end{aligned}$$

By hypothesis, $rF_k + sG_k \equiv rf + sg \equiv 1 \pmod p$, and so $rF_k + sG_k = 1 + qp$ for

some $q \in \Gamma$. Thus

$$\begin{aligned} F_{k+1}G_{k+1} &\equiv F_kG_k + tp^k(1 + qp) \pmod{p^{k+1}} \\ &\equiv F_kG_k + tp^k \pmod{p^{k+1}} \\ &\equiv h \pmod{p^{k+1}}, \end{aligned}$$

and we are done by induction. (The result holds whether or not p is a prime and is independent of the number of variables in the polynomials.)

B-4 (22, 8, 6, 6, 0, 0, 0, 0, 4, 7, 59, 89)

Let m be the largest integer in $N = \{0, 1, \dots\}$ with $r^2 \geqslant m^2$. Let n be the largest integer in N with $(r^2 - m^2)/2 \geqslant n^2$. It follows that $r^2 - m^2 < 2m + 1 \leqslant 2r + 1$ and that

$$\begin{aligned} \frac{r^2 - m^2}{2} - n^2 < 2n + 1 &\leqslant 2\sqrt{\frac{r^2 - m^2}{2}} + 1 \\ &< 2\sqrt{\frac{2r+1}{2}} + 1 = \sqrt{2(2r + 1)} + 1. \end{aligned}$$

Hence $r^2 - m^2 - 2n^2 < 2\sqrt{2}\sqrt{2r + 1} + 2$. Since

$$r^2 - m^2 - 2n^2 = \left(r - \sqrt{m^2 + 2n^2}\right)\left(r + \sqrt{m^2 + 2n^2}\right),$$

$$r - \sqrt{m^2 + 2n^2} = \frac{r^2 - m^2 - 2n^2}{r + \sqrt{m^2 + 2n^2}} < \frac{2\sqrt{2}\sqrt{2r+1} + 2}{r},$$

which $\to 0$ as $r \to \infty$. Hence

$$\lim_{r \to \infty} G(r) = \lim_{r \to \infty} \left|r - \sqrt{m^2 + 2n^2}\right| = 0.$$

B-5 (10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 58, 133)

The assertion is false. Take $p = x$, $q = y$, $r = -z - xy$. We can generate an infinite sequence of counterexamples by noting that if (p, q, r) is a solution then so is $(p, q, -r - pq)$.

B-6 (3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 32, 166)

The conditions of the problem are

(i) $AB^t = (AB^t)^t = BA^t$,

(ii) $CD^t = (CD^t)^t = DC^t$,

(iii) $AD^t - BC^t = I$.

Condition (i) implies $BA^t - AB^t = 0$ (the $n \times n$ zero matrix). Condition (ii) implies $CD^t - DC^t = 0$, and the transpose of condition (iii) is $DA^t - CB^t = I^t = I$. Hence

we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = \begin{pmatrix} AD^t - BC^t & -AB^t + BA^t \\ CD^t - DC^t & -CB^t + DA^t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From this it follows that

$$\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

as well, and the lower right-hand corner of this is $-C^tB + A^tD = I$.

Editor:

Reports of the out-of-printness of *Fundamentals of Linear Algebra*, by K. Nomizu (see review by S. C. Geller, this MONTHLY, January, 1987), are greatly exaggerated. The book is very much alive.

Aaron Galuten
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Chelsea Publishing Co.
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Editor:

In her review of "Real Linear Algebra" by Antal E. Fekete [this MONTHLY, 94 (January) 86–88], Susan C. Geller writes 'There are also a lot of examples of 3×3 orthogonal matrices with rational coefficients.' She, and others, may be interested to see how to generate *all* rational orthogonal matrices.

Take four integers a, b, c and d, not all zero, normalized so that $\gcd(a, b, c, d) = 1$. Set $N = a^2 + b^2 + c^2 + d^2$ and

$$M = M(a, b, c, d)$$
$$= \frac{1}{N}\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(bd + ac) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}.$$

Then M is orthogonal: in fact, $M^t = M^{-1} = M(a, -b, -c, -d)$; and $\det(M) = +1$, so $M \in SO(3, \mathbf{Q})$. Also, M is the identity matrix if and only if $b = c = d = 0$; otherwise the fixed axis of M is the column vector $(b, c, d)^t$, and the angle of rotation θ satisfies $\cos(\frac{1}{2}\theta) = \pm a/\sqrt{(N)}$.

Every matrix in $SO(3,\mathbf{Q})$ arises in this way. More generally, if F is any subfield of $\mathbf{R}$ then

$$SO(3, F) = \{M(a, b, c, d) : a, b, c, d \in F, \text{not all } 0\}$$

with

$$M(a, b, c, d) = M(a', b', c', d') \Leftrightarrow \begin{array}{c} a' = \lambda a, b' = \lambda b, c' = \lambda c, d' = \lambda d \\ \text{for some } \lambda \in F, \lambda \neq 0. \end{array}$$

To see where these facts come from, consider the real algebra of quaternions $\mathbf{H}$. If $q = a + bi + cj + dk \in \mathbf{H}^* = \mathbf{H} \setminus \{0\}$ then the map

$$h \mapsto qhq^{-1}$$

gives an *orthogonal* transformation of $\mathbf{H}$ (identified with $\mathbf{R}^4$) which preserves the 3-dimensional subspace V spanned by i, j, and k. The matrix of this transformation of V with respect to i, j, and k as basis is $M(a, b, c, d)$, as can readily be verified.

This gives a map

$$\mathbf{H}^* \to SO(3, \mathbf{R}),$$

which is a surjective group homomorphism: given an axis of rotation $(b, c, d)^t$ and angle θ one sets $a = \cos(\frac{1}{2}\theta)$ and $q = a + bi + cj + dk$. The kernel is $\mathbf{R}^*$. Identifying S^3, the unit sphere in $\mathbf{R}^4$, with the group of unit quaternions, one obtains the familiar $2:1$ homomorphism $S^3 \to SO(3)$.

To see that the restricted map $\mathbf{H}(F)^* \to SO(3, F)$ is still surjective is a little harder, and can be left as an exercise for the reader!

John Cremona
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NOTES

EDITED BY CAROL G. CRAWFORD, RICHARD LIBERA, AND ANITA E. SOLOW

The Fascination of the Elementary

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Closely based on the 'Hannah Neumann Memorial Lecture' presented at ICME 5 in Adelaide, South Australia, in August, 1984.

Introduction. In this note I plan to share some mathematical ideas which bridge the gap between the classroom and the ivory tower of the professional pure mathematician. The ideas are elementary and, therefore, accessible to the amateur, but they are at the same time challenging, and often require a wide breadth of mathematical knowledge to work through in detail.

We begin with the humble *geoboard*, a teaching aid used (at least) in Australian schools. It is a square board with nails (or golf tees) placed in a square grid formation. Various polygons can be constructed by wrapping a rubber band around the nails (see Figure 1). This illustrates two basic concepts: The *integer lattice* in the plane (or in general dimension) is the set of points having integral coordinates. (More general lattices can be obtained by applying a linear transformation to the integer lattice; many of our results adapt easily to these lattices.) A *lattice polygon* is a polygon which has a lattice point at every vertex.

We now ask: What are the properties of these lattice polygons? Are there any good results to be discovered? Any specializations? any extensions? any generalizations?

Pick's Theorem. Probably the best-known result is that discovered by Georg Pick [**21**] in 1900:

THEOREM . *Let P be a simple (i.e., nonintersecting) lattice polygon, containing B lattice points on its boundary, and I lattice points in its interior. Then the area, $A(P)$, of P is given by*

$$A(P) = \tfrac{1}{2}B + I - 1$$

For example, in Figure 1, $B = 10$, $I = 1$, and $A(P) = 5$.

I first came across Pick's theorem in Coxeter's book *Introduction to Geometry* [3]. The proof given there falls into two parts:

(1) A *primitive triangle* is a lattice triangle with $I = 0$ and $B = 3$ (the vertex points). Then every primitive triangle T satisfies Pick's theorem. That is, $A(T) = 1/2$.

(2) The formula for the area is additive in the sense that if Pick's theorem holds for lattice polygons P_1, P_2, and P_1, P_2 are juxtaposed so that they have a segment

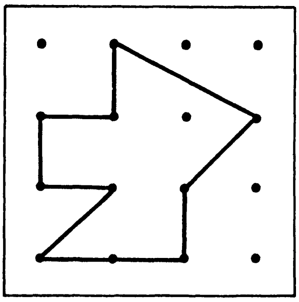


FIG. 1

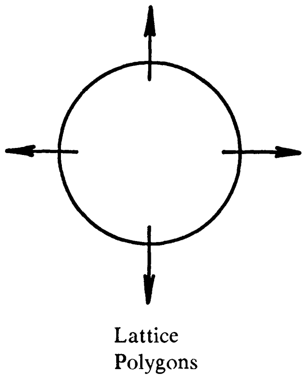


FIG. 2.

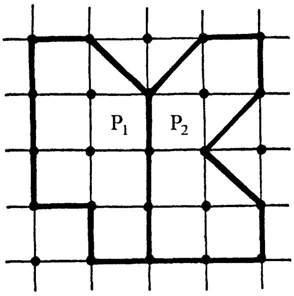


FIG. 3

of boundary in common, then Pick's theorem also holds for the lattice polygon $P_1 + P_2$ obtained by deleting the common boundary (Figure 3).

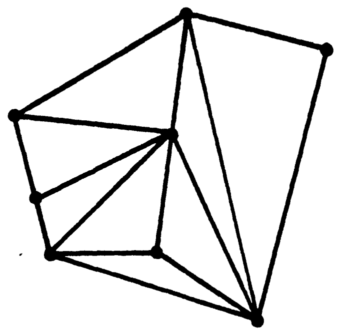
Since every lattice polygon can be built up from primitive triangles, the proof of Pick's theorem now follows inductively from (1) and (2).

If we know that all primitive triangles have area $1/2$, it becomes clear that Pick's theorem is equivalent to showing that the number $N(P)$ of primitive triangles in

polygon P is

$$N(P) = B + 2I - 2.$$

In fact at this stage the lattice becomes redundant, as $N(P)$ counts the number of nonoverlapping triangles which partition a polygon P, for a general triangulation involving B boundary points and I interior points (Figure 4).



$B = 6,\ I = 2,\ N(P) = 8$

FIG. 4

And further, this triangulation formula for $N(P)$ can be shown to be equivalent to Euler's famous formula

$$V - E + F = 2$$

relating the number of vertices, edges and faces of a planar graph.

An interesting trail to follow! Authors who explore these relationships include Funkenbusch [**6**], Gaskell, Klamkin, and Watson [7], Haigh [**10**], Honsberger [**12**], Liu [**16**], Niven and Zuckerman [**19**], Varberg [**29**], and Yaglom and Yaglom [33].

Regular Polygons. A *regular* polygon is a polygon which has all its edges congruent and all its vertices congruent.

The square is the most obvious regular polygon which is also a lattice polygon (Figure 5). An interesting question to ask is:

For what values of $n(\geqslant 3)$ does there exist a regular (convex) lattice n-gon?

The rather surprising answer is given in an ingenious paper by Scherrer [**25**], and described in the book *Combinatorial Geometry in the Plane*—Hadwiger, Debrunner, and Klee [**8**].

We notice that the side length of any lattice polygon is of the form $s = \sqrt{p^2 + q^2}$ where, p, q are integers. Now an area argument quickly eliminates the equilateral triangle. For, a lattice equilateral triangle of side length s would have area $\sqrt{3}\,s^2/4$; this is clearly irrational, since s^2 is an integer. However, by the well-known determinantal formula for the area of a triangle, the area must be rational. Hence no lattice equilateral triangle can exist. A similar argument eliminates the regular lattice hexagon.

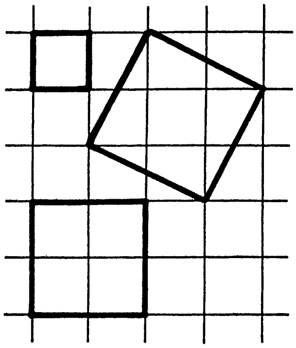


FIG. 5

Next, for given $n \geqslant 5, n \neq 6$, suppose there exists a regular lattice n-gon, and choose one of smallest size with vertices $P_1, P_2, \ldots, P_n$ say. Translate these vertices through lattice vectors $\overrightarrow{P_2P_3}, \overrightarrow{P_3P_4}, \ldots, \overrightarrow{P_1P_2}$, respectively. We thus obtain n new lattice points forming a new smaller regular lattice n-gon (Figure 6). But this contradicts the assumed minimality condition. Hence the square is the only regular lattice polygon.

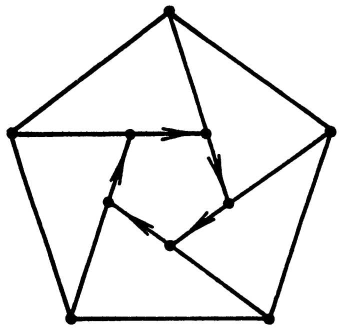


FIG. 6

Parsons and Truran [**20**] give a rotation proof that no equilateral lattice triangle exists. Klamkin [**14**] asks for, and receives, a proof that no lattice square exists with vertices belonging to an equilateral triangular lattice. Ball [**1**] shows that for the integral lattice, a convex equilateral lattice n-gon ($n \geqslant 3$) can be constructed if and only if n is even. Honsberger [**13**] shows that equiangular lattice n-gons can only exist for $n = 4$ and $n = 8$ (see Figure 7). Some results are known about possible angles of lattice polygons (see [**8**] and [**18**]).

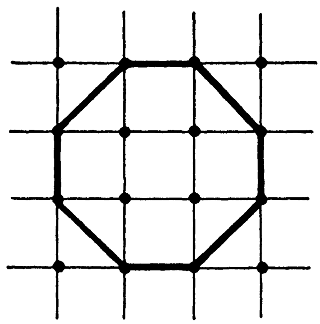


FIG. 7

Convex Polygons. Consider now a convex lattice polygon with B lattice points on the boundary and I ($\geqslant 0$) interior lattice points. If we fix I, and try to vary the polygon in such a way that B is increased, we quickly reach a place where any further increase in B can only be made at the expense of the convexity of the polygon. Thus for convex lattice polygons with given $I(\geqslant 0)$, B is bounded above.

Scott [**26**], and later Coleman [**2**], proved that $B \leqslant 2I + 7$, with equality only for the triangle in Figure 8 (or an equivalent triangle). It is surprising that this simple result remained undetected for so many years. Coleman also makes the following interesting conjecture for a convex lattice polygon having n sides: $B \leqslant 2I + 10 - n$. Setting $n = 3$ we see that the bound is best possible, but it is probably much too large for large values of n. Ehrhart [**4**] shows that any convex lattice polygon having at least five sides must contain an interior lattice point, and it is relatively easy to obtain a sequence of results of the form:

A convex lattice n-gon contains at least $I(n)$ interior lattice points.

Further results on convex lattice polygons have been obtained by Wills [**31**], Weaver [**30**], and Scott [**27**].

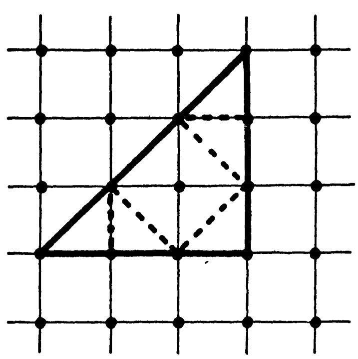


FIG. 8

Nonsimple Polygons. Ehrhart [4] gives a simple extension of Pick's theorem to apply to polygonal regions having polygonal 'holes.' Of rather more interest is an extension to nonsimple polygons given by Reeve [22]. We shall assume that for our polygon P (a) if two edges of P intersect, their intersection is a vertex of each, and so a lattice point; (b) every boundary point belongs to a non-degenerate triangle which is contained in P; (c) the 'area enclosed by P' is the sum of the areas of regions which can be reached from 'outside P' by an odd number of crossings of the boundary (Figure 9).

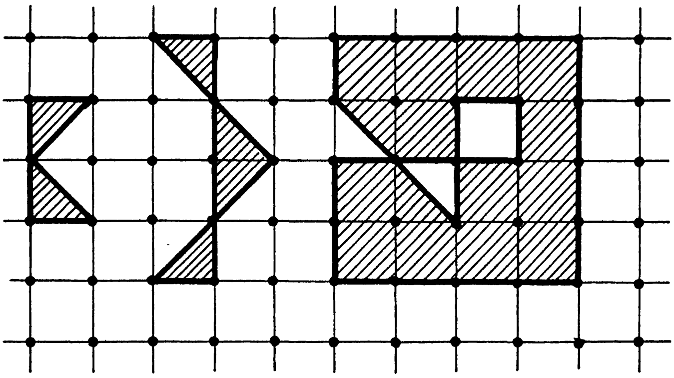


FIG. 9

Now,

$$A(P) = \tfrac{1}{2}B + I + k,$$

where

$$k = -\chi(P) + \tfrac{1}{2}\chi(\partial P),$$

and

$$\chi(P) = V - E + F$$
$$\chi(\partial P) = V - E.$$

The function χ is the *Euler characteristic* of the region and boundary respectively; V, E, and F count the numbers of vertices, edges, and *actual* faces. Thus, for example:

	B	I	$\chi(P)$	$\chi(\partial P)$	k	$A(P)$
Figure 7	8	4	1	0	-1	7
Figure 9(a)	5	0	1	-1	$-\frac{3}{2}$	1
Figure 9(b)	8	0	1	-2	-2	2
Figure 9(c)	22	3	-1	-2	-4	14

Noting that $V = B$, if we substitute the expressions for $\chi(P), \chi(\partial P)$ in the formula for $A(P)$, we obtain the interesting generalization of Pick's theorem:

$$A(P) = \tfrac{1}{2}E + I - F.$$

For simple polygons, $E = B$ and $F = 1$, and Pick's formula is obtained. Hadwiger and Wills [9] and Rosenholtz [23] have explored this area.

Higher Dimensional Analogues. So far we have followed our lattice polygons in four productive directions (Figure 10). It is natural now to ask whether any of our ideas generalize to higher dimensions.

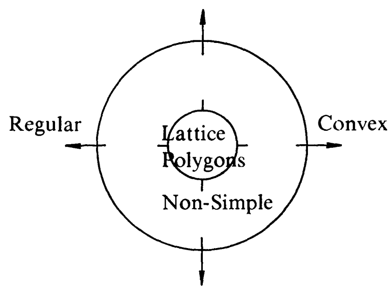


FIG. 10.

There is a pretty counterexample that shows that no simple analogue of Pick's theorem holds in 3-space. For consider the lattice tetrahedron pictured in Figure 11, having vertices (1, 0, 0), (0, 1, 0,), (1, 1, 0), and $(0, 0, k)$; $(k \in Z^+)$. For this tetrahedron, $B = 4$, $I = 0$, but the volume varies according to the choice of k. Hence, there can be no expression for the volume in terms of B and I.

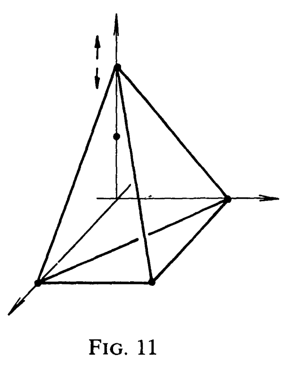


FIG. 11

Reeve [22] overcomes this difficulty in a clever way. Let L denote the integral lattice and let $L_n(n \in Z^+)$ denote the sublattice whose points are x/n, where $x \in L$. For example, L_2 is the set of points $(a/2, b/2, c/2)$, where a, b, c, are integers. If now P is a lattice polyhedron containing $T(T_n)$ boundary and interior points, and $B(B_n)$ boundary points of lattice $L(L_n)$, then the volume $V(P)$ of P is given by

$$2(n-1)n(n+1)V(P) = 2(T_n - nT) - (B_n - nB).$$

As Reeve comments, the existence of such a formula is perhaps of more interest than the formula itself. Reeve gives an extension of his result to include nonsimple polyhedra, and conjectures a formula for four dimensions. MacDonald [17] has since established general results for lattice polyhedra in arbitrary dimension.

The study of regular lattice polygons can be extended in two ways. We can ask if a regular lattice polygon can be embedded in an integral lattice of sufficiently high dimension. Klamkin and Chrestenson [15] find that a necessary and sufficient condition is for the polygon to have 3, 4, or 6 vertices. Alternatively, we can ask about the existence of regular lattice polyhedra in 3-space. Ehrhart [5] shows that the regular icosahedron and dodecahedron cannot occur in this way; the other three regular solids can easily be suitably placed in space (Figure 12). Like the square, the cube can be placed in the integral lattice 'square on' or obliquely. Sárközy [24] determines the possible placings of an $n \times n \times n$ cube for varying values of n. In [28], Scott shows that precisely three of the semiregular polyhedra occur as lattice polyhedra.

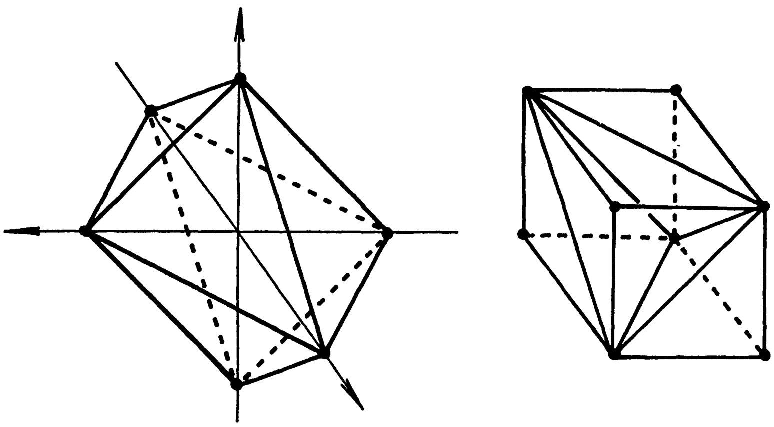


FIG. 12

The inequality $B \leqslant 2I + 7(I > 0)$ for convex lattice polygons has proved to be more difficult to extend to higher dimensions. Hensley [11] has proved the existence of a similar type of inequality for lattice polyhedra in general dimension, and Zaks,

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A Generalization of the Goldbach-Shnirel'man Theorem

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In a letter to Euler in 1742, Goldbach conjectured that every even integer is the sum of at most two primes. In 1930 Shnirel'man [4] proved that there is a number h such that every integer greater than 1 is the sum of at most h primes. The least integer K with this property is called "Shnirel'man's constant." Vaughan [5] recently proved that $K \leqslant 27$ and Deshouillers [1] modified Vaughan's proof to obtain $K \leqslant 26$.

By means of the Selberg sieve, there is now a short proof of the Goldbach-Shnirel'man theorem (see, for example, Hua [2, pp. 518–528]). In his 1930 paper, Shnirel'man actually obtained the following stronger result: If Q is a set that contains a positive proportion of the primes, then there is a number h such that every sufficiently large integer is the sum of at most h primes belonging to Q. In

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By means of the Selberg sieve, there is now a short proof of the Goldbach-Shnirel'man theorem (see, for example, Hua [2, pp. 518–528]). In his 1930 paper, Shnirel'man actually obtained the following stronger result: If Q is a set that contains a positive proportion of the primes, then there is a number h such that every sufficiently large integer is the sum of at most h primes belonging to Q. In

this note I sketch Hua's proof of the Goldbach-Shnirel'man theorem and show how a simple modification of the proof yields Shnirel'man's strengthening of this result. The method is similar to the proof by Rieger [3] of Shnirel'man's generalization of Waring's problem.

Let A be a set of nonnegative integers. The counting function $A(x)$ of the set A is defined by

$$A(x) = \sum_{\substack{a \in A \\ 1 \leqslant a \leqslant x}} 1.$$

The Shnirel'man density $\sigma(A)$ of A is defined by

$$\sigma(A) = \inf\{A(n)/n \mid n = 1, 2, 3, \dots\}.$$

The set A is a *basis of order* h if every nonnegative integer can be written as the sum of h elements of A. Shnirel'man [2], [4] proved that if $\sigma(A) > 0$, then A is a basis of order h for some h.

Let $B = 2A$ be the set of all numbers of the form $a_i + a_j$, where $a_i, a_j \in A$, and let $r_A(n)$ denote the number of representations of n in the form $n = a_i + a_j$, where $a_i, a_j \in A$. The Cauchy-Schwartz inequality implies that

$$\left(\sum_{m=1}^{n} r_A(m)\right)^2 \leqslant B(n) \sum_{m=1}^{n} r_A^2(m),$$

and so

$$\frac{B(n)}{n} \geqslant \frac{\left(\sum_{m=1}^{n} r_A(m)\right)^2}{n \sum_{m=1}^{n} r_A^2(m)}.$$

Denote the right side of the inequality above by $R_A(n)$. If $R_A(n) \geqslant c > 0$ for all $n \geqslant n_0$ and if $B^* = B \cup \{0, 1\}$, then $\sigma(B^*) > 0$ and so B^* is a basis of order h for some h.

Apply this in the special case when $A = \mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$ is the set of all primes. Using the Selberg sieve (Hua [2]), one can prove that for $m = 1, \dots, n$ the average value of $r_{\mathbb{P}}(m)$ is bounded above by $c_0 n/\log^2 n$ and the average value of $r_{\mathbb{P}}^2(m)$ is bounded above by $c_1 n^2/\log^4 n$. Thus,

$$\sum_{m=1}^{n} r_{\mathbb{P}}^2(m) \leqslant \frac{c_1 n^3}{\log^4 n}$$

COROLLARY. *Let $m \geqslant 2$ and $(k, m) = 1$. Then every sufficiently large integer is the sum of a bounded number of primes $p \equiv k(\mathrm{mod}\, m)$.*

Proof. Let $Q = \{p \equiv k(\mathrm{mod}\ m)\}$. By the prime number theorem for arithmetic progressions, $Q(x) \sim \pi(x)/\varphi(m)$, and so Q contains a positive proportion of the primes. The result now follows immediately from the Theorem.

Using probability methods, Wirsing **[6]** proved that there exists a sequence Q of primes such that $\lim_{x \to \infty} Q(x)/\pi(x) = 0$ and every sufficiently large odd number is a sum of three primes belonging to Q. It is an open problem to construct an explicit example of a set Q of primes such that $\lim_{x \to \infty} Q(x)/\pi(x) = 0$ and every $n \geqslant 2$ is a sum of at most h elements of Q.

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A Note on the $3x + 1$ Problem

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Let N^+ be the set of all positive integers. For an x from N^+ put

$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}.$$

The following conjecture, usually called the "$3x + 1$ conjecture," is well known: For every natural x there exists such i that $f^i(x) = 1$, where f^i means the ith iterate of the function f. Many results concerning this conjecture are known: see **[1]**.

Let X, Y be sets of positive integers. We write $X < Y$ if for every $x \in X$, there exists $y \in Y$ and i, j such that $f^i(y) = f^j(x)$. Using this symbol, the $3x + 1$ conjecture can be formulated as follows: $N^+ < \{1\}$.

For $a, m \in N^+$ denote $a(\mathrm{mod}\ m) = \{z \in N^+; z \equiv a(\mathrm{mod}\ m)\}$. The aim of our note is to prove the following.

COROLLARY. *Let $m \geqslant 2$ and $(k, m) = 1$. Then every sufficiently large integer is the sum of a bounded number of primes $p \equiv k(\bmod m)$.*

Proof. Let $Q = \{p \equiv k(\bmod m)\}$. By the prime number theorem for arithmetic progressions, $Q(x) \sim \pi(x)/\varphi(m)$, and so Q contains a positive proportion of the primes. The result now follows immediately from the Theorem.

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A Note on the $3x + 1$ Problem

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Let N^+ be the set of all positive integers. For an x from N^+ put

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}.$$

The following conjecture, usually called the "$3x + 1$ conjecture," is well known: For every natural x there exists such i that $f^i(x) = 1$, where f^i means the ith iterate of the function f. Many results concerning this conjecture are known: see **[1]**.

Let X, Y be sets of positive integers. We write $X < Y$ if for every $x \in X$, there exists $y \in Y$ and i, j such that $f^i(y) = f^j(x)$. Using this symbol, the $3x + 1$ conjecture can be formulated as follows: $N^+ < \{1\}$.

For $a, m \in N^+$ denote $a(\bmod m) = \{z \in N^+; z \equiv a(\bmod m)\}$. The aim of our note is to prove the following.

THEOREM. *Let p be an odd prime such that 2 is a primitive root modulo p^2. Then for all positive integers a and n with $p \nmid a$ we have $N^+ < a(\bmod p^n)$.*

Obviously, if $N^+ < Y$ and the $3x + 1$ conjecture holds for all members of Y, then it is true for all natural numbers. The theorem reduces checking the problem to a set that is of arbitrarily small asymptotical density. Primes which satisfy the assumption of the theorem are, for example, $p = 3, 5, 11, 13, 19, 29, \ldots$.

Proof of Theorem. Let x be a fixed number from N^+. We have to find nonnegative integers i, j, and z such that

$$f^i(a + zp^n) = f^j(x). \tag{1}$$

We shall try to choose j so that $u = f^j(x)$ is not divisible by p. If $p \nmid x$, then we can put $j = 0$; thus $u = x$. If $p|x$, then x can be written in the form

$$x = (2b - 1) \cdot p \cdot 2^{n-1}$$

for some $b, n \in P$. In this case we put $j = n$ and get

$$u = f^n((2b-1) \cdot p \cdot 2^{n-1}) = f((2b-1) \cdot p) = 6bp - 3p + 1,$$

which is not divisible by p.

Hence, obviously, it will be sufficient to find i and z in such a way that

$$a + z \cdot p^n = 2^i \cdot u. \tag{2}$$

If we take a fixed i, then by (2) is z uniquely determined. Because z should be an integer, we have to have

$$a \equiv 2^i \cdot u(\bmod p^n), \tag{3}$$

and in order to have a nonnegative z, the inequality

$$a \leqslant 2^i \cdot u \tag{4}$$

must hold. Since 2 is a primitive root modulo p^2, it is a primitive root modulo p^n as well (see, for example [2], Chapter 6, § 2.*e*). Therefore, since $p \nmid a$, $p \nmid u$, the congruence (3) is solvable and some of its solutions also satisfy (4). Thus for z determined by (2), (1) also holds and the proof of Theorem is finished.

Remark. We have constructed a larger set of residue classes with the same property as those above. For example, we have proved $N^+ < 1(\bmod 4)$; moreover, in this case for every $x \in N^+$, there is an i such that $f^i(x) \in 1(\bmod 4)$. Furthermore, we have also constructed a set X of asymptotical density 0 with the property

$$N^+ < X.$$

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EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

Using Ringers in Teaching Modern Algebra

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In a course on groups and rings, an instructor presents material on quotient structures and kernels and develops the Fundamental Homomorphism Theorem for groups and again for rings. It is hoped that the repetition of concepts in the ring setting reinforces what was learned with groups; but often there is a question of how well the students have grasped these basic ideas. One way to assess the understanding is to give the students another algebraic system and to have them work through these concepts in that new structure. The point of this note is to call attention to a suitable algebraic system, the ringer.

The concept of a ringer is a generalization of that of a ring. Formally, a *ringer* is defined as a system $(N, +, \cdot)$ such that

(1) $(N, +)$ is a group,
(2) $(N, \cdot)$ is a semigroup, and
(3) $a(b + c) = ab + ac$ for a, b, c in N.

Obviously, a ringer that has abelian addition and both distributive laws is a ring. The prototypical ringer is the set $T(G)$ of all functions from a group G (written additively, but not necessarily abelian) to itself; the operations are pointwise addition and composition. (Note that in working with $T(G)$ a functional image is given as $(x)f$.) For various groups G, these $T(G)$ are a rich source of examples of ringers. In fact, any given ringer can be embedded in some $T(G)$. A short verification of this "Cayley theorem" is given in [3]. Incidentally, the left distributive law of a ringer yields $n0 = 0$. However, $T(\mathbf{Z}_2)$ shows that $0n$ need not equal 0 in a ringer.

Now to the main point. If N is a ringer and K is a subringer of N, what are necessary and sufficient conditions on K so that under the usual coset operations, the quotient N/K is a ringer? That is, what conditions on K are needed so that coset addition and coset multiplication are well-defined? Of course, for coset addition the subgroup $(K, +)$ must be normal in the group $(N, +)$. But since $(N, +)$ need not be abelian, normality of a subgroup does not automatically hold as it would in a ring, a point that a student will need to realize. Using the fact that the coset K is the identity for the quotient addition, we can get some information about what is necessary for coset multiplication to be well-defined. Consider that

$$K = [0] = [n0] = [n] \cdot [0] = [n] \cdot [k] = [nk]$$

with n in N, k in K, and $[a]$ designating the coset containing a. Hence a necessary

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Using Commutators to Prove A_5 is Simple

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A standard exercise in introductory group theory courses is to show there are no simple groups of a given order. Recently Gallian [3] showed that the type of reasoning used to solve such problems, namely a combination of various counting arguments and an appeal to the Sylow theorems, could also be used to find a simple group, namely A_5. This note provides an equally elementary proof of the simplicity of A_5 proceeding from the assumption that the student has already shown (or been shown) that there are indeed no nonabelian simple groups of order less than 60.

Recall that the *commutator* of elements x, y in a group G is $[x, y] = x^{-1}y^{-1}xy$. The subgroup of G generated by the commutators is $\delta(G)$. If $G = \delta(G)$ then G is *perfect*. It is well known (and easy to prove) that if N is a normal subgroup of G, then G/N is abelian iff N contains $\delta(G)$. This makes the following result immediate: Every finite perfect group G has a simple nonabelian group as a homomorphic image. For, if H is a maximal proper normal subgroup, then G/H is simple (the normal subgroup correspondence theorem); and if G/H were abelian, then $G = \delta(G)$ would force $G = H$.

Thus it remains only to show that A_5 is perfect. Recall that A_5 is the set of even permutations of $\{1,2,3,4,5\}$. Every element (other than the identity, which is of course always a commutator) is thus either a 5-cycle, a 3-cycle or a product of two disjoint 2-cycles. Then the equations (multiplication is from left to right)

$$\begin{aligned}[(12534),(12)(35)] &= (12345)\\ [(123),(23)(45)] &= (123)\\ [(14)(23),(123)] &= (12)(34)\end{aligned}$$

show that every element of A_5 is a commutator, so that A_5 is perfect.

Note 1. Even relatively weak students should be able to express every element of A_5 as a commutator by trial-and-error, since the above representations are far from unique! Also, it is of course not necessary to know that there are no nonabelian simple groups for *every* order below 60; only nonprime integers dividing 60 need be checked, by Lagrange's theorem and the fact that groups of prime order are abelian.

Note 2. Obviously a nonabelian simple group is perfect, since $\delta(G)$ is normal and $\delta(G) \neq \{e\}$ in a nonabelian group G. We see that the smallest possible order

for a group that is perfect but not simple is 120. There is indeed such a group, namely $SL(2,5)$, the group of 2×2 matrices with entries in $\mathbb{Z}_5$ and determinant 1. The center of this group is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\},$$

so it is not simple; but it is perfect. The center is in fact the only proper normal subgroup, so the resulting quotient group $PSL(2,5)$ must be simple (this is also just an instance of the main structure theorem for the classical groups; see the survey article [5], for example). It is a fairly challenging but manageable exercise for the good student (translate: I could do it with some considerable effort!) to show that every element of $SL(2,5)$ is a commutator, and then to explicitly construct an isomorphism between $PSL(2,5)$ and A_5. The latter can be justified by noting that nonisomorphic simple groups of the same order do exist; again, see [5].

Note 3. In general, $\delta(G)$ and the *set* of commutators of G need not coincide; see [1], [6], [7], and particularly [2] for an elegant example in an infinite group. Hilton [4] in 1908 asked if perfect groups could contain noncommutators. To my knowledge, this question was first answered in the affirmative by Issacs [6] in 1977. Whether a nonabelian simple group (finite or infinite) can contain a noncommutator is, however, apparently still an open question.

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Report of the CUPM Panel on Calculus Articulation: Problems in the Transition from High School Calculus to College Calculus*

I. Introduction. There is a widespread and growing dissatisfaction with the performance in college calculus courses of many students who had studied calculus in high school. In response to this concern, in the fall of 1983 the Committee on the

*This article is based on the report of the CUPM Panel on Calculus Articulation, chaired by Don Small of Colby College. This article differs slightly from the report in that some editorial changes were made by Professor Small. CUPM has approved both the report and the revision, and is grateful for the fine work that the panel has produced.

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Undergraduate Program in Mathematics (CUPM) formed a Panel (the Calculus Articulation Panel) to undertake a three-year study of questions concerning the transition of students from high school calculus to college calculus and submit a report to CUPM detailing the problems encountered and proposals for their solution. Four high school teachers: Gordon Bushaw (Central Kitsap High School, Silverdale, Wash.), Donald J. Nutter (Firestone High School, Akron, Ohio), Ronald Schnackenburg (Steamboat Springs High School, Steamboat Springs, Colo.), and Barbara Stott (Riverdale High School, Jefferson, La.) and three college teachers: John H. Hodges (Univ. of Colorado), Donald R. Sherbert (Univ. of Illinois), and Donald B. Small (Colby College), Chair, constituted the membership of the panel.

The seriousness of the issues involved in the Panel's study is underscored by the number of students involved and their academic ability. During the ten-year period, 1973 to 1982, the number of students in high school calculus courses grew at a rate exceeding 10% annually. Of the 234,000 students who passed a high school calculus course in 1982, 148,600 received a grade of B^- or higher [**2**]. Assuming a continuation of the 10% growth rate and a similar grade distribution, there were approximately 200,000 high school students in the spring of 1985 who received a grade of B^- or higher in a calculus course. Thus possibly a third or more of the 500,000 college students who began their college calculus program (in Calc. I, Calc. II, or Calc. III) in the fall of 1985 had already received a grade of B^- or higher in a high school calculus course.

The students studying calculus in high school constitute a large majority of the more mathematically capable high school students. (In 1982, 55% of high school students attended schools where calculus was taught [**2**].) Students who score a 4 or 5 on an Advanced Placement Calculus examination normally do well in maintaining their accelerated mathematics program during the transition from high school to college. However this is a very small percentage of the students who take calculus in high school. For example, in 1982, of the 32,000 students who took an Advanced Placement Calculus examination, just over 12,000 received scores of 4 or 5, which represents only 6% of all high school students who took calculus that year. The primary concern of the Panel was with the transition difficulties associated with the remaining almost 94% of the high school calculus students.

II. Problem areas. Past studies and the Panel's surveys of high school teachers, college teachers, and State Supervisors suggest that the major problems associated with the transition from high school calculus to college calculus are:

1. High school teacher qualifications and expectations.
2. Student qualifications and expectations.
3. The effect of repeating a course in college after having experienced success in a similar high school course.
4. College placement.
5. Lack of communication between high schools and colleges.

(Copies of the Panel's Report including the surveys and summaries of the responses can be obtained from the Washington Office of the MAA.)

These problems were addressed by first considering accelerated programs in general, high school calculus (successful, unsuccessful), and the responsibilities of the colleges.

III. Accelerated Programs. Accelerated mathematics programs, usually beginning with algebra in eighth grade, are now well established and accepted in most school systems. The success of these programs in attracting mathematically capable students was documented in the 1981–82 testing that was done for the "Second International Mathematics Study." The Summary Report [**9**] states with reference to a comparison between twelfth-grade precalculus students and twelfth grade calculus students in the United States:

> *We note furthermore that in every content area (sets and relations, number systems, algebra, geometry, elementary functions/calculus, probability and statistics, finite mathematics) the end-of-the-year average achievement of the precalculus classes was less (and in many cases considerably less) than the beginning-of-the-year achievement of the calculus students.*

The report continues:

> *It is important to observe that the great majority of U.S. senior high school students in fourth and fifth year mathematics classes (that is, those in precalculus classes) had an average performance level that was at or below that of the lower 25 percent of the countries. The end-of-year performance of the students in the calculus classes was at or near the international means for the various content areas, with the exception of geometry. Here U.S. performance was below the international average.*

Thus those students in accelerated programs culminating in a calculus course perform near the international mean level while their classmates in (non-accelerated) programs culminating in a precalculus course perform in the bottom 25 percent in this international survey. The poor performance in geometry by both the precalculus and calculus students correlates well with the statistic that 38% of the students were never taught the material contained in the geometry section of the test [**9**, p. 59]. The test data underscores the concern expressed by many college teachers that more emphasis needs to be placed on geometry throughout the high school curriculum. This data does not, however, indicate that accelerated programs emphasize geometry less than non-accelerated programs.

The success of the accelerated programs in completing the "normal" four-year high school mathematics program by the end of the eleventh grade presents schools with both an opportunity and a challenge for a "fifth" year program. There are two acceptable options:

1. Offer college level mathematics courses that would continue the students' accelerated program and thus provide exemption from one or two semesters of college mathematics;

2. Offer high school mathematics courses that would broaden and strengthen a student's background and understanding of precollege mathematics.

Not offering a fifth-year course or offering a watered-down college level course with no expectation of students earning advanced placement are not considered to be acceptable options.

A great deal of prestige is associated with offering calculus as a fifth-year course. Communities often view the offering of calculus in their high school as an indication of a quality educational program. Parents, School Board officials, counselors, and school administrators often demonstrate a competitive pride in their school's offering of calculus. This prestige factor can easily manifest itself in strong political pressure for a school to offer calculus without sufficient regard to the qualifications of teachers or students. It is important that this political pressure be resisted and that the choice of a fifth-year program be made by the mathematics faculty of the local school and be made on the basis of the interest and qualifications of the mathematics faculty and the quality and number of accelerated students. School officials should be encouraged to develop public awareness programs to extend the prestige and support that exists for the calculus to acceleration programs in general. This would help diffuse the political pressure as well as broaden school support within the community.

Schools that elect the first option of offering a college level course should follow a standard college course syllabus (e.g., the Advanced Placement syllabus for calculus). They should use placement test scores along with the college records of their graduates as primary measures of the validity of their course.

For schools that elect the second option, a variety of courses is possible. The following course descriptions represent four possibilities.

Analytical Geometry

This course could go well beyond the material normally included in second-year algebra and precalculus. It could include Cartesian and vector geometry in two and three dimensions with topics such as translation and rotation of axes, characteristics of general quadratic relations, curve sketching, polar coordinates, and lines, planes, and surfaces in three-dimensional space. Such a course would provide specific preparation for calculus and linear algebra, as well as give considerable additional practice in trigonometry and algebraic manipulations.

Probability and Statistics

This course could be taught at a variety of levels, to be accessible to most students, or to challenge the strongest ones. It could cover counting methods and some topics in discrete probability such as expected values, conditional probability, and binomial distributions. The statistics portion of the course could emphasize exploratory data analysis including random sampling and sampling distributions, experimental design, measurement theory, measures of central tendency and spread, measures of association, confidence intervals, and significance testing. Such an introduction to probability and statistics would be valuable to all students, and for those who do not plan to study mathematics, engineering, or the physical sciences, probably more valuable than a calculus course.

Discrete Mathematics

This type of course could include introductions to a number of topics that are either ignored or treated lightly within a standard high school curriculum, but which would be stimulating and widely useful for the college-bound high school student. Suggested topics include permutations, combinations, and other counting techniques; mathematical induction; difference equations; some discrete probability; elementary number theory and modular arithmetic; vector and matrix algebra, perhaps with an introduction to linear or dynamic programming; and graph theory.

Matrix Algebra

This course could include basic arithmetic operations on matrices, techniques for finding matrix inverses, and solving systems of linear equations and their equivalent matrix equations using Gaussian elimination. In addition, some introduction to linear programming and dynamic programming could be included. This course could also emphasize three-dimensional geometry.

IV. High school calculus. There are many valid reasons why a fifth-year program should include a calculus course. Four major reasons: (1) calculus is generally recognized as the starting point of a college mathematics program, (2) there exists a (nationally accepted) syllabus, (3) the Advanced Placement program offers a nationwide mechanism for obtaining advanced placement, and (4) there is a large prestige factor associated with offering calculus in high school. Calculus, however, should not be offered unless there is a strong indication that the course will be successful.

Successful Calculus Courses

The primary characteristics of a successful high school calculus course are:

1. A qualified and motivated instructor with a mathematics degree that included at least one semester of a junior-senior real analysis course involving a rigorous treatment of limits, continuity, etc..
2. Administrative support, including provision of additional preparation time for the instructor (e.g., as recommended by the North Central Accreditation Association).
3. A full year program based on the Advanced Placement syllabus.
4. A college text should be used (not a watered-down high school version).
5. Advanced placement for students is a major goal (rather than mere preparation for repeating calculus in college).
6. Course evaluation based primarily on college placement and the performance of its graduates in the next higher level calculus course.
7. Restriction of course enrollment to only qualified and interested students.
8. The existence of an alternative fifth-year course that students may select who are not qualified for or interested in continuing in an accelerated program.

The bottom line of what makes a high school calculus course successful is no surprise to anyone. A qualified teacher with high but realistic expectations, using somewhat standard course objectives, and students who are willing and able to learn

result in a successful transition at any level of our educational process. Problems appear when any of the above ingredients are missing.

Unsuccessful Calculus Courses

Two types of high school calculus courses have an undesirable impact on students who later take calculus in college.

One type is a one semester or partial year course that presents the highlights of calculus, including an intuitive look at the main concepts and a few applications, and makes no pretense about being a complete course in the subject. The motivation for offering a course of this kind is the misguided idea that it prepares students for a *real* course in college. However, such a preview covers only the glory and thus takes the excitement of calculus away from the college course without adequately preparing students for the hard work and occasional drudgery needed to understand concepts and master technical skills. Professor Sherbert has commented: "It is like showing a ten-minute highlights film of a baseball game, including the final score, and then forcing the viewer to watch the entire game from the beginning—with a quiz after each inning."

The second type of course is a year-long, semiserious, but watered-down treatment of calculus that does not deal in depth with the concepts, covers no proofs or rigorous derivations, and mostly stresses mechanics. The lack of both high standards and emphasis on understanding dangerously misleads students into thinking they know more than they really do. In this case, not only is the excitement taken away, but an unfounded feeling of subject mastery is fostered that can lead to serious problems in college calculus courses. Students can receive respectable grades in a course of this type, yet have only a slight chance of passing an Advanced Placement examination or a college administered proficiency examination. Those who place into second-term calculus in college will find themselves in heavy competition with better prepared classmates. Those who elect (or are selected) to repeat first-term calculus believe they know more than they do, and the motivation and willingness to learn the subject are lacking.

V. College programs. Several studies ([1], [3], [5], [6], [7]) have been conducted on the performance in later courses by students who have received advanced placement (and possibly college credit) by virtue of their scores on Advanced Placement Calculus examinations. The studies show that, overall, students earning a score of 4 or 5 on either the AB or BC Advanced Placement Calculus examination do as well or better in subsequent calculus courses than the students who have taken all their calculus in college. It is therefore strongly recommended that colleges recognize the validity of the Advanced Placement Calculus program by the granting of one semester advanced placement with credit in calculus for students with a 4 or 5 score on the AB examination and two semesters of advanced placement with credit in calculus for students with a 4 or 5 score on the BC examination.

The studies reviewed by the Panel do not indicate any clear conclusions concerning performance in subsequent calculus courses by students who have scored a 3 on an Advanced Placement Calculus examination. The treatment of these students

is a very important transition problem since approximately one-third of all students who take an Advanced Placement Calculus examination are in this group and many of them are quite mathematically capable. It is therefore recommended that these students be treated on a special basis in a manner that is appropriate for the institution involved. For example, several colleges offer a student who has earned a 3 on an Advanced Placement Calculus examination the opportunity to upgrade this score to an "equivalent 4" by doing sufficiently well on a Department of Mathematics placement examination. Another option is to give such students one semester of advanced placement with credit for Calculus I upon successful completion of Calculus II. A third option is to give one semester of advanced placement with credit for Calculus I and provide a special section of Calculus II for such students.

Other important transition problems are associated with students who have studied calculus in high school, but have not attained advanced placement either through the Advanced Placement Calculus program or effective college procedures. These students pose an important and difficult challenge to college mathematics departments, namely: How should these students be dealt with so that they can benefit from their accelerated high school program and not succumb to the negative and (academically) destructive attitude problems that often result when a student repeats a course in which success has already been experienced? There are three major factors to consider with respect to these students.

1. The lack of uniformity of high school calculus courses. The wide diversity in the backgrounds of the students necessitates that a large review component be included in their first college calculus course to guarantee the necessary foundation for future courses.

2. The mistaken belief of most of these students that they really know the calculus when, in fact, they do not. Thus they fail to study enough at the beginning of the course. When they realize their mistake (if they do), it is often too late. These students often become discouraged and resentful as a result of their poor performance in college calculus, and believe that it is the college course that must be at fault.

3. The "Pecking Order" syndrome. The better the student, the more upsetting are the understandable feelings of uncertainty about his or her position relative to the others in the class. Although this is a common problem for all college freshman, it is compounded when the student appears to be repeating a course in which success had been achieved the preceding year. This promotes feelings of anxiety and produces an accompanying set of excuses if the student does not do at least as well as in the previous year. The uncertainty of one's position relative to the rest of the class often manifests itself in the student not asking questions or discussing in (or out of) class for fear of appearing *dumb*. This is in marked contrast to the highly confident high school senior whose questions and discussions were major components in his or her learning process.

The unpleasant fact is that the majority of students who have taken calculus in high school and have not clearly earned advanced placement do not *fit* in either the

standard Calculus I or Calculus II course. The students do not have the level of mastery of Calculus I topics to be successful if placed in Calculus II and are often doomed by attitude problems if placed in Calculus I. In modern parlance, this is the *rock and hard place*.

An additional factor to consider is the negative effect that a group of students who are repeating most of the content of Calculus I has on the rest of the class as well as on the level of the instructor's presentations.

What is needed are courses designed especially for students who have taken calculus in high school and have not clearly earned advanced placement. These courses need to be designed so that they:

1. acknowledge and build on the high school experiences of the students,
2. provide necessary review opportunities to ensure an acceptable level of understanding of Calculus I topics,
3. are *clearly different* from high school calculus courses (in order that students do not feel that they are essentially just repeating their high school course),
4. result in an equivalent of a one semester advanced placement.

Altering the traditional lecture format or rearranging and supplementing content seem to be two promising approaches to developing courses that will satisfy the above criteria. For example, Colby College has successfully developed a two-semester calculus course that fulfills the four conditions. The course integrates multivariable with single variable calculus and thereby covers the traditional three-semester program in two semesters [**10**].

Of course, the introduction of a new course entails an accompanying modification of college placement programs. However, providing new or alternative courses should have the effect of simplifying placement issues and easing transition difficulties that now exist.

VI. Recommendations

1. School administrators should develop public awareness programs with the objective of extending the support that exists for fifth-year calculus courses to accelerated programs including all of the fifth-year options.
2. A fifth-year program should offer a student a choice of courses (not just calculus).
3. The choice of fifth-year options should be made by the high school mathematics faculty on the basis of their interest and qualifications and the quality and number of the accelerated students.
4. If a fifth-year course is intended as a college level course, then it should be treated as a college level course (text, syllabus, rigor).
5. A fifth-year college level course should be taught with the expectation that successful graduates (B^- or better) would not repeat the course in college.
6. A fifth-year program should provide an alternative option for the student who is not qualified to continue in an accelerated program.
7. A mathematics degree that includes at least one semester of a junior-senior real analysis course involving a rigorous treatment of limit, continuity, etc. is strongly recommended for anyone teaching calculus.

8. A high school calculus course should be a full year course based on the Advanced Placement syllabus.

9. The instructor of a high school calculus course should be provided with additional preparation time for this course.

10. High school calculus students should take either the AB or BC Advanced Placement Calculus examination.

11. The evaluation of a high school calculus course should be based primarily on college placement and the performance of its graduates in the next level calculus course.

12. Only interested students who have successfully completed the standard four-year college preparatory program in mathematics should be permitted to take a high school calculus course.

13. Colleges should grant credit and advanced placement out of Calculus I for students with a 4 or 5 score on the AB Advanced Placement Calculus examination and credit and advanced placement out of Calculus II for students with a 4 or 5 score on the BC Advanced Placement Calculus examination. Colleges should develop procedures for providing special treatment for students who have earned a score of 3 on an Advanced Placement Calculus examination.

14. Colleges should individualize as much as possible the advising and placement of students who have taken calculus in high school. Placement test scores and personal interviews should be used in determining the placement of these students.

15. Colleges should develop special courses in calculus for students who have been successful in accelerated programs, but have clearly not earned advanced placement.

Colleges have an opportunity and responsibility to develop and foster communication with high schools. In particular:

16. Colleges should establish periodic meetings where high school and college teachers can discuss expectations, requirements, and student performance.

17. Colleges should coordinate the development of enrichment programs (courses, workshops, institutes) for high school teachers in conjunction with school districts and State Mathematics Coordinators.

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PROBLEMS AND SOLUTIONS

*For instructions about submitting **proposed** problems for publication in this department see the inside front cover. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the address given on the inside front cover.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed by Feb. 28, 1988, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3225. *Proposed by Bjorn Poonen, Winchester, Massachusetts.*

Let r be a fixed real number greater than 1, and let $s_n(x)$ and $c_n(x)$ denote the nth Maclaurin polynomials for the sine and cosine functions respectively. If x_n is the smallest positive solution of

$$s_n(x)^2 + c_n(x)^2 = r,$$

show that

$$\lim_{n\to\infty} \frac{x_n}{n} = \frac{1}{e}.$$

E 3226. *Proposed by Stan Wagon, Smith College, Northampton, Massachusetts.*

The standard derivation of the Wallis product for π uses the fact (usually proved by repeated integration by parts) that

$$\lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{n+1} x\,dx}{\int_0^{\pi/2} \sin^n x\,dx} = 1.$$

Show that for any continuous function f: $[a, b] \to [0, +\infty)$ with positive maximum M we have

$$\lim_{n\to\infty} \frac{\int_a^b f(x)^{n+1}dx}{\int_a^b f(x)^n dx} = M.$$

E 3227. *Proposed by Bradley Lucier, Purdue University.*

Assume that $\lambda > 0$, $0 < \Delta < y$, and $m > 2$. (Here m is not necessarily an integer.) Show that

$$\begin{aligned}&\left\{(y+\Delta)^{1/(m-1)} + \lambda\left((y+2\Delta)^{m/(m-1)} - 2(y+\Delta)^{m/(m-1)} + y^{m/(m-1)}\right)\right\}^{m-1}\\ &\quad - \left\{y^{1/(m-1)} + \lambda\left((y+\Delta)^{m/(m-1)} - 2y^{m/(m-1)} + (y-\Delta)^{m/(m-1)}\right)\right\}^{m-1}\\ &\quad \leqslant \Delta.\end{aligned}$$

(This problem arose in studying the regularity of numerical approximations to the so-called porous medium equation:

$$\partial_t u(x,t) - \partial_x^2(u(x,t))^m = 0.)$$

E 3228. *Proposed by David K. Cohoon, Temple University.*

Let S be an m by m matrix over $\mathbb{C}$. It is well known that $S^2 = S$ and trace(S) = 0 imply that S is the zero matrix. For which positive integers $n > 2$ and $m > 1$ does the pair of conditions $S^n = S$ and trace(S) = 0 imply that S is the zero matrix?

E 3229. *Proposed by Gary E. Stevens, Hartwick College, Oneonta, New York.*

Determine the asymptotic behavior of the sequence defined by the recurrence

$$a_0 = 1, \qquad a_n = n^{1/a_{n-1}} \quad \text{for } n > 0.$$

E 3230. *Proposed by William Miller, Le Moyne College, Syracuse, New York.*

If n is a given positive integer, what is the largest possible product of distinct positive integers whose sum is n?

and ε_2 from the representation for b_1' as $b_1 \pm b_2$. We note that when $b_2' \neq b_3$, then $|b_1 \pm b_2| > 1$ for both choices of sign, so that the angle between b_1 and b_2 is between $\pi/3$ and $2\pi/3$ and consequently $|b_1 \pm b_2| < 3^{1/2}$ for both choices of sign. Finally we take $\varepsilon_{j+1} = \varepsilon_j'$ for $j = 3, 4, \ldots, N-1$. Then $|\varepsilon_1 b_1| \leq 1$, $|\varepsilon_1 b_1 + \varepsilon_2 b_2| < 3^{1/2}$, and by our inductive assumption

$$\left|\sum_{j=1}^{n+1} \varepsilon_j b_j\right| = \left|\sum_{j=1}^{n} \varepsilon_j' b_j'\right| < 3^{1/2}$$

for $n = 2, 3, \ldots, N-1$.

A result similar to the above lemma was obtained earlier by Viktor Bergström, Abh. Math. Sem. Hamburgischen Univ., 8(1931) 148–152 and 208–214.

Convergence of $a_{n+1} = a_n^2 - 2$

E 3036 [1984, 140]. *Proposed by F. Lazebnik, University of Pennsylvania and Y. Pilipenko, Kiev University, USSR.*

Define a sequence $\{a_n\}$ by $a_1 = a$ and $a_{n+1} = a_n^2 - 2$. For which values of a does this sequence converge?

Editorial comment. It turns out that, as a number of readers observe, this problem is quite old, and it and its generalizations have already been well studied—indeed, as far back as 1918.

It is easily seen that any limit L must be a fixed point of the function $f(x) = x^2 - 2$, so therefore $L = -1$ or $L = 2$. Moreover, these points are "repulsors," because $|f'(L)| > 1$. Thus, for convergence to occur, the sequence must be eventually constant at -1 or 2. This will occur if and only if the initial term is of the form $2\cos(2\pi k/3 \cdot 2^m)$, k any integer, m any non-negative integer. The convergence is to -1 if $3 \nmid k$ and to 2 if $3 | k$.

M. Ascher and W. F. Smyth called attention to the article by Gaston Julia, "Mémoire sur l'iteration des fonctions rationelles," *Journal de Mathématiques Pures et Appliques*, 7e Série, Tome IV, Fasc 1–2, 47–245 (1918). Smyth, in addition, cites: P. Fatou, "Sur les équations fonctionelles I–III", *Bull. Soc. Math. France*, 47 (1919), 161–271 and 48 (1920), 33–94, 208–314; Hans Brolin, "Invariant sets under iteration of rational functions", *Arkiv för Mathematik* 6 (1965), 103–144; Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman (1982), 180–192. A. A. Jagers (The Netherlands) also mentioned Brolin and Mandelbrot. In the latter, the iteration $f(z) = z^2 - \mu$ is discussed.

C. Georghious (Greece) submitted a complete solution of the case $z^2 - \mu$, and also showed that the more general iteration $b_{n+1} = \alpha b_n^2 + \beta b_n + \gamma$ can be put into this form. H. Waller submitted a complete solution of the general quadratic case.

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sequence is the "Newton's Method" sequence for the roots of the function

$$f(x) = \left(\frac{x+1}{x-2}\right)^{1/3};$$

the points -1 and 2 are discontinuities of $f'(x)$. M. Ascher referred to a paper by herself entitled "Cycling in the Newton-Raphson algorithm," *Int. J. Math. Educ. Sci. Technol.*, vol. 5 (1974) 229–235.

Ascher and others noted the connection to periodic and ultimately periodic sequences. E. Grosswald commented that the finite pre-images of the values of x that satisfy $f^r(x) = x$ for some r are the only ones that will be ultimately periodic. He conjectured that the complement of this set in $(-2, 2)$, which is of second category and measure 4 may contain a subset of first category whose points lead to ultimately almost periodic orbits, and that if this set is also removed, the remaining set is ergodic.

M. Pachter (South Africa) remarked that "we have at hand a dynamical system which exhibits 'sensitivity to initial conditions'. Two trajectories which start close to one another do not remain so," such as 2 and $2 - \varepsilon$. He referred readers to the interesting article J. A. Yorke and E. D. Yorke, "Chaotic behaviour and fluid dynamics," *Hydrodynamic Instabilities and the Transition to Turbulence* (H. L. Swinney, Editor); and to T. P. Golub, *Topics in Applied Physics*, vol. 45, Springer, New York, 1981, where dynamical systems similar to those under discussion are investigated.

P. J. Paúl and E. Freire (Spain) reminded readers of the article by J. Yorke and T. Li, "Period three implies chaos," this MONTHLY, 82 (1975) 985–992. These readers were kind enough to supply a set of interesting graphs illustrating convergence, periodicity, and "chaos"; see FIGURES 1–6.

W. Janous (Austria) referred to this problem as "one of the most prominent in contemporary analysis" and cites P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, Basel-Boston-Stuttgart (1980). He drew from their work the following conclusions for this problem.

(i) For every natural number p there exist sequences of period p.

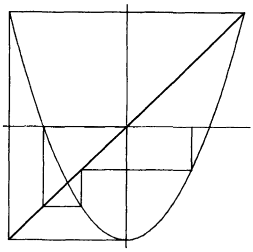
(ii) The set of all initial points for which the resulting sequence is eventually the constant 2 is dense in $[-2, 2]$.

(iii) There exist both a positive ε and an uncountable subset S of $[-2, 2]$ such that for any two sequences $\{a_n\}$ and $\{a'_n\}$ that start from distinct elements of S,

$$\limsup_{n\to\infty} |a_n - a'_n| \geq \varepsilon \quad \text{but} \quad \liminf_{n\to\infty} |a_n - a'_n| = 0.$$

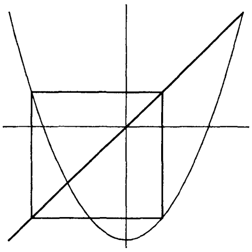
M. Bencze (Romania) quoted a novel generalization of the present problem which is studied in his paper entitled "A Lucás-Féle Rekurziv Sorozatokrol (On the Lucas recursion sequence)", *Matematikai Lapok* (1978) #10, 413–417, Kolotsoai-Napoca, Romania.

M. Woltermann and D. B. Tyler independently investigated the reversed sequence $a_n = \sqrt{(2 + a_{n+1})}$. Woltermann says, suppose that $a_n \to 2$. Let $T_1 = 0$ and



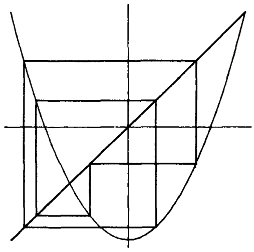
Convergence to 2, $a = \sqrt{2 - \sqrt{2 - \sqrt{2}}}$

FIG. 1



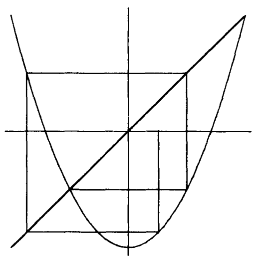
Period 2, $a = (\sqrt{5} - 1)/2$

FIG. 2



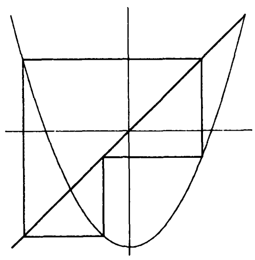
Period 5, $a = 0.471517$

FIG. 3



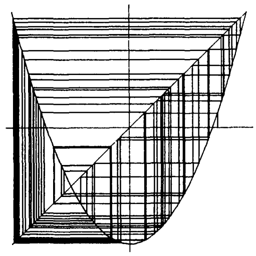
Convergence to -1, $a = 2 - \sqrt{3}$

FIG. 4



Period 3, $a = 1.246979$

FIG. 5



Chaos, $a = 1.5$

FIG. 6

$T_n = \sqrt{2 + T_{n-1}}$ for $n > 1$. Then $T_1 < T_2 < T_3 < \cdots < 2$ and $T_n \to 2$. ($T_n < T_{n+1} < 2 \Rightarrow \sqrt{2 + T_n} < \sqrt{2 + T_{n+1}} < 2 \Rightarrow T_{n+1} < T_{n+2} < 2$.) If some $a_N = T_K$, then $a_{N+1} = T_{K-1}$, and eventually $a_n = 2$; then $a = a_1 = 0, \pm 2, \pm\sqrt{2}, \pm\sqrt{2 \pm \sqrt{2}}, \dots$. If $T_K < a_N < T_{K+1}$, then

$$T_K^2 - 2 < a_N^2 - 2 < T_{K+1}^2 - 2, \quad \text{i.e.,} \quad T_{K-1} < a_{N+1} < T_K,$$

and, therefore, it follows that $\{a_n\}$ can converge to 2 only if some a_N equals some T_K.

Suppose that $a_n \to -1$. Let $U_1 = -\sqrt{3}$ and $U_n = -\sqrt{2 + U_{n-1}}$ for $n > 1$. Then

$$U_1 < U_3 < U_5 < \cdots < -1 < \cdots < U_6 < U_4 < U_2 < 0.$$

$$\Big(-1 < U_{n+2} < U_n < 0 \Rightarrow 1 < \sqrt{2 + U_{n+2}} < \sqrt{2 + U_n} < \sqrt{2}$$

$$\Rightarrow -\sqrt{2} < U_{n+1} < U_{n+3} < -1 \Rightarrow \sqrt{2 - \sqrt{2}} < \sqrt{2 + U_{n+1}} < \sqrt{2 + U_{n+3}} < 1$$

$$\Rightarrow -1 < U_{n+4} < U_{n+2} < -\sqrt{2 - \sqrt{2}} < 0.\Big)$$

If some $a_N = U_K$, then $a_{N+1} = U_{K-1}$, and eventually $a_n = -1$; then

$$a = a_1 = \pm 1, \pm\sqrt{3}, \pm\sqrt{2 \pm \sqrt{3}}, \dots .$$

If a_N is strictly between U_K and U_{K+2}, then calculations as in the parentheses above show that a_{N+2} is strictly between U_{K-2} and U_K, and therefore it follows that $\{a_n\}$ can converge to -1 only if some a_N equals some U_K.

Tyler states the following THEOREM: *If* $-2 \le a \le 2$ *and*

$$\frac{1}{\pi}\cos^{-1}(a/2) = .x_1x_2x_3\dots = x$$

in base 2 (*with* $x_n = 0$ *or* 1 *for all* n) *then*

$$a = s_1\sqrt{2 + s_1s_2\sqrt{2 + s_2s_3\sqrt{2 + s_3s_4\sqrt{2 + \dots}}}}, \qquad *$$

where

$$s_n = \begin{cases} +1 & \text{if } x_n = 0 \\ -1 & \text{if } x_n = 1 \end{cases} \quad \textit{for } n = 1, 2 \dots .$$

Furthermore, $\lim a_n = 2$ if and only if s_ns_{n+1} is ultimately equal to 1 if and only if x is a repeating binary with period 1 and in this case a has exactly two expansions of the form * (just as .385000... = .384999... in base 10). Also $\lim a_n = -1$ if and only if s_ns_{n+1} is ultimately equal to -1 if and only if x is a repeating binary with period (exactly) 2. Finally, the expansion of a in the form * is unique both when $\lim a_n = -1$ and when $\lim a_n$ does not exist.

A. Stenger had an interesting proof that only sequences that are eventually constant can converge: Suppose some value of a gives a sequence which is not eventually constant, i.e., for which $a_{n+1} \neq a_n$ for arbitrarily large n. Consider

$$\frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} = \frac{a_{n+1}^2 - a_n^2}{a_{n+1} - a_n} = a_{n+1} + a_n \to 2L.$$

Since $|2L| \ge 2$, this implies that successive terms of the sequence get farther apart, so the sequence cannot converge.

W. Janous had a nice way of demonstrating that divergence occurs if the initial value a satisfies $|a| > 2$: Set $a = x + 1/x$, where $|x| > 1$. Then note that

$$a_{n+1} = x^{2^n} + x^{2^{-n}},$$

so as $n \to \infty$, a_{n+1} is almost x^{2^n}, and hence diverges.

R. Mazurek (Poland) offered the following generalization of one key step in the analysis of the problem: If $f^n(a) \to b$ as $n \to \infty$ and if f is C^1 in some neighborhood of b with $|f'(b)| > 1$, then the sequence must be constant from some point on.

Solutions were submitted by 82 readers from 23 countries. All but 11 solutions were correct. It is interesting that several of those who submitted incorrect solutions wrote of having done computer experiments. This problem shows clearly the limitations of numerical experimentation, however helpful it

may be in other contexts. Computers found the integer solutions -2, $-1, 0, 1$, and 2, but concluded that all the other initial values produce divergence. The computers correctly determined these five as the only *rational* solutions; for the computers, of course, numbers other than rationals do not exist.

A Homeomorphic Image Made Right

E 3066 [1984, 649]. *Proposed by Bruce Richter, Ohio State University.*

Let A be a subset of $\mathbb{R}^n$ (n-dimensional Euclidean space) that is homeomorphic to $(0,1)$. Let $\bar{A}$ be the closure of A in $\mathbb{R}^n$ and suppose $\bar{A}-A$ contains no homeomorph of $(0,1)$. Show that $\bar{A}$ is homeomorphic to one of $(0,1)$, $[0,1)$, $[0,1]$ and the unit circle.

Editor's note. The problem which appeared in print, where it was stated as above, was not in the form in which the proposer originally submitted it. There were changes introduced into the problem statement during the refereeing and editing process. Unfortunately, those changes resulted in an incorrect printed problem. Counterexamples that were received were by Mark D. Meyerson and John Cobb. Following are: (1) the counterexample submitted by Meyerson, (2) the proposer's original proposal, and (3) the proposer's solution of his original proposal.

Counterexample by Mark D. Meyerson, U.S. Naval Academy. The claim is false. We give a counterexample in R^3 which employs the pseudoarc of Knaster [**1**] (see also Moise [**3**] and Bing [**2**]). The pseudoarc is a limit of increasingly "crooked" arcs which is hereditarily indecomposable (i.e., no subcontinuum is the union of two proper subcontinua). Combining open arcs (for each positive integer n) which are better and better approximations to the pseudoarc and which connect $(0,0,1/n)$ to $(0,1,1/n)$ in the plane $z=1/n$, with the segments from $(0,0,1/n)$ to $(0,0,1/(n+1))$ if n is odd and $(0,1,1/n)$ to $(0,1,1/(n+1))$ if n is even, we can form an open arc A such that $\mathrm{cl}(A)-A$ is the pseudoarc (lying in the plane $z=0$) together with the point $(0,1,1)$. Since the pseudoarc is hereditarily indecomposable it contains no arc. Also, since $\mathrm{cl}(A)$ contains a nonpoint continuum which contains no arc, $\mathrm{cl}(A)$ is not homeomorphic to a segment or to a circle.

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2. R. H. Bing, Snake-like continua, *Duke Math. J.*, 18 (1951) 653–663.
3. E. E. Moise, An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua, *Trans. of the AMS*, 63 (1948) 581–594.

Original statement of the problem. Let A be a subset of R^p, the p-dimensional Euclidean space, that is homeomorphic to R^1. Let $\bar{A}$ be the closure of A in R^p and suppose that $\bar{A}-A$ contains at most countably many points. Show that $\bar{A}$ is homeomorphic to one of R^1, $[0,\infty)$, $[0,1]$, and the unit circle.

Solution by the proposer. Let f: $(0,1)\to A$ be any homeomorphism. We first show that, if $x\in\bar{A}-A$, then either $\lim_{t\to 0}f(t)=x$ or $\lim_{t\to 1}f(t)=x$.

The following facts are easily proved.

Suppose, by way of contradiction, $p_1 \geq n+1$. Therefore, $p_i \geq n+i$ so

$$\frac{\sigma(m)}{m} = \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^{\alpha_1}}\right) \cdots \left(1 + \frac{1}{p_n} + \cdots + \frac{1}{p_n^{\alpha_n}}\right)$$
$$< \left(\sum_{i=0}^{\infty} \frac{1}{p_1^i}\right) \cdots \left(\sum_{i=0}^{\infty} \frac{1}{p_n^i}\right) = \prod_{j=1}^{n}\left(1 + \frac{1}{p_j - 1}\right)$$
$$< \prod_{j=1}^{n}\left(1 + \frac{1}{n+j-1}\right) = \frac{2n}{n} = 2.$$

Therefore, $\sigma(m)/m < 2$, a contradiction. Therefore, the supposition $p_1 \geq n+1$ must be false, i.e., $p_1 \leq n$, as claimed.

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A New Sum for n^2

E 3106 [1985, 590]. *Proposed by Donald E. Knuth, Stanford University.*

Let $S(n)$ be the set of all positive integers k such that the fractional part of n/k is $1/2$ or more. For example,

$$S(17) = \{2, 3, 6, 9, 10, 11, 18, 19, 20, \ldots, 34\}.$$

Prove that

$$\sum_{k \in S(n)} \phi(k) = n^2,$$

where ϕ is Euler's totient function.

Solution by Pamela Y. C. Fong (student), Occidental College, Los Angeles, CA. Let $\{x\} = x - [x]$ denote the fractional part of x. Then $k \in S(n)$ if and only if $\{n/k\} \geq \frac{1}{2}$. If $k > 2n$, then $n/k < \frac{1}{2}$, so $k \notin S(n)$. Therefore $k \in S(n)$ implies

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so

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When $f(n) = \Lambda(n)$, Mangoldt's function, we have

$$g(n) = \sum_{k=1}^{n} \Lambda(k)\left[\frac{n}{k}\right] = \log n!.$$

So

$$g(2n) - 2g(n) = \log\frac{(2n)!}{n!n!} = \log\binom{2n}{n},$$

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Partitioning the Unit Square

E 3129. *Proposed by R. Kannan, Carnegie-Mellon University; D. J. Kleitman, Massachusetts Institute of Technology; and J. C. Lagarias, AT & T Bell Laboratories.*

Let $f(n)$ denote the maximum number of disjoint rectangles that the unit square in $\mathbb{R}^2$ can be partitioned into such that any horizontal line in $\mathbb{R}^2$ intersects the interior of at most n rectangles and any vertical line in $\mathbb{R}^2$ intersects the interior of at most n rectangles.

(a) Show that

$$3 \cdot 2^{n-1} - 2 \le f(n) < 3^{n+3}.$$

(b)* Prove or disprove that $f(n) = 3 \cdot 2^{n-1} - 2$, i.e., that $f(n)$ is determined by $f(1) = 1$ and the recurrence

$$f(n) = 2f(n-1) + 2.$$

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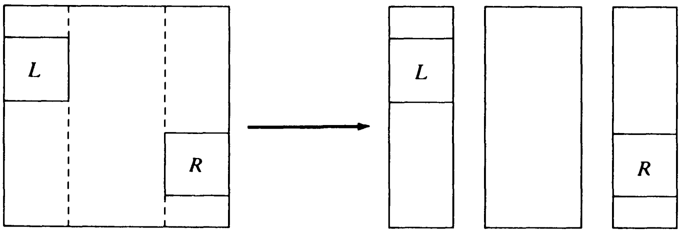

FIG. 1

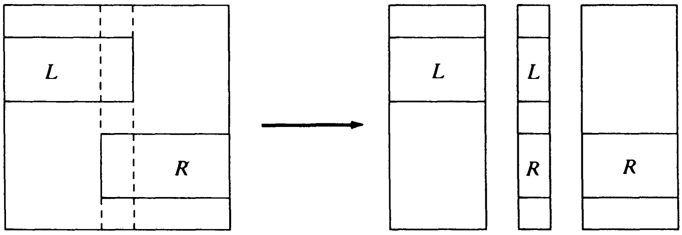

FIG. 2

Case b) $w(L) + w(R) > 1$. Again we define left, right, and middle pieces by the extremes of L and R; this time L and R each extend across the middle piece (cf. FIG. 2).

As in the preceding case, the number of rectangles in the left piece or right piece is bounded by $f(m-1, n-1) + 1$. The middle satisfies (m, n), however if we collapse both the remnants of L and R the middle satisfies $(m, n-2)$. However, the 2 collapsed remnants were already counted as part of L and R in the left and right pieces. Hence we have $f(m, n) \leq 2f(m-1, n-1) + 2 + f(m, n-2)$ in this case. Thus the lemma is proved.

Editorial Note: The lower bound construction also extends to the general (m, n) condition, yielding $f(m, n) \geq 2f(m-1, n-1) + 2$, which has the solution $f(m, n) \geq (m-n+3)2^{n-1} - 2$, given $m \geq n$ and $f(m, 1) = m$. Furthermore, it is easy to prove $f(m, 2) = 2m$.

Professor Kleitman has informed us that for large n it is possible to partition the unit square into more than $(6/5)^n$ disjoint rectangles (with sides parallel to those of the unit square) in such a way that any line whatsoever in the plane intersects the interior of at most n of these rectangles.

Also solved by the proposers.

ADVANCED PROBLEMS

6554. *Proposed by Boo Rim Choe (student), University of Wisconsin, Madison.*

Let E be the Cantor set. It is well-known that $E + E = [0, 2]$, where $E + E = \{x + y\colon\ x \in E,\ y \in E\}$. (Cf. R. P. Boas, Jr., *A Primer of Real Functions*, § 20.)

Define h: $[0,1] \to [0,1]$ as follows:

$$h(s) = \sup\{y\colon y + x = 2s;\ x, y \in E\}.$$

(1) Prove that h is a Borel function and compute its mean value.
(2) Given $s \in E$, determine the value of $h(s)$ in terms of its ternary expansion.
(3) Find all points of continuity of h.
(4) Prove that h is not of bounded variation.

6555. *Proposed by Moshe Laub, Jerusalem, Israel.*

Let B be the set of natural numbers n such that $\sigma(2n+1) \geqslant \sigma(2n)$, where $\sigma(k)$ is the sum of the positive integral divisors of k. Let $\beta(x)$ be the number of elements of B not exceeding x. Prove that there exist constants λ and μ with $0 < \lambda < \mu < 1$ such that

$$\lambda x < \beta(x) < \mu x$$

for all large positive x.

6556. *Proposed by N. J. Fine, Deerfield Beach, Florida.*

(a) Consider a random walk around the edges of a square, where the probability of moving from a given vertex to either of the two adjacent vertices is $1/2$. Suppose the walk stops as soon as all edges have been traversed. Find the expected path-length.

*(b) Consider a random walk around the edges of a cube, where the probability of moving from a given vertex to any one of the three adjacent vertices is $1/3$. Find the expected path-length needed to traverse all edges.

*(c) Similarly with the frame of the n-dimensional cube, where each probability is $1/n$.

SOLUTIONS OF ADVANCED PROBLEMS

6507 [1986, 65]. *Proposed by David Callan, University of Bridgeport, Connecticut.*

Let P be an $r \times r$ stochastic matrix. It is known that P^n is Cesàro summable to a matrix A, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k = A, \text{ and } Z = (I - (P - A))^{-1}$$

exists.

Find $\min\{\operatorname{tr}(Z)\}$ and when it occurs.

Solution by Thomas N. Delmer, Advanced Computer Solutions, La Jolla, California, and Western Maryland College Problem Group, Westminster, Maryland (independently). The minimum value for $tr(Z)$ is $(r+1)/2$, and is attained if and

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6509 [1986, 133]. *Proposed by Boo Rim Choe, University of Wisconsin at Madison.* Let $D = \{r_i\}_1^\infty$ be a countable dense subset of $\mathbb{R}$ and $\{a_n\}_1^\infty$ a sequence of positive real numbers. If $\lim_{n\to\infty} a_n = 0$, show that a sequence f_n: $\mathbb{R} \to \mathbb{R}$ of positive continuous functions can be constructed such that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} +\infty & \text{if } x \notin D, \\ a_m^{-1} & \text{if } x = r_m \in D. \end{cases}$$

Also, what can be said if a_n does not converge to zero?

Solution by San Bernardino Problem Solving Group, California State University at San Bernardino. Since $a_n \to 0$, the function

$$g(x) = \begin{cases} 0 & \text{if } x \notin D \\ a_m & \text{if } x = a_m \in D \end{cases}$$

is in Baire class 1. Thus, we can pick a sequence of continuous functions $g_n \to g$. Let $\{h_n\}$ be the sequence defined by $h_n(x) = \max\{g_n(x), 1/n\}$. Then $f_n = 1/h_n$ is the desired sequence. If $\{a_n\}$ does not converge to 0, the function g above is in Baire class 2, so we can pick the functions f_n to be in Baire class 1.

What if we remove the hypotheses that D is dense, and that $a_n \to 0$? Here James Munkres proved a penetrating result:

THEOREM. *Let A denote the set $\{a_n\}_1^\infty$. Given $\varepsilon > 0$, let*

$$D_\varepsilon = \{r_i | a_i > \varepsilon\}.$$

The following statements are equivalent:

(a) The desired sequence of functions f_n exists.

(b) For each $\varepsilon > 0$ with ε not belonging to A, the set D_ε is a G_δ set in $\mathbb{R}$.

(c) There is a sequence ε_n of positive numbers converging to zero, such that for each n, the set D_{ε_n} is a G_δ set in $\mathbb{R}$.

Now if we assume a_n converges to zero, then D_ε is finite for each $\varepsilon > 0$ and it follows that the sequence f_n exists. On the other hand, if D is dense in $\mathbb{R}$ and if $a_n > \varepsilon$ for all n, then D_ε is not a G_δ set in $\mathbb{R}$ and the sequence f_n does not exist.

In connection with Baire classes, the text *Real Functions*, by Casper Goffman (Holt, Rinehart, Winston, 1964), was cited by several readers.

Also solved by Robert Bernstein, F. S. Cater, George Crofts, A. K. Desai (India), Ellen Hertz, Roger Kraft, O. P. Lossers (The Netherlands), A. Meir (Canada), Victor Pambuccian (Romania), Nicholas Passell, Daniel M. Rosenblum, Joanne R. Snow, R. Vitale, David M. Wells, and the proposer.

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Let G be a finite group with nontrivial cyclic Sylow 2-groups. Prove that the product of all the elements of G never equals the identity. (Generalization of Wilson's Theorem.)

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Let G be a finite group with nontrivial cyclic Sylow 2-groups. Prove that the product of all the elements of G never equals the identity. (Generalization of Wilson's Theorem.)

Solution by Duane Broline, Eastern Illinois University, Charleston, IL. We prove the following generalization. (The special case when G is abelian is familiar.)

THEOREM. *Let G be a finite group with nontrivial cyclic Sylow 2-groups. Let j be an involution in G. If g is an element obtained by taking the product of all the elements of G in some order, then $g \in jG'$, where G' is the commutator subgroup of G.*

Proof. Let S be a Sylow 2-subgroup of G containing j. By a theorem of Burnside (see D. Gorenstein, *Finite Groups*, Theorem 7.6.1) there is a normal subgroup N of G such that $G = NS$ and $N \cap S = 1$. In particular $|N| = [G : S]$ is odd. Since G/N is isomorphic to S and is therefore abelian, $G' \subseteq N$. Hence $jG' \neq G'$ and $1 \notin jG'$.

Since all Sylow 2-subgroups are cyclic, each one contains a unique involution. Since all Sylow 2-subgroups are conjugate, all involutions in G are conjugate. Let n be the number of involutions in G, i.e., the number of conjugates of j. Then $n = [G : C_G(j)]$, where $C_G(j)$ is the centralizer of j in G. But $S \subseteq C_G(j)$ and so n divides $[G: S] = |N|$. Thus n is odd. Let $A = \{x \in G: o(x) \neq 2\}$ and let B be the set of all involutions in G, where we have just seen that $|B| = n$. Suppose g is the product of the elements of G in some order. Since G/G' is abelian and A is closed under taking inverses, we have

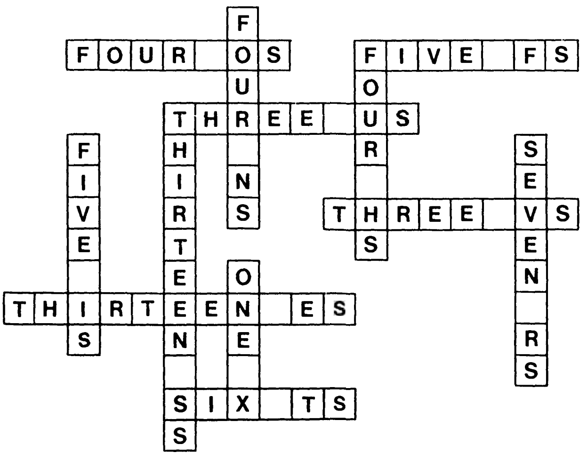
$$gG' = \prod_{x \in G}(xG') = \prod_{x \in A}(xG')\prod_{x \in B}(xG') = \prod_{x \in B}(xG').$$

Since $y^{-1}jyG' = jG'$ for any y in G, we have $\prod_{x \in B}(xG') = (jG')^n = j^nG' = jG'$. Thus $gG' = jG'$, so that $g \in jG'$, as asserted.

Also solved by F. Rudolf Beyl, Ze-Li Dou, Thomas Jager, Michael Josephy (Costa Rica), David W. Koster, O. P. Lossers (The Netherlands), I. D. Macdonald, David E. Manes, Daniel Neuenschwander (Switzerland), Harvey Schmidt, Jr., Z. Z. Uoiea, Gary L. Walls, Ken Yanosko, and the proposer.

SOLUTION TO SELF-ENUMERATING CROSSWORD

(Aug.-Sept. MONTHLY, p. 666)



REVIEWS

EDITED BY JOSEPH KONHAUSER
Department of Mathematics, Macalester College, St. Paul, MN 55105

Abstract Algebra. By I. N. Herstein. Macmillan Publishing Company, 1986. vii + 289 pp.

GEORGIA BENKART
Mathematics Department, University of Wisconsin, Madison, WI 53706

Ever since it was first published twenty-three years ago, I. N. Herstein's *Topics in Algebra* has set a standard for texts on abstract algebra. A whole generation of textbooks and an entire generation of mathematicians, myself included, have been profoundly influenced by that text. (I never would have dreamt in my undergraduate days as I struggled in frustration with some of the book's exercises that years later, as a practicing algebraist, I would be asked to review its descendant, *Abstract Algebra.*) Since 1964 there have been a host of texts written on the subject of abstract algebra. Too many of those books have diluted the subject to the point that students arrive for graduate work ill-prepared to tackle the rigors of a graduate algebra course. Another recent trend has been to include some of abstract algebra's beautiful applications to codes, lattices, geometries, and crystallographic groups to help instructors and students answer the proverbial question, "What's all this theory good for?" and to meet the growing demand for training in applied algebra. The book under review, Herstein's *Abstract Algebra*, follows neither of these trends. It is abstract algebra—rigorous and pure.

Abstract Algebra is neither a remodeling nor a revision of the classic, *Topics in Algebra*. Rather, it is a new text on groups, rings, and fields intended to be less inclusive and more informal than *Topics*. The tone is more chatty, paralleling in its presentation a classroom discussion. Clear, concise exposition, which has been the hallmark of Herstein texts, is at the heart of this new one. The student is sure to sense that behind all the theory presented is a mathematician trying above all else to communicate his subject and in the process to convey his great enthusiasm for it. Years of experience teaching algebra courses undoubtedly have made the author very sensitive to the difficulties that the student is likely to encounter. Care is taken to head mistakes off at the pass by presenting nonexamples and discussing common misconceptions. The reader receives stern warnings such as the exhortation, "Avoid a mathematical stomachache later by assimilating this section well," but frequently sympathy is extended, too.

Nowadays the umbrella of modern algebra covers a wide diversity in both course content and level. To achieve greater flexibility, *Abstract Algebra* is designed for courses at three different levels of difficulty, and an outline of what might be included in each course is provided in the Instructor's Manual. To accommodate this scheme further, problems are divided into three categories—easier, middle-level, and harder—except for an occasional section where the exercises are intentionally not rated in order to give students experience in working problems of an unknown

algorithm is not presented for polynomials or for integers, the reader is left to wonder if every greatest common divisor must be so divined. The useful formula for computing the product $\tau\sigma\tau^{-1}$ for τ, σ in the symmetric group is relegated to the special topics section at the end, where it might be omitted for lack of time. Such concrete calculations would nicely complement the text material, have a practical value, and afford a respite from the abstract theory.

These points aside, it should be stated that throughout the text care is taken to achieve interesting, applicable, and significant results about each of the fundamental algebraic systems studied—groups, rings, and fields. Constructibility and the special topics at the end, especially the irrationality of π, provide entertaining sidelights to the main topics. A real attempt has been made to prepare the ground for new concepts, to give many varied examples, and to present a wide range of challenging problems (662, by the author's count). There is more than enough material for a one-semester course. The material on fields could be postponed for a second course, where it could be supplemented with topics from linear algebra such as matrices, vector spaces, transformations, and bilinear forms. No background in calculus or linear algebra is assumed, but certainly the level of sophistication of the text and its problems demands much mathematical maturity from its readers.

Perhaps before closing I should issue a few warnings of my own. The definition of left coset in a group is the one usually given for right coset—a departure from the convention in *Topics*, for example. Another departure is that an integer is defined to be a prime number $p > 1$ if for any integer a, either $p|a$ or p is relatively prime to a. From this vantage, the definition of prime depends on the existence of the greatest common divisor. Although it is pointed out that every such prime is a prime in the conventional sense, the converse is not discussed, nor is there any explanation of why this unusual approach is taken. This, however, is really atypical of the text, which makes every effort to alert its audience to what lies ahead.

Abstract Algebra has much to offer in its readability and clarity. Both the author and the publisher, Macmillan, should be commended on that score. The examples and exercises enhance and enrich the fine exposition and are valuable resources in themselves. There are a few logistical difficulties which undoubtedly will be ironed out by the time the second edition rolls off the press, as I expect this Herstein book, too, will be around a long time to influence a new generation of mathematicians and another generation of textbooks.

The Fascination of Statistics. Edited by Richard J. Brook, Gregory C. Arnold, Thomas H. Hassard, and Robert M. Pringle. Marcel Dekker, Inc., New York and Basel, 1986. xi + 433 pp.

JOSEPH GANI
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In their preface, the editors of this book comment that the theory and practice of statistics is essentially a 20th-century phenomenon. They state that it was the wide

algorithm is not presented for polynomials or for integers, the reader is left to wonder if every greatest common divisor must be so divined. The useful formula for computing the product $\tau\sigma\tau^{-1}$ for τ, σ in the symmetric group is relegated to the special topics section at the end, where it might be omitted for lack of time. Such concrete calculations would nicely complement the text material, have a practical value, and afford a respite from the abstract theory.

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In their preface, the editors of this book comment that the theory and practice of statistics is essentially a 20th-century phenomenon. They state that it was the wide

range of applications of statistics which motivated them to invite 35 experts, not all of them statisticians, to write for nonexperts on the various aspects of statistics which they found most fascinating.

The authors are from six English-speaking countries: one from Australia, four from Britain, one from Canada, 21 from New Zealand, one from South Africa, and seven from the USA; 13 of the 21 New Zealanders hail from Massey University, four of the seven Americans from North Carolina State University, and all those in Britain from the University of London—hardly, one would think, a balanced collection. Despite this, the book gives a judicious and readable account of the application of statistics to a wide spectrum of problems, most of them accessible to the nonexpert numerate reader.

The contents are divided into seven sections, each corresponding to an important aspect of modern statistics: these are I. Probability, II. Condensing complex data, III. Hypothesis testing, IV. Estimation, V. Experimental, VI. Prediction, and VII. Modeling. In each, several down-to-earth articles, liberally illustrated with graphs, tables, and cartoons, are presented; among them, by way of examples (in the order of the sections), are the following:

I. "Probabilities, meeting and mating," by Barry Singer, a humorous probabilistic account of the difficulties encountered in the search for a marriage partner;

II. "Finders keepers," by Alan Tyree, a statistical study of the similarity of legal "finders cases" in which a dispute arises as to the rights of the finder of an object, as against those of the owner of the premises where the object was found;

III. "Unraveling DNA information," by Bruce Weir, a discussion of the variation in 45 out of 900 sites in nucleotide sequences of mitochondrial DNA in seven individuals;

IV. "Who still comes to the races," by Richard LeHeron, a description of the sample survey carried out for the New Zealand Racing Authority to account for the decline in the popularity of race meetings;

V. "Randomization: a historic controversy," by Gregory Arnold, a survey of the arguments about randomization, illustrated by the correspondence between A. W. Hudson, a crop experimentalist in New Zealand, W.S. Gosset, the statistician better known as "Student," and K. Mather, then of the Galton Laboratory;

VI. "Predicting earthquakes," by David Rhoades, a preliminary analysis of the predictive value of the method of precursory swarms for earthquakes;

VII. "Forecasting visitor arrivals to New Zealand: an elementary analysis," by Peter Thomson, a time series forecasting exercise on the number of arrivals to be expected in New Zealand between 1982 and 1986, based on month-to-month visitor data from 1956 to 1982.

The editors have successfully achieved their objective of providing students and teachers of statistics, statisticians and numerate readers, with an overview of the many uses of statistics, the variety of statistical techniques, and the scope of statistical thinking. Everyone will find something of interest in this collection; its easy style and clear explanations could (dare one suggest it?) well serve as a model to writers of more technical statistical papers. The contents of the book raise two

important questions: the first is a matter of education, the second concerns the further development of statistics in the next decade.

If, as the book clearly indicates, statistical thinking permeates practically every area of daily life, how are we to inculcate the ordinary citizen with the elements of statistics and their practical use? Clearly some effort must be made to include statistics in primary and secondary education, in much the same way as arithmetic and elementary mathematics are. Statistical societies and teachers' associations in many countries have tried to achieve this end; it has also been one of the aims of the International Statistical Institute since 1949, through its Statistical Education Committee, and in particular the Committee's Task Force on the teaching of statistics at the school level. The endeavor is a major one, requiring the cooperation of parents, teachers, and educational authorities, but some progress has been made over the past thirty years.

Pamphlets and textbooks on statistics suitable for various children's age-groups have been produced by concerned individuals and groups in diverse countries. A survey, "The Teaching of Statistics in Schools Throughout the World," edited by V. Barnett, was also published by the International Statistical Institute in 1982. There have already been two international conferences on the teaching of statistics, ICOTS 1 in 1982 held at Sheffield, England, and ICOTS 2 in 1986 held at Victoria, BC, Canada; both have drawn about 500 international participants, and ICOTS 3 is planned for Dunedin, New Zealand, in 1990. It will be no easy matter to incorporate statistics into the school curriculum, particularly since only a few teachers of mathematics or the physical and social sciences have the necessary training to teach the subject. Nevertheless, the logic of necessity is making itself felt, and statistics, like computing, will gradually find its way into the school curriculum.

The second question is the direction in which one might expect new statistical developments to occur. To this there are probably as many answers as there are statisticians; I shall avail myself of this opportunity to put forward a personal conjecture. Fisher, in his "Statistical Methods for Research Workers" (1925), characterized statistics as the study of populations, variation, and methods for the reduction of data. When he began his research, and for several decades after, most statistical data (in agriculture, for example) was assumed to have been collected at a single instant in time; the main exception to this was time series data. The statistical methods developed to reduce and analyze such data, and the inferential procedures used, reflected this assumption.

Much statistics today remains based on the selfsame assumption, as the majority of the articles in the book under review indicate. The development of stochastic process models, with their emphasis on variation in time, came later as a post-Fisherian phase, and it was their mathematical properties which were first investigated. Only recently have inferential methods been applied systematically to this field. My belief is that time-dependent processes (including time series) have become an increasingly important focus of research over the past 20 years, and statistical developments will, therefore, concentrate increasingly on stochastic modelling and on inference for stochastic processes. Classical statistics, while retaining

its value, will gradually give way to the study of time-dependent models; in the present book, for example, such models are described for pattern development, weather trends, sequential experimentation, record times, suicide prediction, earthquake prediction, visitor forecasts, epidemic models, the stock market, and floods.

Naturally, the established methods of classical statistics are, in part, extendable to the new models, but other methods are also emerging, of particular relevance to the special stochastic processes to which they are applied. It is interesting to note that, in recent years, many new ideas in time series analysis have been contributed by electrical engineers interested in very specific models, and that advances in diffusion processes have been made by geneticists studying the fixation of a gene. We must remain alert to these new and important scientific developments and not allow our comfortable familiarity with classical statistics to blind us to the importance of novel ideas. Many of these are being triggered by the wealth of stochastic processes studied in biology, engineering, physical science, and the social sciences.

I shall look forward to seeing whether statistical trends between 1987 and 1997 verify my conjecture; but irrespective of whether I am right or not, *Fascination of Statistics* deserves to be read by all who enjoy statistics and its multitudinous applications.

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatibilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

Mathematics Appreciation, S*(13-15), L*. *Knotted Doughnuts and Other Mathematical Entertainments.* Martin Gardner. WH Freeman, 1986, xiii + 278 pp, $10.95 (P). [ISBN: 0-7167-1799-9] The eleventh collection of reprints of Gardner's well-known columns in *Scientific American.* Includes Newcomb's paradox, worm paths, Laffer curves, polycubes, Gray codes, and fifteen other fascinating topics. As usual, each selection contains an addendum with commentary based in part on correspondence generated by the original columns. LAS

Precalculus, T(13: 1). *College Algebra, Third Edition.* Bernard J. Rice, Jerry D. Strange. Prindle Weber & Schmidt, 1986, viii + 408 pp. [ISBN: 0-87150-043-4] Emphasizes the function concept and graphing throughout; new to this edition is an emphasis on finite mathematics; addition of matrix algebra, expansion and rearrangement of the material on sequences, counting, probability and interest, more business-oriented problems, and more review exercises. (TR, *First Edition,* August-September 1978.) JNC

Precalculus, T(13: 1). *College Algebra.* David Cohen. West, 1986, ix + 513 pp. [ISBN: 0-314-93164-3] Conversational style with an emphasis on using the language of algebra as a tool in problem solving. Outstanding collection of exercises ranging in difficulty from very easy (carefully correlated with worked examples in that section), to more challenging precalculus or calculator types. Review exercises and end-of-chapter tests. A final chapter on topics from discrete mathematics (sequences, counting). LCL

Logic, T(18: 2), P. *Algebraic Recursion Theory.* L.L. Ivanov. Math. & Its Applic. Halsted Pr, 1986, 256 pp, $59.95. [ISBN: 0-470-20725-6] Introduction to fundamentals of modern recursion theory expanding on the work of Skordev and Kleene. The central object of study is the iterative operative space (IOS), a partially-ordered semigroup (e.g., monotonic unary maps over a Skordev space), with certain binary operations, and, most importantly, a μ-induction axiom. This allows one to generalize the definition of effective computability. Generally detailed exposition with some exercises. MR

Foundations, T*(16-18: 1, 2), S, P, L. *Introduction to Mathematical Logic, Third Edition.* Elliott Mendelson. Wadsworth, 1987, ix + 341 pp, $38.95. [ISBN: 0-534-06624-0] This edition represents fine-tuning in exposition and pedagogy (e.g., changes in emphasis, arrangement of topics), plus some new material (e.g., new sections on semantic trees, axiomatic set theories, enlarged sections on recursion theory). There are more exercises, and the bibliography has been expanded. (*Second Edition,* TR, April 1980.) LCL

Foundations, S(16-18), P, L. *An Introduction to the Theory of Surreal Numbers.* Harry Gonshor. Math. Soc. Lect. Note Ser., V. 110. Cambridge U Pr, 1986, 192 pp, $19.95 (P). [ISBN: 0-521-31205-1] Surreal numbers, discovered and used by J.H. Conway to think about games, form a proper class which contain the real numbers and the ordinal numbers among other things. This book provides a systematic introduction to surreal numbers, based on foundations that are familiar to most mathematicians (equivalence classes of inductively defined objects). LCL

Linear Algebra, T(14-15). *Introduction to Linear Algebra and Differential Equations.* John W.

Dettman. Dover, 1986, xi + 404 pp, $8.95 (P). [ISBN: 0-486-65191-6] Well-written, concise introduction to linear algebra (over C or R) and differential equations. Includes optional sections on infinite dimensional vector spaces, Jordan forms, existence and uniqueness theorems for differential equations and Green's functions. Examples, exercises, and selected solutions. (1974 McGraw-Hill edition, TR, May 1974.) LC

Algebra, T(15-16: 2), L. *Undergraduate Algebra: A First Course.* C.W. Norman. Clarendon Pr, 1986, xiii + 416 pp, $39.95. [ISBN: 0-19-853249-0] Not really comparable to any current American text for a similar audience, the treatment here develops the elementary theory of rings and fields followed by a fairly thorough approach to linear algebra. Includes chapters on diagonalization and duality, Euclidean and unitary space. Some exercises, no answers, index. JS

Algebra, T(15-18), S, P, L*. *Introduction to Finite Fields and Their Applications.* Rudolf Lidl, Harald Niederreiter. Cambridge U Pr, 1986, viii + 407 pp, $29.95. [ISBN: 0-521-30706-6] A textbook edition of the authors' monograph *Finite Fields* which appeared in 1983 as Volume 20 of the *Encyclopedia of Mathematics and Its Applications* (TR, April 1987). This edition has a stronger emphasis on applications (cryptography, coding theory, pseudorandom sequences), and less material on theoretical topics (exponential sums, equations over finite fields). Especially suitable as a textbook for a senior seminar. LCL

Algebra, T(16: 1), L. *A Course in Galois Theory.* D.J.H. Garling. Cambridge U Pr, 1986, xiii + 167 pp, $39.50; $14.95 (P). [ISBN: 0-521-32077-1; 0-521-31249-3] Introduction to Galois theory, beginning with field extensions, automorphisms and then the fundamental theorem of Galois theory. Also covers finite fields, solubility by radical, and transcendence. Assumes familiarity with groups, rings, and fields although a brief "refresher" course is given in the first 36 pages. LC

Calculus, T*(14: 1, 2). *Calculus of Several Variables, Third Edition.* Serge Lang. Undergrad. Texts in Math. Springer-Verlag, 1987, xii + 598 pp, $48. [ISBN: 0-387-96405-3] For one or two semesters to follow author's *A First Calculus Course* (or equivalent text). Excellent balance between touches of rigor and appeals to intuition. Sufficient linear algebra for course needs. This edition contains additional worked-out examples and expanded answers to some exercises. Lean but complete. Clear and uncluttered appearance. Unintimidating but authoritatively written with clarity and style. (*First Edition,* TR, October 1974; *Second Edition,* TR, June-July 1979.) JK

Calculus, S(13). *Calculus, Exam File.* Ed: Eric M. Lederer, *et al.* Engineering Pr, 1986, $9.95 each (P). *I,* v + 250 pp [ISBN: 0-910554-61-7]; *II,* v + 282 pp [ISBN: 0-910554-62-5]; *III,* v + 282 pp. [ISBN: 0-910554-63-3] Each volume consists of over 300 problems, which are given exactly as they appeared on an exam, and the solution in the teacher's own handwriting. The three volumes are designed to accompany the standard three-semester sequence. About 25 teachers have collaborated on each volume. CEC

Real Analysis, T(16: 1, 2), S, L. *Real and Functional Analysis, Part A: Real Analysis, Second Edition.* A. Mukherjea, K. Pothoven. Math. Conc. & Meth. in Sci. & Eng., V. 27. Plenum Pr, 1984, viii + 364 pp. [ISBN: 0-306-41557-7] A beginning graduate-level introduction to measure-theoretic real analysis through theorems relating integral and derivatives. Begins with very complete introduction to topology and measure theory. Readable and detailed, yet not verbose. Many exercises. (*First Edition,* TR, October 1978.) PZ

Complex Analysis, T(18: 2, 3), S, P. *Holomorphic Functions and Integral Representations in Several Complex Variables.* R. Michael Range. Grad. Texts in Math., V. 108. Springer-Verlag, 1986, xix + 386 pp, $49.50. [ISBN: 0-387-96259-X] A general introduction to complex analysis of several variables, emphasizing integral formulas and their applications to global behavior of complex analytic objects. Self-contained—basic local theory is developed fully in the text. Readable and inviting, with extensive historical commentary throughout. Many exercises. PZ

Differential Equations, P. *The Lefschetz Centennial Conference: Proceedings on Differential Equations.* Ed: A. Verjovsky. Contemp. Math., V. 58.III. AMS, 1987, ix + 253 pp, $29 (P). [ISBN: 0-8218-5064-4] Final volume of the three-part Lefschetz conference held in Mexico City in December 1984. First two (TRs May 1987 and October 1987) feature algebraic geometry and algebraic topology. LAS

Partial Differential Equations, P. *Lecture Notes in Mathematics-1218: Schrödinger Operators, Aarhus 1985.* Ed: E. Balslev. Springer-Verlag, 1986, 222 pp, $19.40 (P). [ISBN: 0-387-16826-5] A collection of papers based on lectures given at a symposium on Schrödinger operators held at the Institute of Mathematics at Aarhus University, October 2-4, 1985. JAS

Partial Differential Equations, P. *Introduction to Microlocal Analysis.* M. Kashiwara. L'Enseignement Math, 1986, 37 pp, (P). Survey lectures given at the University of Bern in June 1984. JAS

Numerical Analysis, S(17-18), P. *Matrizen und ihre Anwendungen für Angewandte Mathematiker, Physiker und Ingenieure, Teil 2: Numerische Methoden.* Rufold Zurmühl, Sigurd Falk. Springer-Verlag, 1986, xv + 476 pp, DM118. [ISBN: 0-387-15474-4] Second volume of *Fifth Edition.* Extensive treatment

of numerical methods for solving systems of linear equations and finding characteristic values. JD-B

Numerical Analysis, T(15: 1), S, P, L. *Microcomputer Modelling by Finite Differences.* Gorden Reece. Halsted Pr, 1986, x + 126 pp, $18.95 (P). [ISBN: 0-470-20739-6] Solutions of specific first and second order differential equations through Basic and finite differences. The book culminates with a program which solves the transient heat conduction problem. This program is then applied to other classical physics problems. CEC

Functional Analysis, T(18: 2, 3), S, P. *Fundamentals of the Theory of Operator Algebras, Volume II: Advanced Theory.* Richard V. Kadison, John R. Ringrose. Pure & Appl. Math., V. 100-II. Academic Pr, 1986, xiv + 675 pp, $79.50; $39.50 (P). [ISBN: 0-12-393302-1] Theory of algebras of linear operators (bounded and unbounded). Builds on basics developed in *Volume I.* Comprehensive and self-contained. With hundreds of exercises, many of which "constitute small...research projects." PZ

Functional Analysis, S(18), P. *Stone Spaces.* Peter T. Johnstone. Stud. in Adv. Math., V. 3. Cambridge U Pr, 1986, xxi + 370 pp, $24.95 (P). [ISBN: 0-521-33779-8] Paperback edition of the 1982 edition (TR, February 1985). BH

Functional Analysis, P. *Sobolev Spaces of Infinite Order and Differential Equations.* Julij A. Dubinskij. Math. & Its Applic. D Reidel, 1986, 161 pp, $48. [ISBN: 90-277-2147-5] Development and applications of boundary value problems of infinite order and the theory of Sobolev spaces of infinite order which are the "energy" spaces of the corresponding problems. Up-to-date with a good bibliography. JK

Functional Analysis, S(15-17), P, L. *A Short Course on Functional Equations.* J. Aczél. Theory & Dec. Lib., Ser. B. D Reidel, 1987, vi + 169 pp, $49.50. [ISBN: 90-277-2376-1] An introduction to the most important functional equations and methods of solution, each introduced and based on applications to the social and behavioral sciences. Requires a mathematically mature reader, but otherwise a minimal mathematical prerequisite (multivariable calculus, linear algebra, intermediate real analysis). LCL

Analysis, T(17-18: 1-3), S, P, L. *Real and Complex Analysis, Third Edition.* Walter Rudin. McGraw-Hill, 1987, xiv + 416 pp, $44.95. [ISBN: 0-07-054234-1] This *Third Edition* retains familiar features of earlier editions: stress on connections between real and complex analysis; brisk, lucid exposition; challenging exercises. Treatment of differentiation differs from earlier versions, beginning from definition and properties of "maximal functions" of complex measures. (*Second Edition,* TR, April 1974.) PZ

Analysis, P. *K-Theory for Operator Algebras.* Bruce Blackadar. Math. Sci. Res. Inst. Pub., V. 5. Springer-Verlag, 1986, 338 pp, $28. [ISBN: 0-387-96391-X] Exposition of the basic theory which has been developed over the last ten years. Extensive bibliography including basic texts and a reasonable index. Emphasis is on basic concepts with only a brief summary of applications. JAS

Algebraic Geometry, P. *Diagonal Complexes and F-gauge Structures.* Torsten Ekedahl. Hermann, 1986, 120 pp, 160F (P). [ISBN: 2-7056-6045-3] Solutions to several problems in crystalline cohomology of algebraic varieties. JAS

Geometry, S, P, L. *The Ancient Tradition of Geometric Problems.* Wilbur Richard Knorr. Birkhauser Boston, 1986, ix + 411 pp, $69. [ISBN: 0-8176-3148-8] Focuses on efforts to solve the three "classical problems" while surveying the "nature and development of this ancient tradition of analysis;" emphasizes the mathematical, historical, and philosophical aspects of the ancient writings. JNC

Geometry, S, P, L. *A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller.* Amy C. Edmondson. Birkhauser Boston, 1987, xxvi + 302 pp, $37.50. [ISBN: 0-8176-3338-3] The first in the Design Science Collection which will explore three-dimensional space from the perspectives of the designer, artist, and scientist. This text introduces and presents the tools required for further exploration of synergetics (the study of spatial complexity). Requires no mathematical background. JNC

Algebraic Topology, P. *The Lefschetz Centennial Conference: Proceedings on Algebraic Topology.* Ed: S. Gitler. Contemp. Math., V. 58.II. AMS, 1987, ix + 137 pp, $20 (P). [ISBN: 0-8218-5063-6] Second of three volumes containing papers from the three sections of the December 1984 Lefschetz conference held in Mexico City. The first volume featured algebraic geometry (TR, May 1987), the third (TR, October 1987) differential equations. LAS

Optimization, P. *An Algorithmic Theory of Numbers, Graphs and Convexity.* László Lovász. CBMS-NSF Reg. Conf. Ser. in Appl. Math. SIAM, 1986, iii + 91 pp, $14 (P). [ISBN: 0-89871-203-3] Begins with the basic problem of rounding numbers and the implications of the simultaneous Diophantine approximation algorithm, then covers the ellipsoid method and other algorithms for problems involving convex sets. Ends with survey of some applications to combinatorics (e.g., flows, optimization in perfect graphs, etc.). Emphasis of notes is on ideas and examples, not technical details. LC

Control Theory, P. *Lecture Notes in Mathematics-1216: Bifurcation of Extremals in Optimal Control.* Jacob Kogan. Springer-Verlag, 1986, viii + 106 pp, $12.80 (P). [ISBN: 0-387-16818-4] "The purpose of this work is to investigate under what conditions an extremal trajectory branches or bifurcates

in time." Overview, with examples, followed by full proofs. LCL

Control Theory, P. *A Stochastic Maximum Principle for Optimal Control of Diffusions.* U.G. Haussmann. Res. Notes in Math. Ser., V. 151. Longman Scientific & Technical (US Distr: Wiley), 1986, 109 pp, $31.95 (P). [ISBN: 0-470-20786-8] An organized presentation of results scattered through the literature in addition to some new material; useable in the classroom at an advanced level. LCL

Probability, P. *Lecture Notes in Mathematics-1180: École d'Été de Probabilités de Saint Flour XIV—1984.* R. Carmona, H. Kesten, J.B. Walsh. Springer-Verlag, 1986, x + 439 pp, $32.50 (P). [ISBN: 0-387-16441-3] Three long papers: "Random Schrödinger Operators," by R. Carmona; "Aspects of First Passage Percolation," by H. Kesten; and "An Introduction to Stochastic Partial Differential Equations" by J.B. Walsh. JAS

Probability, P. *Lecture Notes in Mathematics-1203: Stochastic Processes and Their Applications.* Ed: K. Itô, T. Hida. Springer-Verlag, 1986, 222 pp, $18.50 (P). [ISBN: 0-387-16773-0] Proceedings of the international conference held in Nagoya, July 2-6, 1986. JAS

Statistics, P*. *Regression Analysis with Applications.* G. Barrie Wetherill, *et al.* Mono. on Stat. & Prob. Chapman & Hall, 1986, xi + 311 pp, $34.50. [ISBN: 0-412-27490-6] Arising out of work done to write a user-friendly regression program containing validity checks and diagnostics (U-REG), this volume pays particular attention to problems that may occur. Among topics discussed in detail are multicollinearity, outliers, and testing for transformations, normality, and lack of homoscedasticity. RSK

Statistics, P, L.** *Artificial Intelligence and Statistics.* Ed: William A. Gale. Addison-Wesley, 1986, xiv + 418 pp, $38.95. [ISBN: 0-201-11569-7] Primarily contains papers from a workshop held in April, 1985 in Princeton, New Jersey. Deals with both artificial intelligence in statistics, e.g., the use of expert systems, and statistics in artificial intelligence. Excellent overview of the interactions between these two areas. RSK

Statistics, P. *Lecture Notes in Statistics-37: Advances in Order Restricted Statistical Inference.* Ed: R. Dykstra, T. Robertson, F.T. Wright. Springer-Verlag, 1986, viii + 295 pp, $31.70 (P). [ISBN: 0-38-96419-3] Proceedings of a symposium held in Iowa City, Iowa, September 11-13, 1985. JAS

Statistics, S(16), P*. *Akaike Information Criterion Statistics.* Y. Sakamoto, M. Ishiguro, G. Kitagawa. Math. & Its Applic. D Reidel, 1986, xix + 290 pp, $89. [ISBN: 90-277-2253-6] Translation of the original 1983 Japanese edition. Unified new approach to elementary statistical analysis in which the goodness of an assumed model is measured by the Akaike Information Criterion, an extension of the concept of likelihood introduced by H. Akaike in 1971. Note price! RSK

Statistics, T(16-17: 1, 2), P. *The Sequential Statistical Analysis of Hypothesis Testing, Point and Interval Estimation, and Decision Theory.* Z. Govindarajulu. Ser. in Math. & Manage. Sci., V. 5. American Sciences Pr, 1987, ix + 680 pp, $48.50 (P). [ISBN: 0-935950-17-6] Corrected version of the original 1981 edition. Comprehensive treatment, covering both testing and estimation procedures when the number of observations is not fixed in advance. Good set of references. RSK

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Applications (Biology), P. *Positive Feedback in Natural Systems.* D.L. DeAngelis, W.M. Post, C.C. Travis. Biomathematics, V. 15. Springer-Verlag, 1986, xii + 290 pp, $63. [ISBN: 0-387-15942-8] Negative feedback systems (i.e., ones tending to stability) have received considerable study, e.g., in cybernetics. The authors, asserting the importance of positive feedback dynamics, aim here to redress the balance. From the preface: "The present book draws together many...scattered examples to...present a coherent picture of the role of positive feedback in nature." PZ

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Macmillan, 1987, xv + 479 pp. [ISBN: 0-02-318010-2] An attractive, well-organized introductory text on basic mathematical principles of signal processing, especially filtering. The first half is on deterministic signals; the second half, following 100 pages of basic probability theory, treats nondeterministic (noisy) signals. Exercises at the end of each chapter. BC

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Applications (Engineering), P. *Mathematical and Computational Methods in Seismic Exploration and Reservoir Modeling.* Ed: William E. Fitzgibbon. SIAM, 1986, xiv + 277 pp, $32.50. [ISBN: 0-89871-205-X] Ten plenary lectures and summaries of eight mini-symposia from a January 1985 conference in Houston on mathematical methods in petroleum engineering—the first conference of its kind. Contributed papers are not included. LAS

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Applications (Medicine), P. *Recent Developments in Structured Continua.* Ed: D. De Kee, P.N. Kaloni. Res. Notes in Math. Ser., V. 143. Longman Scientific & Technical (US Distr: Wiley), 1986, 283 pp, $49.95 (P). [ISBN: 0-470-20364-1] Eight invited lectures from a May 1985 conference at the University of Windsor dealing with transport phenomena, polymer mechanics, biological fluids, and structured fluids. LAS

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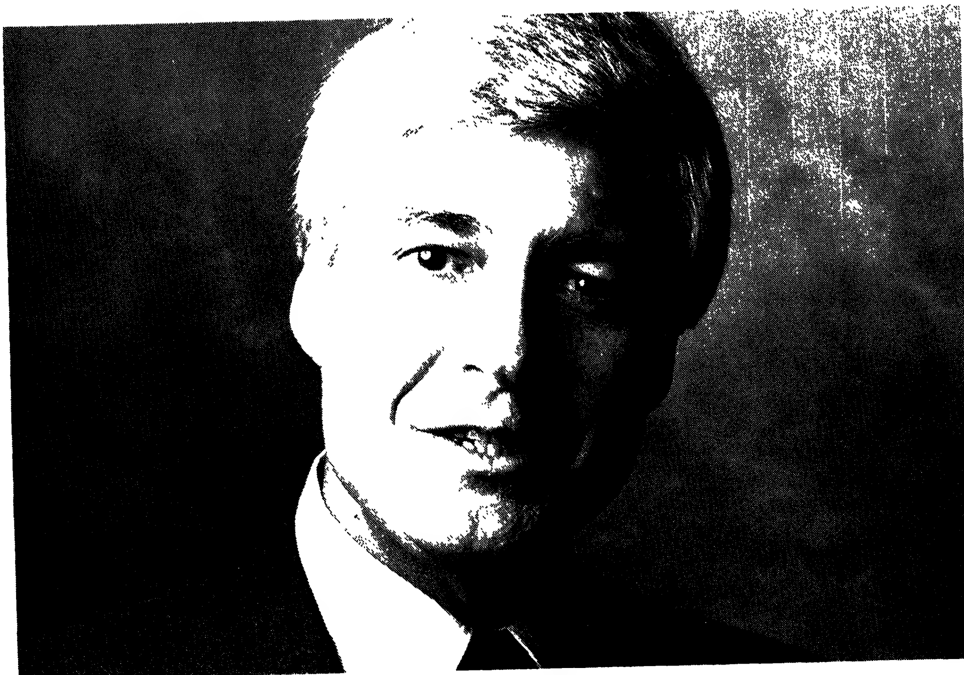

THE AMERICAN
MATHEMATICAL MONTHLY

Volume 94, Number 9 November 1987

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(ISSN 0002-9890)

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Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

Hyperbolic Motions of Conics

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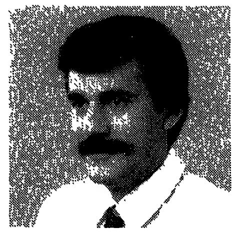
CLAUDIO MENDES studied under Professor Favaro at the University of São Paulo where he received his Ph.D. in 1981. He does research in Singularity Theory and its applications.



MARIA RUAS studied under T. Gaffney at Brown University. After returning to Brazil in 1979, she received her Ph.D. from the University of São Paulo, at São Carlos, under L. Favaro. Her research interests are in Singularity Theory and its applications.

The geometry associated with Einstein's principle of relativity is Lorentzian geometry. In the simplest case, the Lorentzian plane is $\mathbb{R}^2$ with the Lorentz bilinear form $\langle x, y\rangle = x_1y_1 - x_2y_2$. Lorentzian plane geometry consists of the study of properties invariant by the action of the group, called hyperbolic motions, which preserve this bilinear form. In general, a Lorentz geometry is the geometry of an affine space resulting from the existence of a symmetric bilinear form with signature $(+,+,\ldots,+,-)$. We will follow the standard custom of calling the bilinear form above an inner product even though it is possible for $\langle x, x\rangle = 0$ but x not be zero.

The references dealing with elementary properties of this geometry are rare and brief. An overall view of Lorentz geometry is found in the book of Beem and Ehrlich [**1**]. In a recent article, Birman and Nomizu [**2**] develop Lorentzian trigonometry. A natural question would be to look at the action of the hyperbolic group on conics. A modern exposition of the Euclidean motions of conics, using matrix techniques, can be found in [**4**]. We shall follow the development of that paper.

By analogy with the Euclidean case, we present the hyperbolic group and analyse its action on a conic defined by $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$. We show that certain polynomials in the coefficients of the conic are invariant under the hyperbolic group and that these are all of the invariants. We then study the geometry of the space of orbits of this action and normal forms are found.

1. A Group Action

1.1. *The Hyperbolic Rotations*. The Euclidean group is the group generated by rotations (special orthogonal matrices) and translations of $\mathbb{R}^2$. A special orthogonal matrix is one which takes the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ. To obtain the hyperbolic analogy we shall first discuss hyperbolic rotations which are obtained by replacing the trigonometric functions by the hyperbolic ones in the above formula.

A 2×2 matrix N is called *special hyperbolic* if there exists θ such that

$$N = N(\theta) = \begin{bmatrix} \operatorname{ch}\theta & \operatorname{sh}\theta \\ \operatorname{sh}\theta & \operatorname{ch}\theta \end{bmatrix}, \qquad \theta \in \mathbb{R},$$

where we are writing ch and sh for hyperbolic functions cosh and sinh.

The set G of all such matrices, with the usual multiplication constitutes a group, called the *group of hyperbolic rotations*, or proper *Lorentz group* $SO^+(1,1)$.

Each $N \in G$ defines a linear transformation

$$N : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \end{bmatrix} = N \begin{bmatrix} x \\ y \end{bmatrix}$$

which preserves the Lorentz inner product, defined by $\langle \vec{v}, \vec{w} \rangle = x_1x_2 - y_1y_2$, where $\vec{v} = (x_1, y_1)$, $\vec{w} = (x_2, y_2)$.

It follows easily that the eigenvectors of N are $e_1 = (1, -1)$ and $e_2 = (1,1)$ and that the corresponding eigenvalues $\lambda_1 = \operatorname{ch}\theta - \operatorname{sh}\theta$ and $\lambda_2 = \operatorname{ch}\theta + \operatorname{sh}\theta$ are positive real numbers.

The sets I, II, III, IV, V, and VI in Figure 1 remain G invariant, that is: if $(x, y) \in A$, then $N(x, y) \in A$, $\forall N \in G$ (where A denotes any one of these sets).

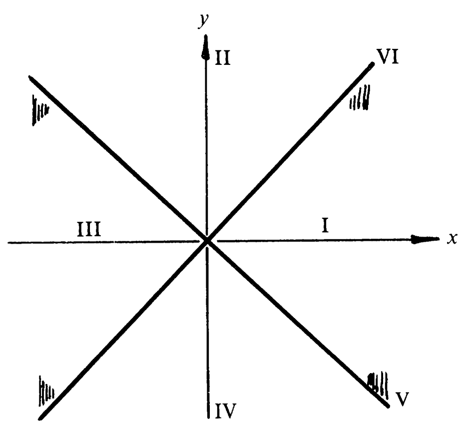


FIG. 1.

To see, for instance, that I is invariant, let $\vec{v} = (x, y)$ be any element in I. Then $\vec{v} = \alpha e_1 + \beta e_2$, where α and β are positive. Hence, $N\vec{v} = N(\alpha e_1 + \beta e_2) = (\alpha\lambda_1)e_1 + (\beta\lambda_2)e_2$ with $\alpha\lambda_1$, $\beta\lambda_2$ positive and so $N\vec{v} \in$ I.

Vectors (x, y) in I or III are called *spacelike*, in II or IV are *timelike*, and in V or VI, *null* [**2**].

Furthermore, the points of the invariant sets I, II, III, IV, slide along the hyperbolas $x^2 - y^2 =$ constant. Now, $\lambda_1\lambda_2 = 1$ for all θ. If $\theta > 0$, we have $\lambda_2 > 1$ and $\lambda_1 < 1$, and in this case, the plane shrinks λ_1 times to the straight line $x = -y$ and stretches in the orthogonal direction away from $x = y$ as shown in Figure 2. When $\theta < 0$, we just reverse the directions of stretching and compression, and the direction of motion of the points along the hyperbolas [**3**].

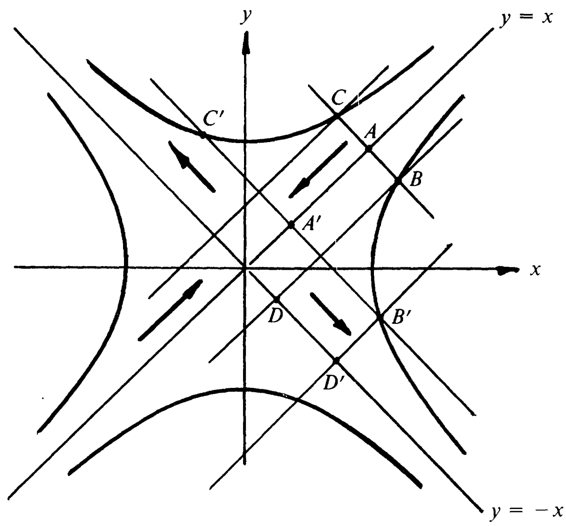


FIG. 2.

1.2. *The Hyperbolic Group.* In analogy with the Euclidean case (see Grosshans [**4**]), we introduce the group $\mathscr{H}$ consisting of all 3×3 real matrices of the form:

$$h(\theta) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{N(\theta)} & B \\ \hline 0 & 0 & 1 \end{array}\right], \tag{1}$$

where $N(\theta)$ is a special hyperbolic matrix, and B is a 2×1 column matrix.

We can identify the plane $\mathbb{R}^2$ with the plane $z = 1$ in $\mathbb{R}^3$ that is,

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Each matrix $h \in \mathscr{H}$ gives rise to a motion M_h, where $M_h(p) = hP = Np + B$, that is a hyperbolic rotation N followed by a translation B. The 2-dimensional Lorentzian geometry is the study of properties invariant by these motions.

In this article, we shall be interested in the action of M_h on the conics $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$, a, b, c do not simultaneously vanish.

1.3. *The Action of $\mathscr{H}$ on V.* For a moment, let G be any group and V be a set. An *action* of G on V is given by a mapping

$$G \times \mathrm{V} \to \mathrm{V} \quad \text{denoted by } (g, v) \to g \cdot v$$

such that (i) $(hg) \cdot v = h \cdot (g \cdot v)$ and (ii) $1 \cdot v = v$ for all h, g in G and v in V. The *orbit* of an element v in V is

$$G \cdot v = \{g \cdot v | g \in G\}.$$

In our case, V will be the vector space of 3×3 symmetric real matrices

$$Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix},$$

and we shall be interested in the following action of $\mathscr{H}$ on V:

$$\text{Given } h \in \mathscr{H} \text{ and } Q \in V, \quad h \cdot Q = {}^t h^{-1} Q h^{-1}. \tag{2}$$

We can identify a matrix $Q \in \mathrm{V}$ with the polynomial $Q(p) = Q(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$. Note that ${}^tPQP = Q(p)$.

Let $Q' = h \cdot Q = {}^t h^{-1} Q h^{-1}$ and let C and C' be the conics associated to Q and Q'; that is $C = \{p : Q(p) = 0\}$ and $C' = \{p : Q'(p) = 0\}$.

The equation

$$Q(p) = {}^tPQP = {}^tP\,{}^thQ'hP = {}^t(hP)Q'(hP) = Q'(hP)$$

shows that p lies on C if and only if $hP = M_h(p)$ is on C'.

Hence, M_h sends C to C'. Thus, if Q' is in the orbit of Q under the action described above, there is a hyperbolic motion carrying the conic defined by Q to the conic defined by Q'. This means that the conics are equivalent under the action.

To study the orbit under $\mathscr{H}$ of an element Q in V, we shall denote a typical element in V by:

$$Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} = \left[\begin{array}{c|c} A(Q) & D(Q) \\ \hline {}^tD(Q) & f(Q) \end{array}\right], \tag{3}$$

where

$$A(Q) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad D(Q) = \begin{bmatrix} d \\ e \end{bmatrix} \quad \text{and} \quad f(Q) = f.$$

To simplify notation, the matrix (1) will be denoted by h^{-1}. The corresponding matrix Q' in the orbit of Q, $Q' = {}^t h^{-1} Q h^{-1}$ is obtained as follows:

$$\begin{aligned} A(Q') &= NA(Q)N \\ D(Q') &= NA(Q)B + ND(Q) \\ f(Q') &= {}^tBA(Q)B + 2\,{}^tD(Q)B + f(Q). \end{aligned} \tag{4}$$

The following more explicit expression for $A(Q')$, will be useful later:

$$A(Q') = \begin{bmatrix} a\,\mathrm{ch}^2\theta + b\,\mathrm{sh}\,2\theta + c\,\mathrm{sh}^2\theta & ((a+c)/2)\mathrm{sh}\,2\theta + b\,\mathrm{ch}\,2\theta \\ ((a+c)/2)\mathrm{sh}\,2\theta + b\,\mathrm{ch}\,2\theta & a\,\mathrm{sh}^2\theta + b\,\mathrm{sh}\,2\theta + c\,\mathrm{ch}^2\theta \end{bmatrix}. \tag{5}$$

1.4. *$\mathscr{H}$-Invariant Polynomials.* Let $R = \mathbb{R}[z_1, z_2, z_3, z_4, z_5, z_6]$ be the polynomial ring in six indeterminates over the reals. We can identify R with the ring of polynomial functions on V by defining $P(Q) = P(a, b, c, d, e, f)$, where $P = P(z_1, z_2, \ldots, z_6)$ is in R and Q is an element of V.

A polynomial P in R is called *invariant* with respect to $\mathscr{H}$ if $P(h \cdot Q) = P(Q), \forall h \in \mathscr{H}, \forall Q \in$ V. The set of such invariant polynomials, denoted by $\mathscr{I}$, is an algebra over $\mathbb{R}$.

Now, we define three polynomials in R and show that they are $\mathscr{H}$ invariant: Let

$$\delta(Q) = \det A(Q) = ac - b^2, \qquad \Delta(Q) = \det Q \quad \text{and} \quad \tilde{\tau}(Q) = a - c.$$

LEMMA. *δ, Δ and $\tilde{\tau}$ are $\mathscr{H}$-invariant.*

Proof.

(i) $\delta(Q') = \det(A(Q')) = \det(NA(Q)N) = \det(A(Q)) = \delta(Q)$, since $\det N = 1$;

(ii) $\Delta(Q') = \det(Q') = \det({}^th^{-1}Qh^{-1}) = \det Q$, since $\det h = 1$;

(iii) it follows from (5) that $\tilde{\tau}(Q') = a - c = \tilde{\tau}(Q)$.

Remark. Two nonequivalent conics may have the same invariant polynomials δ, $\tilde{\tau}$, and Δ.

For instance, let $C_1 : x^2 + 2bxy = 1$ and $C_2 : x^2 - 2bxy = 1$, $b > 0$ as in Figure 3. It follows from the invariance of the sets I, II, III and IV in Figure 1 that a motion M_h cannot send C_1 onto C_2.

2. Canonical Forms

2.1. *Normal Forms Under Rotations.* It is well known that under a Euclidean motion (rotations and translations) most conics (when $\delta \neq 0$) admit reduction to a sum of squares. This is not always the case for the action of the Hyperbolic group.

As before, let

$$Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \quad \text{and} \quad A(Q) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

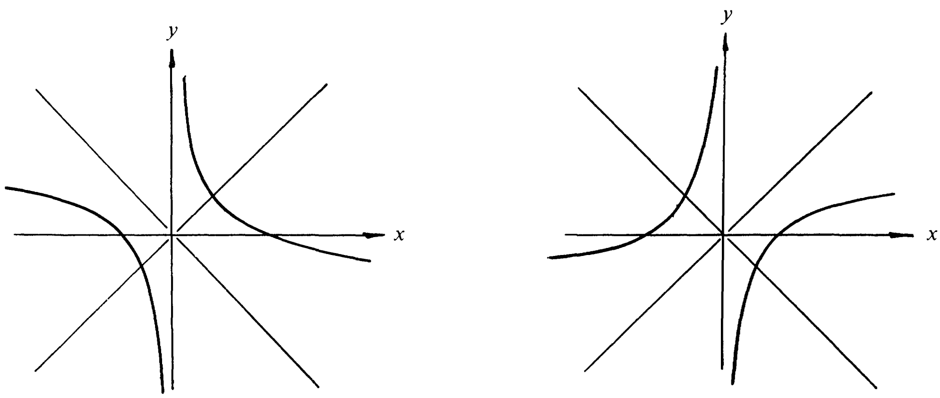


FIG. 3.

The first step is to get rid of b, that is, we want to find conditions to reduce $A(Q)$ to a diagonal form. As we shall see the possibility of this reduction will depend upon the sign of

$$\Lambda(Q) = [\tau(Q)]^2 + 4\delta(Q) = (a-c)^2 + 4(ac - b^2) = (a+c)^2 - 4b^2.$$

So a Lemma follows.

LEMMA. *If $b \neq 0$, $\Lambda(Q) > 0$ if and only if $|(a+c)/2b| > 1$.*

PROPOSITION 1. *If $\Lambda(Q) > 0$, then in the orbit of Q there exists a Q' which is of the following form*

$$Q' = \left[\begin{array}{cc|c} a' & 0 & d' \\ 0 & c' & e' \\ \hline d' & e' & f' \end{array}\right].$$

Proof. We can assume $b \neq 0$.
From (5) we need θ such that

$$b' = \frac{a+c}{2} \cdot \operatorname{sh} 2\theta + b \operatorname{ch} 2\theta = 0,$$

or equivalently

$$\cot h 2\theta = \frac{a+c}{-2b}.$$

This is possible if and only if $|(a+c)/2b| > 1$, or by the Lemma, $\Lambda(Q) > 0$.
Now, to obtain Q', we take

$$h^{-1} = \left[\begin{array}{c|c} N(\theta) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0\ 0 & 1 \end{array}\right], \quad \text{as in (1)}.$$

The diagonal elements a' and c' are solutions of the system:

$$\begin{cases} a' - c' = \tilde{\tau}(Q) = a - c \\ a' \cdot c' = \delta(Q) = ac - b^2. \end{cases}$$

Now, from (5) we can deduce $(a')^2 - (c')^2 = \dfrac{(a-c)}{(a+c)}\Lambda \operatorname{ch} 2\theta$. Since $\Lambda > 0$, it follows that the sign of $(a')^2 - (c')^2$ is equal to the sign of $a^2 - c^2$. These conditions determine a' and c' uniquely.

PROPOSITION 2. *If $\Lambda(Q) < 0$ ($b \neq 0$), then Q is $\mathcal{H}$-equivalent to*

$$Q' = \left[\begin{array}{cc|c} \tilde{\tau}(Q) & \pm\sqrt{-\delta(Q)} & d' \\ \pm\sqrt{-\delta(Q)} & 0 & e' \\ \hline d' & e' & f \end{array}\right]$$

(*where the sign of the square root is equal to the sign of b*).

Proof. We want to make $c' = 0$.

It follows from (5) that we need θ such that

$$\begin{cases} a' = a \operatorname{ch}^2\theta + b \operatorname{sh} 2\theta + c \operatorname{sh}^2\theta \\ b' = \dfrac{a+c}{2} \operatorname{sh} 2\theta + b \operatorname{ch} 2\theta \\ 0 = a \operatorname{sh}^2\theta + b \operatorname{sh} 2\theta + c \operatorname{ch}^2\theta. \end{cases} \tag{6}$$

The invariance of $\tilde{\tau}$ and δ imply $a' = a - c = \tilde{\tau}(Q)$ and $b' = \pm\sqrt{-\delta(Q)}$. We shall be assuming that b' has the same sign as b, that is, $bb' > 0$.

Now, it is easy to see that the first and the last equation of (6) are linearly dependent. Hence, we obtain the equivalent system:

$$\begin{cases} b' = \dfrac{a+c}{2} \operatorname{sh} 2\theta + b \operatorname{ch} 2\theta \\ 0 = a \operatorname{sh}^2\theta + b \operatorname{sh} 2\theta + c \operatorname{ch}^2\theta, \end{cases}$$

or

$$\begin{cases} 2b' = (a+c)\operatorname{sh} 2\theta + 2b \operatorname{ch} 2\theta \\ a - c = 2b \operatorname{sh} 2\theta + (a+c)\operatorname{ch} 2\theta. \end{cases} \tag{7}$$

For a moment, let us consider the system:

$$\begin{cases} 2b' = (a+c)y + 2bx \\ a - c = 2by + (a+c)x. \end{cases} \tag{8}$$

Since the determinant $\Lambda(Q) = (a+c)^2 - 4b^2 < 0$, (8) has unique solutions

$$X = \frac{a^2 - c^2 - 4bb'}{\Lambda(Q)} \quad \text{and} \quad Y = \frac{2b'(a+c) - 2b(a-c)}{\Lambda(Q)}.$$

It is not hard to show that $X^2 - Y^2 = 1$. To finish the proof, we must show that $X \geqslant 0$. In fact, since $\Lambda(Q) < 0$, it is only necessary to show that $(a^2 - c^2) - 4bb' < 0$.

Now,

$$\Lambda(Q) = [\tilde{\tau}(Q)]^2 + 4\delta(Q). \tag{9}$$

From the invariance of $\delta(Q)$, we also have:

$$\Lambda(Q) = [\tilde{\tau}(Q)]^2 - 4(b')^2. \tag{10}$$

From (9) and (10), it follows that the condition $\Lambda(Q) < 0$ is equivalent to each one of the inequalities:

(i) $|(a + c)| < 2|b|$

(ii) $|a - c| < 2|b'|$.

By multiplying (i) and (ii) side by side, we obtain: $a^2 - c^2 \leqslant |a^2 - c^2| < 4|b||b'|$ or, since $bb' > 0$, $a^2 - c^2 - 4bb' < 0$, as desired.

PROPOSITION 3. *if* $\Lambda(Q) = 0$ *then*:

(i) *If* $a^2 - c^2 > 0$, Q *is* $\mathscr{H}$-*equivalent to*

$$Q' = \left[\begin{array}{cc|c} \tilde{\tau}(Q) & \pm|\tilde{\tau}(Q)|/2 & d' \\ \pm|\tilde{\tau}(Q)|/2 & 0 & e' \\ \hline d' & e' & f \end{array}\right] \qquad \begin{array}{l}\text{plus sign if } b > 0 \\ \text{minus sign if } b < 0.\end{array}$$

(ii) *If* $a^2 - c^2 < 0$, Q *is* $\mathscr{H}$-*equivalent to*

$$Q'' = \left[\begin{array}{cc|c} 0 & \pm|\tilde{\tau}(Q)|/2 & d' \\ \pm|\tilde{\tau}(Q)|/2 & -\tilde{\tau}(Q) & e' \\ \hline d' & e' & f \end{array}\right] \qquad \begin{array}{l}\text{plus sign if } b > 0 \\ \text{minus sign if } b < 0.\end{array}$$

(iii) *If* $a = c$, *then* $b = \pm a$ *and*

$$Q = \left[\begin{array}{cc|c} a & \pm a & d \\ \pm a & a & e \\ \hline d & e & f \end{array}\right].$$

(iv) *If* $a = -c$, *then* $b = 0$ *and*

$$Q = \left[\begin{array}{cc|c} a & 0 & d \\ 0 & -a & e \\ \hline d & e & f \end{array}\right].$$

Proof. First, let us suppose $a^2 - c^2 > 0$. In this case, we shall be assuming that b' has the same sign as b.

As in the proof of Proposition 2, we obtain the system:

$$\begin{cases} 2b' = (a + c)y + 2bx \\ a - c = 2by + (a + c)x. \end{cases} \tag{11}$$

The determinant $\Lambda(Q) = (a + c)^2 - 4b^2$ is equal to zero. The system (11) admits a line of solutions, with slope $m = \pm 1$, which intercepts the x-axis in $x_0 = (a - c)/(a + c) > 0$.

Thus, this line intercepts the hyperbola $x^2 - y^2 = 1$, $x > 0$ and we can take θ such that $\begin{cases} x = \operatorname{ch} 2\theta \\ y = \operatorname{sh} 2\theta. \end{cases}$

When $a^2 - c^2 < 0$, we shall make $a' = 0$. In this case, we proceed as before, just solving the system

$$\begin{cases} 2b' = (a + c)y + 2bx \\ c - a = 2by + (a + c)x \end{cases}.$$

Of course, (iii) and (iv) follow directly from the hypothesis.

2.2. *Completing the Reduction*. Up to this point, we analyzed the action of a hyperbolic rotation on the matrix Q. Now, we shall be using translations to bring Q to a normal form.

Case 1. If $\delta(Q) \neq 0$ and

$$\begin{cases} \text{(i)}\ \Lambda(Q) > 0 & \text{or} \\ \text{(ii)}\ \Lambda(Q) = 0, & \text{with } b = 0, \end{cases}$$

then the orbit of Q contains the matrix:

$$\begin{bmatrix} a' & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & \Delta(Q)/\delta(Q) \end{bmatrix}. \tag{12}$$

In (ii), we also have $c' = -a'$.

Proof.

(i) Since $\Lambda(Q) > 0$, the orbit of Q contains, from Proposition 1, the matrix:

$$Q' = \begin{bmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{bmatrix}. \tag{13}$$

Since $a' \cdot c' = \delta(Q') = \delta(Q) \neq 0$, there exist real numbers s and t such that:

$$a's + d' = 0 \quad \text{and} \quad c't + e' = 0.$$

Next, we take h^{-1} in the form (1) with $N = I$ and ${}^tB = (s, t)$. Then $Q'' = h \cdot Q'$ is diagonal, with entries a', c' and f', say. From $\Delta(Q) = \Delta(Q'') = a'c'f' = \delta(Q) \cdot f'$, it follows that Q'' is as we wanted.

(ii) When $b = 0$ and $\Lambda(Q) = 0$, it also follows that $c' = -a'$.

Case 2. If $\delta(Q) \neq 0$ and $\Lambda(Q) < 0$, then the orbit of Q contains the matrix

$$\begin{bmatrix} \tilde{\tau}(Q) & \pm\sqrt{-\delta(Q)} & 0 \\ \pm\sqrt{-\delta(Q)} & 0 & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix}$$

(sign $\sqrt{\ }$ equals sign of b).

Proof. Let us suppose $b > 0$ (the case $b < 0$ is analogous). The orbit of Q contains, from Proposition 2, the matrix

$$Q' = \begin{bmatrix} \tilde{\tau}(Q) & \sqrt{-\delta(Q)} & d' \\ \sqrt{-\delta(Q)} & 0 & e' \\ d' & e' & f \end{bmatrix}.$$

Since $\delta(Q) \neq 0$, there exist real numbers s and t such that

$$\tilde{\tau}(Q)s + \sqrt{-\delta(Q)}\,t + d' = 0 \quad \text{and} \quad \sqrt{-\delta(Q)}\,s + e' = 0.$$

Now, we take h^{-1} in the form (1), with $N = \mathrm{I}$ and ${}^{t}B = (s, t)$. Then, $Q'' = h \cdot Q'$ is as we wanted.

Case 3. If $\delta(Q) \neq 0$ and $\Lambda(Q) = 0$, then

(i) If $a^2 - c^2 > 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} \tilde{\tau}(Q) & \pm|\tilde{\tau}(Q)|/2 & 0 \\ \pm|\tilde{\tau}(Q)|/2 & 0 & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix} \quad \begin{matrix} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0. \end{matrix}$$

(ii) If $a^2 - c^2 < 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} 0 & \pm|\tilde{\tau}(Q)|/2 & 0 \\ \pm|\tilde{\tau}(Q)|/2 & -\tilde{\tau}(Q) & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix} \quad \begin{matrix} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0. \end{matrix}$$

The proof is analogous to the proof of Case 2, and we shall omit it.

Case 4. If $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$ (hence, $\Lambda(Q) > 0$) and $\Delta(Q) \neq 0$, then

(i) If $\tilde{\tau}(Q) \cdot \Delta(Q) < 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & e'' \\ 0 & e'' & 0 \end{bmatrix}, \quad \text{where } e'' = \pm\sqrt{\frac{-\Delta(Q)}{\tilde{\tau}(Q)}}.$$

(ii) If $\tilde{\tau}(Q) \cdot \Delta(Q) > 0$, the orbit of Q contains the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & d'' \\ 0 & -\tilde{\tau}(Q) & 0 \\ \hline d'' & 0 & 0 \end{array}\right], \quad \text{where } d'' = \pm\sqrt{\frac{\Delta(Q)}{\tilde{\tau}(Q)}}.$$

Proof.

Case i. The hypotheses imply that $\Lambda(Q) > 0$.
Since $\delta(Q) = 0$, and $\tilde{\tau}(Q) \cdot \Delta(Q) < 0$, we may assume in (13) that:

$$Q' = \begin{bmatrix} \tilde{\tau}(Q) & 0 & d' \\ 0 & 0 & e' \\ d' & e' & f \end{bmatrix}.$$

We note that $0 \neq \Delta(Q) = \Delta(Q') = -(e')^2 \cdot \tilde{\tau}(Q)$, so that $e' \neq 0$. Then, we can find s and t such that

$$\tilde{\tau}(Q) \cdot s + d' = 0 \quad \text{and} \quad \tilde{\tau}(Q) \cdot s^2 + 2(d's + e't) + f = 0.$$

To obtain the normal form, let $Q'' = h \cdot Q'$, where h^{-1} has the form (1) with $N = \mathrm{I}$ and ${}^tB = (s, t)$.

Case ii. Follows analogously.

Case 5. If $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$ (hence $\Lambda(Q) > 0$), and $\Delta(Q) = 0$ then, the orbit of Q contains either the matrix

(i) $$\left[\begin{array}{cc|c} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when} |c| < |b| \text{ or } (b = 0 \text{ and } a \neq 0),$$

or

(ii) $$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -\tilde{\tau}(Q) & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when } |c| > |b| \text{ or } b = 0 \text{ and } c \neq 0.$$

In either case,

$$f' = f - \frac{d^2 - e^2}{\tilde{\tau}(Q)}.$$

Proof. From Proposition 1, we may assume

$$Q = \begin{bmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{bmatrix},$$

where $a' \cdot c' = 0$, since $\delta = 0$.
If $c' = 0$, $\Delta = 0$ implies $e' = 0$.
Now, choosing

$$h^{-1} = \left[\begin{array}{cc|c} 1 & 0 & s \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

where $\tilde{\tau}(Q) \cdot s + d' = 0$, we verify that

$$Q' = h \cdot Q = \begin{bmatrix} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f' \end{bmatrix}.$$

With the same procedure, when $a' = 0$, we obtain (ii).

Using the expression for $f' = f(Q')$ as in (4) and the invariance of $d^2 - e^2$ with respect to a hyperbolic rotation, we can find f'.

The decision about formulas (i) or (ii), can be made as follows:

Under the hypothesis, the conic C associated to Q is a pair of parallel lines or a line counted twice (or empty). An easy exercise shows that a normal vector to these lines is (b, c). Now, the invariance of the sets I to VI in Figure 1 gives the answer (Figure 4).

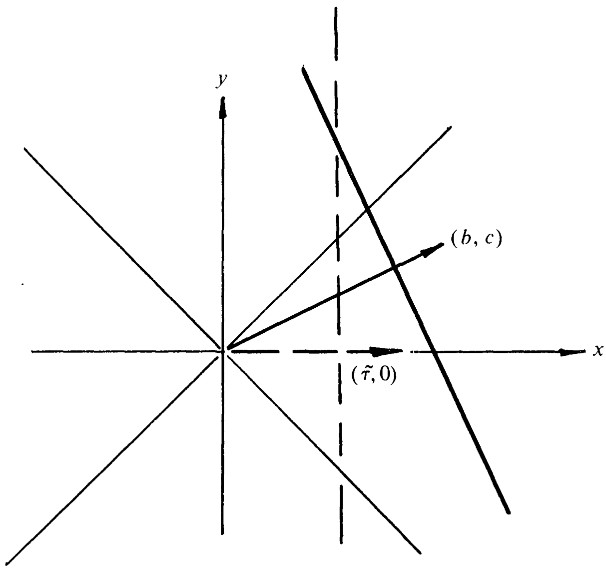


FIG. 4.

Case 6. If $\delta(Q) = \tilde{\tau}(Q) = 0$ (hence, $\Lambda(Q) = 0$), then
(i) If $b = a$, the orbit of Q contains either the matrix

$$\left[\begin{array}{cc|c} a & a & d' \\ a & a & 0 \\ \hline d' & 0 & 0 \end{array}\right] \quad \text{when } d \neq e,$$

or the matrix

$$\left[\begin{array}{cc|c} a & a & 0 \\ a & a & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when } d = e.$$

Furthermore, $d' = d - e$ and $f' = f - d^2/a$.

(ii) If $b = -a$, the orbit of Q contains either the matrix

$$\left[\begin{array}{cc|c} a & -a & d' \\ -a & a & 0 \\ \hline d' & 0 & 0 \end{array}\right] \quad \text{when } d \neq -e,$$

or the matrix

$$\left[\begin{array}{cc|c} a & -a & 0 \\ -a & a & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when } d = -e.$$

Furthermore, $d' = d + e$ and $f' = f - d^2/a$.

The proof is analogous to the previous cases, and we shall omit it.

Remark. The description of the canonical forms shows that the equation of a standard conic can be found in terms of the coefficients a, b, c, d, e, and f of $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$.

For instance, if $\delta(Q) \neq 0$ and $\Lambda(Q) < 0$ (with $b \neq 0$), the standard equation is

$$\tilde{\tau}(Q)x^2 + 2\sqrt{-\delta(Q)}\,xy - \frac{\Delta(Q)}{\delta(Q)} = 0, \quad \text{if } b > 0,$$

or

$$\tilde{\tau}(Q)x^2 - 2\sqrt{-\delta(Q)}\,xy - \frac{\Delta(Q)}{\delta(Q)} = 0, \quad \text{if } b < 0.$$

3. Geometric Interpretation

3.1. *The Sign of Λ and the Hyperbolas.* As we saw before, according to $\Lambda(Q) > 0$ or $\Lambda(Q) < 0$, a given hyperbola may or may not be reduced to one of the form $a'x^2 + c'y^2 + f' = 0$. Now, we want to find a geometric interpretation for this fact.

We recall here that the group of hyperbolic rotations leaves invariant the sets I to VI of Figure 1, in Section 1.1. To simplify notation, we shall denote by $S = \mathrm{I} \cup \mathrm{III}$, the set of spacelike vectors and by $T = \mathrm{II} \cup \mathrm{IV}$ the set of timelike vectors.

PROPOSITION 1. *If $\delta(Q) < 0$ and $\Lambda(Q) > 0$, then both asymptotes of the hyperbola C associated to Q are timelike or both are spacelike. More precisely, if $a^2 - c^2 > 0$, they are both in T and if $a^2 - c^2 < 0$, they are in S.*

Proof. Since S and T are invariant, we may analyse the position of the asymptotes of the reduced form.

First, let's assume $a^2 - c^2 > 0$. The standard equation of C after a hyperbolic rotation is $a'x^2 + c'y^2 = -\Delta(Q)/\delta(Q)$, where $(a')^2 - (c')^2 > 0$ (Case 1, §2). Then, the asymptotes of this reduced conic are the lines L_1 and L_2, where

$$L_1 : \sqrt{|a'|}\, x + \sqrt{|c'|}\, y = 0$$
$$L_2 : \sqrt{|a'|}\, x - \sqrt{|c'|}\, y = 0.$$

L_1 and L_2 are symmetric with respect to the x-axis.

Since $a'^2 - c'^2 > 0, |a'|/|c'| > 1$ and it follows that L_1 lies in T (the same holding with L_2), (Figure 5).

The case $a^2 - c^2 < 0$ is analogous. In this case, L_1 and L_2 are in S (Figure 6).

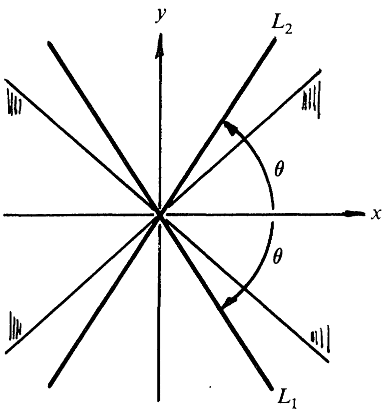


FIG. 5.

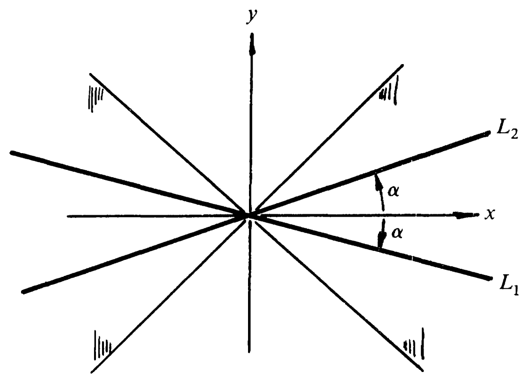


FIG. 6.

PROPOSITION 2. *If $\delta(Q) < 0$ and $\Lambda(Q) < 0$, the hyperbola C has one asymptote which is timelike and one which is spacelike.*

Proof. By Case 2, §2, it follows that the standard equation of C is

$$\text{(i)} \qquad \tilde{\tau}(Q)x^2 + 2\sqrt{-\delta(Q)}\,xy = \frac{\Delta(Q)}{\delta(Q)},$$

or

$$\text{(ii)} \qquad \tilde{\tau}(Q)x^2 - 2\sqrt{-\delta(Q)}\,xy = \frac{\Delta(Q)}{\delta(Q)}.$$

Let's assume that (i) holds. (Case (ii) is analogous.) The asymptotes are the lines:

$$L_1 : x = 0 \quad \text{and} \quad L_2 : \tilde{\tau}(Q)x + 2\sqrt{-\delta(Q)}\,y = 0.$$

To finish the proof, we just observe that the slope of L_2 is in the interval $(-1, 1)$.

Proposition 3. *If $\delta(Q) < 0$ and $\Lambda(Q) = 0$, at least one of the lines $y = x$ or $y = -x$ is an asymptote of C (maybe both).*

Proof.

(i) Let's assume $b = 0$. Then, from Case 1 of §2, the standard equation is

$$a'x^2 - a'y^2 = -\frac{\Delta(Q)}{\delta(Q)},$$

which has $y = x$ and $y = -x$ as its asymptotes.

Since these lines are themselves invariant, the result is also true for the original C;

(ii) Let $b \neq 0$. Then, $\Lambda(Q) = 0 \Leftrightarrow \tilde{\tau}(Q) = \pm 2\sqrt{-\delta(Q)}$. From Case 3, §2, it follows that the standard equation is

$$\tilde{\tau}(Q)x^2 \pm \tilde{\tau}(Q)xy = \frac{\Delta(Q)}{\delta(Q)} \tag{14}$$

or

$$\pm\tilde{\tau}(Q)xy - \tilde{\tau}(Q)y^2 = \frac{\Delta(Q)}{\delta(Q)}. \tag{15}$$

If, for instance, (14) holds, then the asymptotes are

$$L_1 : x = 0 \quad \text{and} \quad L_2 : y = x \quad \text{or} \quad y = -x.$$

Of course, the asymptotes of the original C inherit the properties: L_1 lies in S and L_2 is $y = x$ or $y = -x$.

3.2. *The Geometry of Hyperbolic Rotations.* Two conics equivalent by a hyperbolic motion do not "look the same" in the Euclidean sense. We shall complete the study of §2, by giving geometric constructions relating a given conic with its standard form.

First, we note that a hyperbolic motion is given by a orthogonal matrix, hence preserves area of a plane figure.

Let us consider an ellipse C, with center at the origin. To obtain the standard ellipse C' equivalent to C, we shall determine its vertices by the following procedure: take the points of tangency between C and an invariant hyperbola $x^2 - y^2 =$ const, and slide them along this hyperbola until they meet the coordinate axis (Figure 7). Furthermore, C and C' have the same area.

A special case occurs when the axes of the ellipse C coincide with the invariant lines $y = \pm x$. In this case, C' is always a circle (Figure 8).

Now, let C be a hyperbola with its center at the origin and such that both of its asymptotes are timelike or both are spacelike.

As above, we find a hyperbola $x^2 - y^2 =$ const tangent to C. Sliding the point P of tangency we determine the vertex P' of the standard hyperbola C'. The point Q, obtained by the intersection of $x^2 - y^2 =$ const with one of the asymptotes of C, will be sent to a point Q' in the corresponding asymptote of C'. This point Q' is

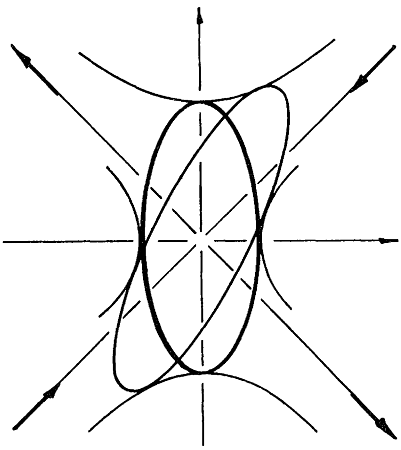


FIG. 7.

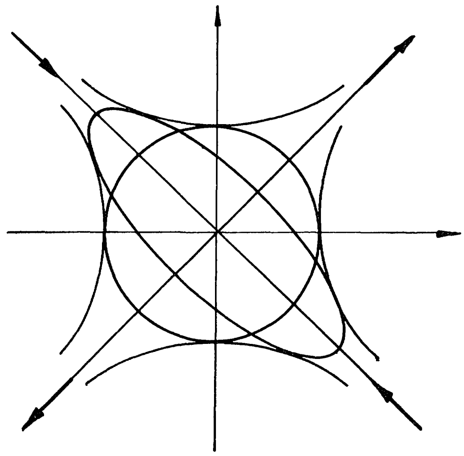


FIG. 8.

uniquely determined by the condition that the triangles OPQ and $OP'Q'$ have the same area. The other asymptote of C' is symmetric with respect to the x-axis (Figure 9).

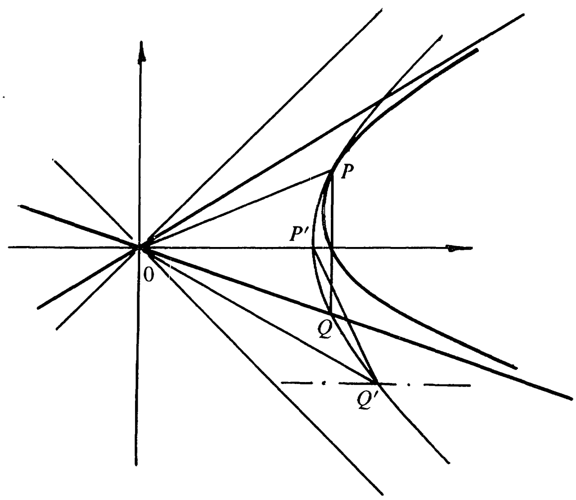


FIG. 9.

When C has one asymptote which is timelike and one which is spacelike the standard hyperbola C' has the axis $x = 0$ as one of its asymptotes. In this case, sliding the point P given by the intersection of any hyperbola $x^2 - y^2 = k$ $(k < 0)$ with the asymptote of C, we find its image P' in the line $x = 0$.

Also, the intersection of C with the line $y = x$ determines a point Q which slides to Q' along this diagonal. As above, this point Q' is uniquely determined by an area argument.

We can find the other asymptote of C', by observing that the triangles OQR and $OQ'R'$ have the same area, where R is the point in the intersection of any hyperbola $x^2 - y^2 = k$ $(k > 0)$ with the other asymptote of C (Figure 10).

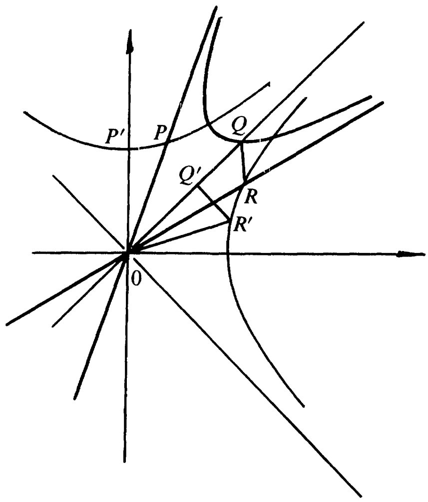


FIG. 10.

Finally, we note that a parabola C with vertex at the origin can be transformed by a hyperbolic rotation in a parabola C' with axis parallel to the coordinate axis, but with its vertex out of the origin. With similar argument, we can determine C'.

3.3. *Some Geometry of Orbits*. In this section we shall be studying the geometry of the equivalence classes, not on the whole space of conics, but rather in the subspace H^2 of all quadratic forms in two variables.

Now, a quadratic form can be written $ax^2 + 2bxy + cy^2$, so it can be identified with the point (a, b, c) in $\mathbb{R}^3$.

We say that a set X in $\mathbb{R}^3$ is invariant, when X is a union of G-orbits.

Figure 11 shows how the cone $\delta = ac - b^2 = 0$ and the pair of planes $\Lambda = (a - 2b + c) \cdot (a + 2b + c) = 0$ separate $\mathbb{R}^3$ in invariant sets.

If $\delta > 0$ (hence $\Lambda > 0$), then (a, b, c) is in the interior of the cone $ac - b^2 = 0$, and all such points are of elliptic type (Case 1, §2).

If $\delta < 0$, then (a, b, c) is in the exterior of the cone and all points are of hyperbolic type. More precisely, they fall in three cases:

(i) $\Lambda > 0$. By Case 1, §2 it follows that these points correspond to hyperbolas that admit reduction to a sum of squares. They determine the invariant set B of Figure 11.

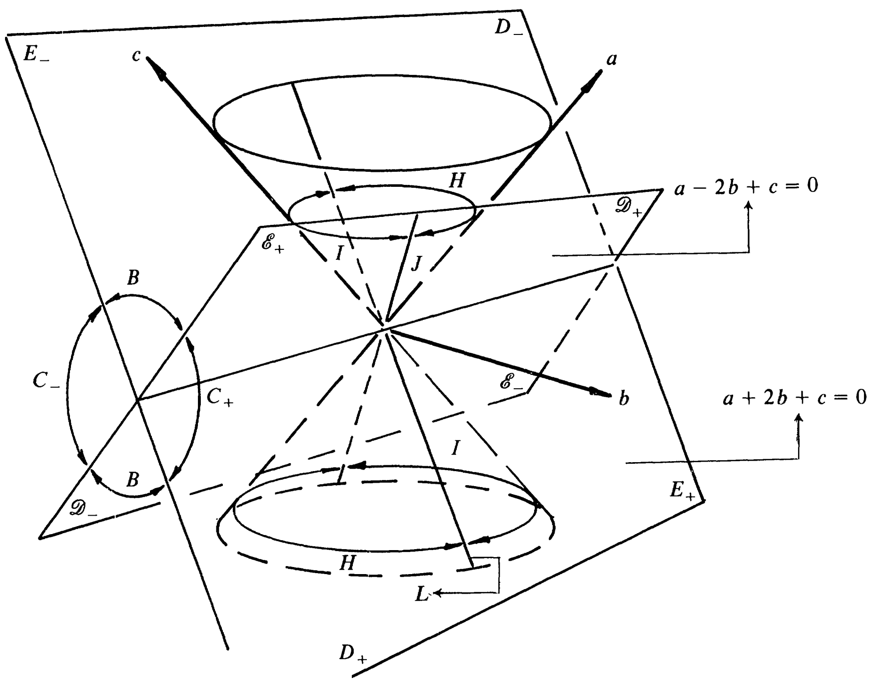


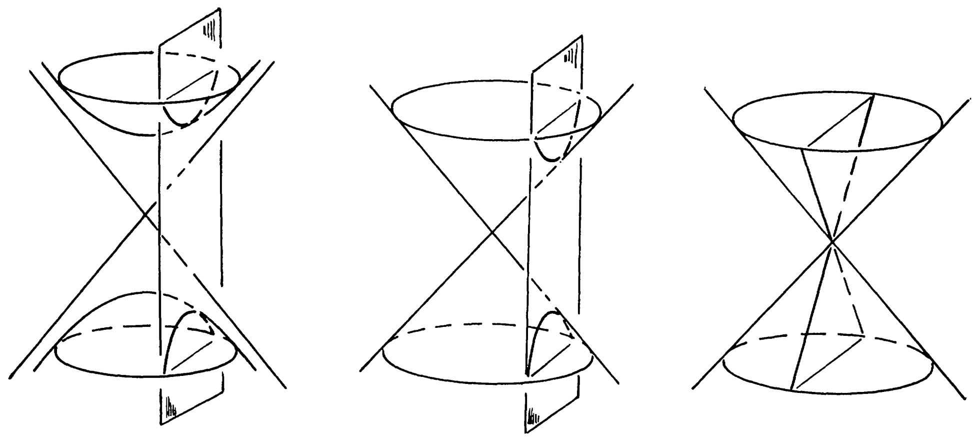
FIG. 11.

(ii) $\Lambda < 0$ corresponds to the hyperbolas that do not admit reduction to a sum of squares, as we saw in Case 2, §2. These are the sets C_+ and C_- in Figure 11.

(iii) $\Lambda = 0$. This is the limit case of the previous ones. From Case 3, §2, it follows that D_+, D_-, E_+, and E_- in Figure 11 are invariant sets.

If $\delta = 0$, the points lie on the cone. When $\Lambda > 0$, the subsets H and I of the cone are invariant, as we saw in Case 5, §2. When $\Lambda = 0$, the lines L and J, the intersections of the cone with the planes $a + 2b + c = 0$ and $a - 2b + c = 0$ are also invariant sets (Case 6, §2).

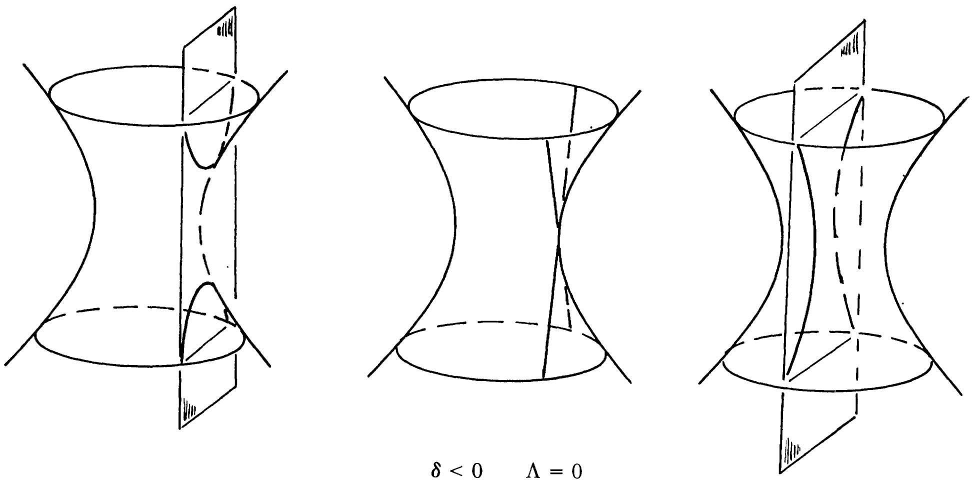
To complete our description, we observe that each orbit is contained in the curve intersection of the hyperboloid $ac - b^2 = \delta$ (or cone, when $\delta = 0$) with the plane $a - c = \tilde{\tau}$. Of course, this curve can be a hyperbola or a pair of concurrent lines. Their intersection with the invariant sets of Figure 11, determine the orbits. The several possibilities are shown in Figure 12.



$\delta > 0$
(only one orbit)

$\delta = 0 \quad \Lambda \neq 0$
(two orbits)

$\delta = 0 \quad \Lambda = 0$
(four orbits)



$\delta < 0 \quad \Lambda > 0$
(only one orbit)

$\delta < 0 \quad \Lambda = 0$
(four orbits lying respectively in D_+, D_-, E_+, E_-)

$\delta < 0 \quad \Lambda < 0$
(two orbits)

FIG. 12.

4. The Ring of Invariants

4.1. *Proof of Theorem* 1. The exposition of this paragraph is devoted to showing that any polynomial in $R = \mathbb{R}[z_1, z_2, z_3, z_4, z_5, z_6]$ invariant under the action of the hyperbolic group $\mathscr{H}$ is a polynomial in $\tilde{\tau}$, δ and Δ. In fact, we prove:

THEOREM 1. *If* $\mathbb{R}[\tilde{\tau}, \delta, \Delta]$ *denotes the subalgebra of* R, *generated by* $\tilde{\tau}$, δ *and* Δ, *and* $\mathscr{I}$ *is the algebra of invariant polynomials, then* $\mathscr{I} = \mathbb{R}[\tilde{\tau}, \delta, \Delta]$. *Moreover* $\tilde{\tau}$, δ *and* Δ *are algebraically independent over* $\mathbb{R}$.

To simplify notation, we shall denote a matrix

$$Q = \left[\begin{array}{cc|c} a & b & d \\ b & c & e \\ \hline d & e & f \end{array}\right] \quad \text{by } (a, b, c, d, e, f).$$

We recall that R can be identified with the ring of polynomial functions over V, as we saw in (1.4).

Now, we define the following subsets of V:

$$X_1 = \{(a, b, 0, 0, 0, f);\ b \neq 0,\ a^2 - 4b^2 < 0\}$$
$$X_2 = \{(a, 0, c, 0, 0, f);\ ac \neq 0,\ a \neq -c\}$$
$$U_1 = \{Q \in \mathrm{V} : \delta(Q) < 0 \quad \text{and} \quad \Lambda(Q) < 0\}$$
$$U_2 = \{Q \in \mathrm{V} : \delta(Q) \neq 0 \quad \text{and} \quad \Lambda(Q) > 0\}.$$

Any element in U_1 is transformed by $\mathscr{H}$ in an element in X_1. Also, any element in U_2 is $\mathscr{H}$-equivalent to an element in X_2.

We need the following lemma:

LEMMA. *Any invariant polynomial is even in the variable b.*

Proof. First, we observe that if $(a, b, c, d, e, f) \in U_2$, then $(a, -b, c, d, e, f) \in U_2$.

Hence, given any invariant polynomial P, we define in U_2 the polynomial:

$$\tilde{P}(a, b, c, d, e, f) = P(a, b, c, d, e, f) - P(a, -b, c, d, e, f).$$

Now, we saw that any element (a, b, c, d, e, f) in U_2 is equivalent to an element $(a, 0, c, 0, 0, f)$. The invariance of P implies $P(a, b, c, d, e, f) = P(a, 0, c, 0, 0, f)$.

Hence, $\tilde{P} \equiv 0$ in U_2. Since U_2 is an open set in V, it follows that $\tilde{P} \equiv 0$ in V, that is, P is even in b.

We divide the proof of Theorem 1 into three main steps:

Step 1. Let $\mathscr{B}$ be the ring of the polynomial functions over X_1, and, as before, $\mathscr{I}$ be the set of invariant polynomials.

We define a mapping $H : \mathscr{I} \to \mathscr{B}$ by

$P \to H(P) = P | X_1$, restriction of P to X_1, that is $P(a, b, c, d, e, f) \mapsto P(a, b, 0, 0, 0, f)$.

It follows that H is a one-to-one homomorphism. In fact, from the invariance of P, $H(P) = P | X_1 = 0$ implies $P | U_1 = 0$. Since U_1 is open in V, the polynomial P must be identically zero in V.

We conclude this step by noting that:

$$H(\tilde{\tau}) = a, \quad H(\delta) = -b^2, \qquad H(\Delta) = -b^2 f.$$

Step 2. Let P be an invariant polynomial. There are unique polynomials $g_0(a, b), \ldots, g_m(a, b)$, such that

$$H(P) = g_0(a, b) f^m + \cdots + g_m(a, b).$$

From the previous Lemma, P is even in the variable b. Hence

$$H(P) = h_0(a, -b^2)f^m + \cdots + h_m(a, -b^2).$$

Next, let us consider the invariant polynomials $\delta^m P$ and $h_0(\tilde{\tau}, \delta)\Delta^m + h_1(\tilde{\tau}, \delta)\delta\Delta^{m-1} + \cdots + h_m(\tilde{\tau}, \delta)\delta^m$. The image by H of each one of these polynomials is equal to

$$(-b^2)^m H(P).$$

As we saw in Step 1, H is 1–1, and we have the equality:

$$\delta^m P = H_0(\tilde{\tau}, \delta)\Delta^m + h_1(\tilde{\tau}, \delta)\delta\Delta^{m-1} + \cdots + h_m(\tilde{\tau}, \delta)\delta^m. \tag{16}$$

Step 3. This part of the Proof follows as in [**4**]. We repeat it here for completeness.

We rewrite the right-hand side of (16) as:

$$\delta^m P = \delta^n\big(j_0(\tilde{\tau}, \Delta)\delta^k + \cdots + j_k(\tilde{\tau}, \Delta)\big), \quad \text{where } j_k(\tilde{\tau}, \Delta) \neq 0.$$

If $n \geqslant m$, it follows that P is a polynomial in $\tilde{\tau}$, δ and Δ.

Now, assuming $n < m$, we obtain a contradiction. In fact,

$$\delta^{m-n} P = j_0(\tilde{\tau}, \Delta)\delta^k + \cdots + j_k(\tilde{\tau}, \Delta), \quad \text{where } m - n > 0. \tag{17}$$

Since $j_k(\tilde{\tau}, \Delta) \neq 0$, there are real numbers α and β such that $j_k(\alpha, \beta) \neq 0$ and $\alpha \cdot \beta < 0$. Now, we take $\gamma = \sqrt{-\beta/\alpha}$ and $Q = (\alpha, 0, 0, 0, \gamma, 0)$.

Evaluating both sides of (17) in Q, gives

$$0 = j_k(\alpha, \beta) \neq 0.$$

Finally, $\tilde{\tau}$, δ, Δ are algebraically independent since their images under H are.

4.2. *Final Remarks.* 1. If Q_1 is not in the orbit of Q_2, we may ask if there is a polynomial $P \in \mathscr{I}$ such that $P(Q_1) \neq P(Q_2)$. The remark at the end of section 1.4 together with Theorem 1 implies that this is not always possible. For the action of the Euclidean Group on the nondegenerated conics, the answer is affirmative ([**4**].)

2. Another interesting plane geometry is the Galilean geometry, whose "motions" are the Galilean transformations of classical kinematics. A very complete account of this geometry can be found in [**5**]. (A brief reference for the orbits of the action of the Galileo Group on conics appears on page 300 of this book.)

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Rational Representations of Finite Groups: The Story of γ

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DENNIS KLETZING: I received my Ph.D. from Dartmouth College in 1973 working under Ernst Snapper. After teaching for a year in Vermont I moved to Florida, where I am currently Professor of Mathematics at Stetson University. My mathematical interests are mainly in the representation theory of finite groups and the history of mathematics.

Mathematics is full of examples that illustrate the principle of "extend and conquer." Put simply, mathematical results become clearer and more complete when the framework of the discussion is enlarged. The fundamental theorem of algebra is a good example. For it is usually difficult, if not impossible, to describe the number of integers or rational numbers that are solutions to a polynomial equation; but by enlarging the scope of the discussion to include complex numbers and by agreeing to count multiple solutions separately, we obtain the simple and elegant statement that every polynomial equation of positive degree n has exactly n complex solutions. This article discusses an example of the "extend and conquer" principle that arises in the representation theory of finite groups—the Artin induction theorem. This theorem shows that every linear representation of a finite group over the field of rational numbers extends to a permutation representation of the group and, as such, provides a link between representing the elements of a finite group as linear transformations of a vector space and representing them as permutations of a set. But more importantly, when the Artin theorem is stated in terms of group characters, it provides a precise way to measure, numerically, how close the rational representations of the group are to being permutation representations. We denote this numerical measure by the symbol γ. The purpose of this article is to discuss the history of γ, its existence and known values, and to indicate briefly the local algebraic methods for determining this invariant.

1. Introduction. Let Q stand for the field of rational numbers and let G be a finite group. A *rational representation* of G is a homomorphism $\rho: G \to GL(V)$ from G into the group $GL(V)$ of nonsingular linear transformations of a finite-dimensional vector space V over Q. By choosing a coordinate system for V, a rational representation may be regarded as a way of associating a non-singular matrix with rational number entries to each element in G. The *character* of ρ is the function $\chi_\rho: G \to Q$ defined by setting $\chi_\rho(\sigma) = \operatorname{trace} \rho(\sigma)$ for each element σ in G, where trace $\rho(\sigma)$ means the sum of the diagonal entries in any matrix representation of $\rho(\sigma)$; recall that all matrix representations of a linear transformation are similar and thus have the same trace ([**8**], p. 328). If V has a basis that is permuted by every

linear transformation $\rho(\sigma)$, $\sigma \in G$, then ρ is called a *permutation representation* of G and its character χ_ρ is called a *permutation character*.

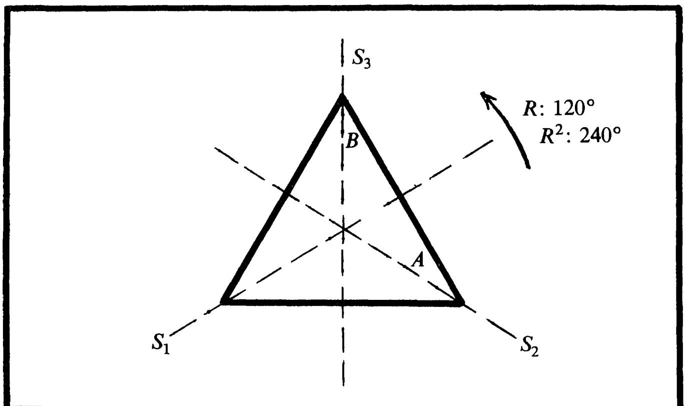
Consider, for example, the symmetric group $\mathrm{Sym}(n)$ of all permutations of the set $\{1, \ldots, n\}$, where n is a positive integer. Let Q^n stand for the vector space of n-tuples over Q and define the function $\rho: \mathrm{Sym}(n) \to GL(Q^n)$ as follows: if σ is a permutation in $\mathrm{Sym}(n)$, let $\rho(\sigma): Q^n \to Q^n$ stand for the mapping defined by setting $\rho(\sigma)(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all n-tuples $(x_1, \ldots, x_n) \in Q^n$. Then $\rho(\sigma)$ is a nonsingular linear transformation that permutes the coordinate entries of every vector in Q^n according to σ, and it follows easily that ρ is a rational representation of the group $\mathrm{Sym}(n)$. Now, for $i = 1, \ldots, n$, let e_i stand for the vector all of whose entries are zero except the ith entry which is equal to 1. Then $\{e_1, \ldots, e_n\}$ is a basis for Q^n and we find that $\rho(\sigma)(e_i) = e_{\sigma(i)}$ for $i = 1, \ldots, n$ and all elements $\sigma \in \mathrm{Sym}(n)$. Thus, Q^n has a basis that is permuted by every transformation $\rho(\sigma)$, and therefore ρ is a permutation representation of $\mathrm{Sym}(n)$.

For a second example of a rational representation let us turn to Galois theory. Let K be a Galois extension of Q with Galois group G. Then every element in G is an automorphism of the field K and is, therefore, a non-singular linear transformation of K as a vector space over Q. Hence, the mapping $\rho: G \to GL(K)$ defined by $\rho(\sigma) = \sigma$ for all elements $\sigma \in G$ is a rational representation of G. Now, the Normal Basis Theorem in Galois theory states that K, when viewed as a vector space over Q, has a basis of the form $\{\sigma(x) | \sigma \in G\}$ for some element $x \in K$ ([**8**], p. 229). Since $\rho(\sigma)(\tau(x)) = \sigma\tau(x)$ for all elements σ, τ in G, ρ permutes this basis and is, therefore, a permutation representation of G. For the character χ_ρ of ρ we find that $\chi_\rho(\sigma) = 0$ for every nonidentity element σ in G while $\chi_\rho(1) = |G|$, the order of G.

As a final example of a rational representation, one which is not a permutation representation, let us turn to geometry. Let T stand for the symmetry group of an equilateral triangle, that is, the group of rigid motions of the Euclidean plane that map an equilateral triangle to itself; T has order 6 and contains three rotations and three reflections. Here the underlying vector space is the real plane R^2 although we may just as easily consider it to be the rational plane Q^2 because each transformation in T has a matrix representation whose entries lie in Q, as illustrated in Figure 1. As such, each element in T is a non-singular linear transformation of Q^2 and the mapping $\rho: T \to GL(Q^2)$ defined by setting $\rho(\sigma) = \sigma$ for all $\sigma \in T$ is a rational representation of T.

We claim, however, that ρ is not a permutation representation of T. For suppose that Q^2 has a basis $\{A, B\}$ that is permuted by all transformations in T. Then only two permutations are possible: the identity permutation and the permutation that interchanges A and B. But there are six different rigid motions in T and each is uniquely determined by the corresponding permutation of A and B. It follows, therefore, that no such basis for Q^2 exists and hence that ρ is not a permutation representation of T.

But let us not give up so easily—let us "extend and conquer!" Enlarge the plane to three dimensions and select a basis $\{A, B, C\}$ for R^3 as indicated in Figure 2. We extend the representation ρ of T to a 3-dimensional representation ρ^* of T as



Matrix representation relative to basis $\{A, B\}$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

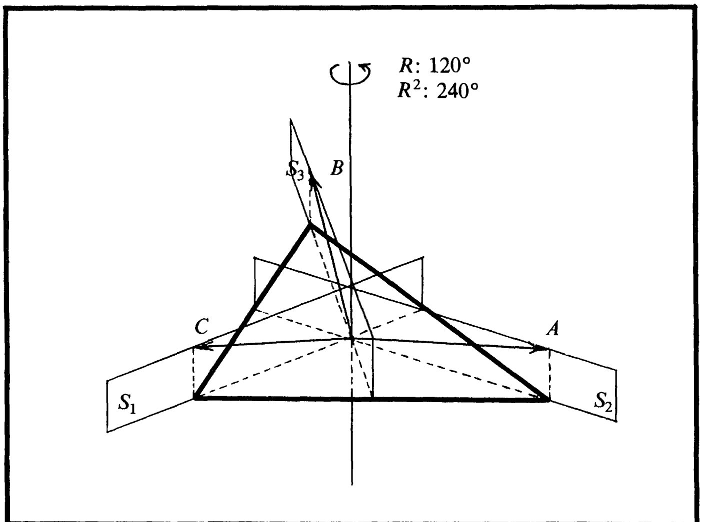
$$R \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad S_2 \leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$R^2 \leftrightarrow \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad S_3 \leftrightarrow \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

FIG. 1

follows. Extend the three rotations of the triangle to the three rotations of R^3 about the axis perpendicular to the plane of the triangle and passing through its center; these rotations then permute the base vectors A, B, C. Now, extend the three reflections in T to the three symmetries of R^3 whose reflecting planes are perpendicular to the plane of the triangle and which contain the base vectors A, B, C; these symmetries also permute the vectors A, B, C. The extended representation ρ^* is, therefore, a permutation representation of T. Figure 2 summarizes the matrices and permutations for ρ^*. Thus, while there is no basis for Q^2 that is permuted by the transformations in T, the representation may be extended to one that has such a basis.

In general, it is not an easy matter to describe explicitly the rational representations of a finite group. The situation for permutation representations is different, however, as they may be described in terms of subgroups and their cosets. Let H be any subgroup of a finite group G and let $G/H = \{A_1, \dots, A_n\}$ stand for the collection of left cosets of H in G, where $n = [G : H]$ is the index of H in G. Let $Q[G/H]$ stand for the vector space over Q having the left cosets in G/H as basis; a typical element in $Q[G/H]$ has the form $c_1 A_1 + \cdots + c_n A_n$, where the coefficients



Matrix representation relative to basis $\{A, B, C\}$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} \qquad S_1 \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

$$R \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \qquad S_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$

$$R^2 \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \qquad S_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

FIG. 2.

$c_1, \ldots, c_n$ are rational numbers. If $A = \tau H$ is a typical left coset in G/H and σ is any element in G, let $\sigma A = \sigma\tau H$, and define the mapping $\rho_H\colon G \to GL(Q[G/H])$ by setting $\rho_H(\sigma)(c_1A_1 + \cdots + c_nA_n) = c_1(\sigma A_1) + \cdots + c_n(\sigma A_n)$ for all vectors $c_1A_1 + \cdots + c_nA_n$ in $Q[G/H]$ and all $\sigma \in G$. Then ρ_H is a rational representation of G that permutes the left coset basis $\{A_1, \ldots, A_n\}$ and is, therefore, a permutation representation of G. Let $\pi_H\colon G \to Q$ stand for the character of ρ_H. Then, using the left coset basis for $Q[G/H]$, we find that

$$\pi_H(\sigma) = \frac{\text{Number of conjugates of } \sigma \text{ lying in } H}{\text{Total number of conjugates of } \sigma \text{ in } G}[G\colon H]$$

for every element σ in G. The permutation characters π_H associated with the subgroups H of G are important because they generate all permutation characters

of G. More precisely, if π is the character of a permutation representation of G, then there are subgroups $H_1, \ldots, H_s$ and positive integers $N_1, \ldots, N_s$ such that $\pi = N_1\pi_{H_1} + \cdots + N_s\pi_{H_s}$; that is, every permutation character of G is a linear combination of permutation characters of the form π_H with positive integer coefficients. For a given permutation representation, the subgroups H that occur in this decomposition are just the stabilizers associated with each orbit of the representation. This, then, provides an effective computational procedure for determining all permutation characters of a finite group.

Unfortunately, the characters of rational representations are not as easily described. There is, however, a fundamental relationship between these two types of characters and this is what the Artin induction theorem is all about.

ARTIN INDUCTION THEOREM. *Let G be a finite group of order $|G|$ and let χ be the character of a rational representation of G. Then there are cyclic subgroups $H_1, \ldots, H_s$ and integers $N_1, \ldots, N_s$ such that $|G|\chi = N_1\pi_{H_1} + \cdots + N_s\pi_{H_s}$.*

Thus, the character of a rational representation of G may always be written as a linear combination of permutation characters with cyclic stabilizers and in which the coefficients are *rational* numbers.

It is not our intention to prove the Artin theorem since one may be found in any standard book on representation theory such as ([**2**] p. 378), but rather to interpret this theorem geometrically. Let χ_ρ be the character of a rational representation $\rho\colon G \to GL(V)$ and let $|G|\chi_\rho = N_1\pi_{H_1} + \cdots + N_s\pi_{H_s}$, where $H_1, \ldots, H_s$ are subgroups of G and $N_1, \ldots, N_s$ are integers, not necessarily positive. By collecting those terms on the right side of this equation that have the same sign, we may write the equation in the form $|G|\chi_\rho = \pi^* - \pi$, where π and π^* are permutation characters, or, equivalently, as $\chi_\rho + [(|G| - 1)\chi_\rho + \pi] = \pi^*$. This last equation, when expressed in terms of representations, says that the representation ρ has an extension to a permutation representation $\rho^*\colon G \to GL(V^*)$ whose character is π^*. That is, V is a subspace of V^* and each of the transformations $\rho(\sigma)$ extends to the transformation $\rho^*(\sigma)$, and V^* has a basis that is permuted by each of the transformations $\rho^*(\sigma)$, $\sigma \in G$.

Said more informally, it means that the representation space V extends to a representation space V^* having a basis that is permuted by every element in the group. We call this the geometric interpretation of the Artin induction theorem. Thus, no matter how badly a rational representation of a group may act, it is always possible to extend the representation in such a way that there is a basis permuted by the group elements.

Let us now turn to the main topic of this article: using the Artin theorem to measure precisely how far the rational representations of a group differ from being permutation representations.

2. The Invariant γ. Let G be a finite group. Any expression of the form $\sum N_H\pi_H$, where the summation is over arbitrary subgroups H of G and where the coefficients N_H are integers, is called a *generalized permutation character* of G. We emphasize

that the subgroups H in this summation do not have to be cyclic and that the coefficients N_H do not have to be positive.

Now, the Artin induction theorem shows that if χ is the character of any rational representation of G, then $|G|\chi$ is a generalized permutation character of G. Consequently there is a smallest positive integer $\gamma(G)$ with the property that $\gamma(G)\chi$ is a generalized permutation character for χ the character of any rational representation of G. At worst, $\gamma(G) = |G|$. In general, $\gamma(G)$ divides $|G|$ and $\gamma(G) = 1$ if and only if the character of every rational representation of G is a generalized permutation character. Thus, the invariant γ measures in a global sense how close the rational representations of a group are to being permutation representations.

Values of γ have been known ever since the German mathematician G. Frobenius (1849–1917) explicitly constructed the rational representations of the symmetric groups in 1900 ([**5**]). Moreover, Frobenius showed that the character of every such representation is a generalized permutation character and hence that $\gamma(\mathrm{Sym}(n)) = 1$ for every positive integer n.

The year 1900 also saw the first of a nine part series of papers by the English clergyman and mathematician Alfred Young (1873–1940). In part five of "On Quantitative Substitutional Analysis," published in 1930, Young deals with the representation theory of the groups of signed and even signed permutations. A signed permutation is any permutation σ of the set $\{\pm 1, \dots, \pm n\}$ with the property that $\sigma(-i) = -j$ whenever $\sigma(i) = j$. The collection of signed permutations form a group which we denote by $\mathrm{Sign}(n)$ and which is frequently called the n-dimensional hyperoctahedral group because it is isomorphic to the symmetry group of a hyperoctahedron in real n-space. The subgroup $\mathrm{Sign}^+(n)$ of even signed permutations consists of those signed permutations that are even as permutations of the set $\{\pm 1, \dots, \pm n\}$. Young explicitly constructs the representations of these groups, and it follows from his work that $\gamma(\mathrm{Sign}(n)) = \gamma(\mathrm{Sign}^+(n)) = 1$ for every positive integer n.

The results of Frobenius and Young deal with specific groups and were obtained, as mentioned above, by intricate constructive methods. The first result of a general nature was the Artin induction theorem, proved by E. Artin (1898–1962) in 1931, which shows that the invariant $\gamma(G)$ in fact exists for any finite group G and divides the order of G. After Artin, the problem of determining γ remained dormant until 1972, at which time there was a flurry of activity centered around finding $\gamma(G)$ when G is a p-group. Working independently, G. Segal, J. Ritter, and J. Rasmussen showed that $\gamma(G) = 1$ for every p-group ([**12**], [**11**], [**9**]); Segal used methods of algebraic K-theory while Ritter and Rasmussen used standard group-theoretic methods. Rasmussen extended the result in 1974 by showing that γ is either 1 or 2 for every nilpotent group and, in addition, gave necessary and sufficient conditions under which each case occurs ([**9**]). Also, in 1973 L. Solomon announced that $\gamma(\mathrm{PSL}(2, q)) = 1$ for q the power of an odd prime ([**13**]). Finally, in 1977 Rasmussen determined $\gamma(G)$ when G is a faithful metacyclic group ([**10**]). Such a group has a standard presentation in terms of generators and relations and, using this presentation, Rasmussen gave an explicit formula for γ. It follows from this work

that $\gamma = 1$ for every dihedral group and that, for every positive integer n, there is a finite group G for which $\gamma(G) = n$.

The flurry of activity during the 1970's that led to the determination of γ for various groups was important from another point of view, for out of it emerged a uniform approach to the study of γ. This approach, which we call the local theory of γ, uses the local methods of commutative algebra and is akin to the local techniques of algebraic geometry. Briefly, the local theory identifies certain subsets of the group called local classes, which are sets of elements whose p-regular parts generate conjugate cyclic subgroups, p a prime, and associates with each local class V a local invariant γ_V. γ then turns out to be the least common multiple of the local invariants γ_V. To find γ_V, one first associates a local subgroup G_V with each local class V and then uses the characters of G_V to determine γ_V. We refer the reader to [**6**] for the details of this approach. The important point here is that the local theory not only characterizes γ in terms of local invariants, it also provides an effective computational procedure for determining the local invariants in terms of the local structure of the group.

The local theory is especially useful in determining $\gamma(G)$ when G is a group all of whose representations over the field of complex numbers have rationally valued characters. Such a group is called a *Q-group* and is characterized internally by the property that two elements are conjugate if and only if they generate conjugate cyclic subgroups. Many well-known groups are Q-groups. The symmetric groups, for example, are Q-groups, as well as the groups $\mathrm{Sign}(n)$ and $\mathrm{Sign}^+(n)$ of signed and even signed permutations. More generally, the Weyl group of any semisimple complex Lie algebra is a Q-group. The Weyl groups fall into seven isomorphism types which are labeled by the symbols A_n, B_n, D_n, F_4, E_6, E_7, E_8. Those of type A_n, B_n and D_n are called the classical Weyl groups and are isomorphic to $\mathrm{Sym}(n+1)$, $\mathrm{Sign}(n)$ and $\mathrm{Sign}^+(n)$, respectively; the fact that they are Q-groups follows from the work of Frobenius and Young. The groups of type F_4, E_6, E_7 and E_8 are the exceptional Weyl groups, and the fact that they are Q-groups follows from the explicit construction of their irreducible complex characters by T. Kondo ([**7**]) and J. S. Frame ([**3**], [**4**]). When the local theory of γ is applied to the Weyl groups, we find that $\gamma(A_n) = \gamma(B_n) = \gamma(D_n) = 1$ while $\gamma(F_4) = \gamma(E_6) = \gamma(E_7) = 2$ and $\gamma(E_8) = 4$. Let us mention, incidentally, that in 1972 M. Benard showed that every complex representation of an exceptional Weyl group is equivalent to a rational representation ([**1**]).

The local methods used to determine $\gamma(G)$ when G is a Q-group also reveal a criterion on the internal structure of the Q-group under which $\gamma(G) = 1$. Let G be a Q-group, p a prime, and σ an element in G whose order is not divisible by p. Let $N(\sigma) = \{\tau \in G | \tau(\sigma)\tau^{-1} = (\sigma)\}$ and $C(\sigma) = \{\tau \in G | \tau\sigma\tau^{-1} = \sigma\}$ stand for the normalizer and centralizer of σ, respectively, and let $N(\sigma)_p$ stand for a Sylow p-subgroup of $N(\sigma)$. Then $C(\sigma)_p = N(\sigma)_p \cap C(\sigma)$ is a Sylow p-subgroup of $C(\sigma)$. We say that G is *locally split* at the pair (p, σ) if there is a subgroup K of $N(\sigma)_p$ such that $N(\sigma)_p = C(\sigma)_p \times K$, the direct product of $C(\sigma)_p$ and K.

LOCAL SPLITTING CRITERION. *Let G be a Q-group. If G is locally split at the pair (p, σ) for every element $\sigma \in G$ whose order is not divisible by the prime p, then $\gamma(G)$ is not divisible by p. If G is locally split at all such pairs (p, σ) for all primes p, then $\gamma(G) = 1$.*

The local splitting criterion is a very effective way to determine γ since it not only provides conditions under which $\gamma = 1$ but also identifies those primes that may divide γ. But determining whether or not a Q-group is locally split at a given pair (p, σ) is usually a tedious process. Frequently the group may have a global structure from which one may deduce local splitting without resort to explicit calculation. This happens, for example, with the classical Weyl groups; these groups are everywhere locally split, thus giving a new interpretation to the fact that $\gamma = 1$ for these groups. For the exceptional Weyl groups, however, the situation is more complicated. These groups are not everywhere locally split and consequently one must analyze each of the pairs (p, σ) for which $N(\sigma)_p$ does not split over $C(\sigma)_p$.

This completes our discussion of the invariant γ. Although the local theory characterizes γ and is useful in calculating and understanding this invariant, there are questions that still remain unanswered. Why, for example, is $\gamma(G)$ always a power of 2 when G is a Q-group?; the empirical evidence suggests that this is always true, but so far the local theory has failed to indicate why. More generally, is it possible to characterize internally those groups for which $\gamma = 1$?

3. Conclusion. The underlying theme of this article has been the Artin induction theorem. When interpreted from a geometric point of view, this theorem reveals an unexpected relationship between the rational representations of a group and the permutation representations of the group. When expressed in terms of group characters, it provides a concise numerical way to measure this relationship by means of the invariant γ. But perhaps the most striking aspect of this theorem is the circumstance surrounding its discovery, for it does not stem from group theory as one might expect, but rather from Artin's generalization of the classical notion of an L-series. Every character of the Galois group of an algebraic number field has associated with it an infinite series called an L-series. By studying the analytic properties of the L-series, one obtains information about the distribution of primes in the number field as well as information about the class number of the field.

The concept of an L-series was first introduced by Dirichlet to investigate the distribution of primes in an arithmetic sequence and was later extended by Weber to those algebraic number fields whose Galois group is abelian—the abelian L-series. Artin then generalized the notion of an L-series to completely arbitrary number fields. By expressing the original character of the Galois group as a linear combination of permutation characters, it would follow that his new L-series could be written as the product of classical abelian L-series and thus make it possible to derive properties of the new series from known properties of the abelian series. It was for this reason that Artin began the search for a relationship between the

characters of rational representations and permutation characters, a search which eventually led to his induction theorem in 1931.

In keeping with the spirit of "extend and conquer," let us conclude by noting that the Artin induction theorem itself was extended by R. Brauer in 1947 to include characters of all complex representations of a finite group and not just of rational representations, and that Brauer's theorem, in turn, was extended to include characters of representations over any field by E. Witt in 1952 and S. D. Berman in 1956.

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The Editor's Corner: Strings, Substrings, and the 'Nearest Integer' Function

HERBERT S. WILF

The use of 'the nearest integer' function can tidy up a good many otherwise complicated formulas. Let's use $\langle x \rangle$ to denote the integer nearest* to x, so, for instance, $\langle 2.7 \rangle = 3$, $\langle 5.1 \rangle = 5$, and so forth.

As an example of the many talents of this function, consider the following formula for the number of permutations of n letters that have no fixed points:

$$D_n = n! \sum_{j=0}^{n} (-1)^j/j! \qquad (n \geq 1).$$

Obviously this number is pretty close to $n!/e$ because the sum is a truncation of the infinite series for e^{-1}. Now it will take only a moment or two to convince yourself from this formula that the difference between D_n and $n!/e$ is less than $1/2$. Since D_n is an integer, it must be the nearest integer to $n!/e$. Thus we have a much simpler formula for D_n, namely,

$$D_n = \langle n!/e \rangle \qquad (n \geq 1).$$

For our second example we take the Fibonacci numbers. Does anyone know an explicit formula for F_n that clearly reveals, at a casual glance, the fact that F_n is an integer, and how big an integer it is? For example, the nth Fibonacci number is given by the formula

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right\},$$

which doesn't reveal a whole lot to a casual glance. However, the second term approaches 0 rapidly. So if n is large enough, F_n will be the nearest integer to the first term. It happens that $n = 0$ is already large enough, and, therefore, we have the slightly simpler expression

$$F_n = \left\langle \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \right\rangle \qquad (n \geq 0),$$

which, whatever its cosmetic demerits may be, at least shows instantly that F_n is an integer, and shows its size too.

This brings me to the main topic of today's tour, which is about counting certain kinds of words.

*If $x = m + 1/2$, for integer m, put $\langle x \rangle = m$.

Of the 26^n possible strings of n English letters, how many do not contain CAT as a substring? More generally, suppose we have an alphabet of A letters, and that we are given a string σ of letters over that alphabet. We want to know how many of the A^n n-letter strings do not contain σ as a substring. By 'contain' we will always mean contain as a consecutive block of letters.

Speaking personally, my own interest in such problems was awakened by questions that were asked (and later answered) by a former doctoral student of mine, Judith Dayhoff. She was working on her thesis in biophysics, and was studying, in a certain crayfish, the action of closing the claw. The question was to identify, if possible, the specific sequence of nerve impulses that cause the claw to close.

She studied reams of experimental data that showed the various spike trains marching down the nerves, and then codified it into the form of words. To do that, the vertical axis of intensity of the signal was divided up into a modest number of subranges, and each of these was made to correspond to a certain letter. The entire time sequence of nerve spike train propagation was in that way changed into a long word over a certain alphabet.

Now the question was one about strings and substrings of letters. By looking at the sequence of letters that preceded an observed claw closing, she became interested in knowing if there was anything unusual about it, such as containing some substring unusually often. Before dealing with that problem, one first needs to know what is 'unusually often.' That leads next to analyzing the probability that a word of given length contains some given number of copies of the distinguished substring σ, and finally to analyzing the question posed at the beginning of this column, about the probability of containing no copies at all of σ.

I don't want to tell any more of the story that is hers to tell, so if you want to know more about how the biophysical-mathematical work turned out (I will say that it was quite interesting), please refer to [**1**] or [**2**] below.

Here I would like to discuss some of the older theory of excluded substrings of strings, though perhaps from a different viewpoint. In a subsequent column some newer results will be reviewed.

Let $f(n) = f(n; \sigma)$ be the number of n-letter words over an alphabet of A letters that do not contain a given string σ as a substring. Imagine that the whole list of $f(n)$ such words is sitting in front of you. Form a new list of exactly $Af(n)$ words, each of length $n + 1$, by adjoining to each word on that list, in turn, each one of the A possible letters, as a new first letter.

Some of the words on this new list of $Af(n)$ words of $n + 1$ letters will, however, contain σ as a substring. We want to count those, subtract the number from $Af(n)$, and, thereby, get a recurrence relation for the unknown function $f(n)$, of the form $f(n + 1) = Af(n)-?$

Suppose $\sigma =$ 'CAT'. A word on the new list that contains 'CAT' must actually begin with 'CAT' and, therefore, it consists of a word from the list of CAT-free words of length $n - 2$ plus the prefix 'CAT'. Hence, the number of words on the

new list that contain 'CAT' is precisely $f(n-2)$, and we have the recurrence

$$f(n+1; CAT) = Af(n; CAT) - f(n\text{–}2; CAT) \tag{1}$$

valid for $n \geq 0$. This together with the obvious initial values $f(j; CAT) = 0$ if $j < 0$ and $f(0; CAT) = 1$ define the function f completely.

Things can be more delicate than the above, though. Suppose, for instance, that $\sigma =$'PUP'. Consider a word on the new list that contains σ. It is no longer true that σ must appear right at the beginning. For example, the word might be 'ZZPUPxxx...xxx' ($n - 4$ x's), in which the 'UPxxx...xxx' part is a PUP-free word of length $n - 2$. The essential difference is that the word 'PUP' has an initial substring 'P' that is also a final substring. Words of that kind, such as 'MAGMA', 'INGRATIATING', 'BERIBERI', and so forth (can you think of a legal English word that has initial and final proper substrings of length 5 or more that are the same?) require a subtler analysis, of which we will say more later. For the moment, we return to the simpler case since there are some interesting points already there.

So let σ be a word of length L over an alphabet of A letters, and suppose that no proper initial string of σ is also terminal. Then as in (1) we find that

$$f(n; \sigma) = Af(n-1; \sigma) - f(n-L; \sigma) \qquad (n \geq 1)$$
$$(f(0; \sigma) = 1, f(j; \sigma) = 0 \qquad (j < 0)). \tag{2}$$

This is a linear difference equation with constant coefficients, and so we look for solutions of the form (we suppress the 'σ' now) $f(n) = r^n$, where r is to be determined. If we substitute in (2) and cancel the common factor we find that, as usual, r must be a root of the algebraic equation

$$r^L = Ar^{L-1} - 1 \tag{3}$$

of degree L. Let $r_1, \ldots, r_L$ be the roots of this equation, and suppose that they are simple. Then the general solution of the recurrence formula (2) is

$$f(n) = c_1 r_1^n + c_2 r_2^n + \cdots + c_L r_L^n. \tag{4}$$

Now here is one point of interest. If our alphabet has more than two letters in it ($A > 2$) then, of the L roots of equation (3), *all but one lie inside the unit circle*, while the other one, r_1, is positive and real. What that means is that while (4) has L terms in it, as soon as n gets at all large, the total contribution of the last $L - 1$ terms will be less than, say, $1/2$ in absolute value.

It follows that the number of n-letter σ-free words is exactly $\langle c_1 r_1^n \rangle$ for all sufficiently large n. A little closer analysis, that we won't go into here, shows that 'all sufficiently large n' can be replaced by 'all n' if $A > 2$.

Let's see why only one root of (3) lies outside the unit disk. By Rouché's theorem, the two functions $z^L - Az^{L-1}$ and $z^L - Az^{L-1} + 1$ will have the same number of zeros inside a disk of radius R centered at the origin, provided only that $1 < |z^L - Az^{L-1}|$ whenever $|z| = R$. But if $|z| = R$ then

$$|z^L - Az^{L-1}| \geq AR^{L-1} - R^L = R^{L-1}|A - R|,$$

which is obviously >1 if $R = 1$ and $A > 2$, for instance. Since $z^L - Az^{L-1}$ has exactly $L - 1$ roots inside the unit disk, the same must be true of the polynomial that we are interested in. ■

Next, since $c_1 r_1^n$ is giving us such an astonishingly good approximation to $f(n)$, we suddenly find that our interest in the constant c_1 is perking up. Can we find c_1 in some simple form? Indeed we can.

In fact, c_1 is determined by requiring that the solution (4) satisfy the initial conditions contained in (3). More precisely, we require that

$$c_1 r_1^j + c_2 r_2^j + \cdots + c_L r_L^j = \begin{cases} 0, & \text{for } j = -1, -2, \dots, -(L-1); \\ 1, & \text{for } j = 0. \end{cases}$$

This shows that c_1 is the $(1,0)$ entry of the inverse of the $L \times L$ Vandermonde matrix $\{r_i^j\}_{i=1,L;\, j=0,-(L-1)}$. It is, therefore, the ratio of two immense products in which almost everything cancels out, leaving just $c_1 = r_1/(Lr_1 - (L-1)A)$. This all proves*

THEOREM 1. *Let σ be a fixed word of L letters chosen from an alphabet of $A > 2$ letters, and suppose that no proper initial substring of σ is also a terminal substring. For each n, let $f(n)$ be the number of n-letter words over that alphabet that do not contain the string σ. Then for every $n = 0, 1, 2, \dots,$*

$$f(n) = \left\langle \frac{r_1^{n+1}}{Lr_1 - (L-1)A} \right\rangle,$$

where r_1 is real and positive, and is the unique root of (3) *that has modulus greater than* 1.

Example. How many 4-letter words, formed from an alphabet X, Y, Z, do not contain 'XY'? Here $n = 4$, $A = 3$, $L = 2$, and the root r_1 of (3) is $2.618\dots$. Hence, the number of 'XY'-free words is

$$\left\langle \frac{(2.618\dots)^5}{2.236\dots} \right\rangle = \langle 55.03 + \rangle.$$

We expect 55 such words, therefore. It's easy to see that this is correct, for among the 81 possible words there are 9 of the type 'XY**', 9 more like '*XY*', and 9 more '**XY'. That would make 27 words that *do* contain 'XY', except that we counted 'XYXY' twice, so there are 26 of the 81 possible words that have 'XY', leaving 55 that don't. ■

In general, is there any hope of 'finding' the root r_1 in some explicit form? Indeed there is, and not just an approximation either. What we have to do is

*This argument would have been shorter if we used generating functions, but the method we have chosen seems a little more elementary.

explicitly solve the polynomial equation (3) of degree L. The secret weapon is the Lagrange inversion formula, one version of which I will now quote ([**3**, p. 133]; see [**6**] for a recent application of the same idea).

THEOREM 2. *Let $\phi(z)$ be analytic in some region containing a contour C surrounding the origin, and let t be such that the inequality $|t\phi(z)| < |z|$ is satisfied on the boundary of C. Then the equation $\zeta = t\phi(\zeta)$ has one root ζ inside C and that root is given by*

$$\zeta = \sum_{n=1}^{\infty} \frac{t^n}{n!}\alpha_n, \tag{5}$$

where, for each $n \geq 1$,

$$\alpha_n = \left[\frac{d^{n-1}}{dz^{n-1}}\{\phi(z)^n\}\right]_{z=0}. \tag{6}$$

We can easily apply this to the polynomial equation (3). After dividing by Ar^L that equation becomes

$$\frac{1}{r} = \frac{1}{A}\left\{1 + \frac{1}{r^L}\right\},$$

which is exactly of the type considered in Lagrange's theorem, with $t = 1/A$, $\zeta = 1/r$, and $\phi(\zeta) = 1 + \zeta^L$. If we just substitute in (5) and (6) we get, after a little calculation, a formula for the principal root of (3), to wit

$$\begin{aligned} \frac{1}{r_1} &= \sum_{s\geq 0}\binom{sL+1}{s}\frac{1}{(sL+1)}A^{-sL-1} \\ &= \frac{1}{A} + \frac{1}{A^{L+1}} + \frac{L}{A^{2L+1}} + \cdots. \end{aligned} \tag{7}$$

This infinite series is both convergent and asymptotic. It shows that the interesting root r_1 is very close to A. The combination of the 'nearest integer' formula of theorem 1 above, together with the formula (7) for the principal root r_1 gives quite an explicit solution to the question that we asked at the beginning.

Here's another way to think about the answer. The number of all n-letter words is A^n. The number of words that don't contain the given substring is nearly cr_1^n. We might say that the exclusion of the given substring forces the alphabet to act as if it had only r_1 letters in it instead of A letters. In view of (7), the alphabet 'effectively contains about $A - A^{-(L-1)}$ letters' in some information-theoretic sense. For instance, the 'CAT'-free words on the English alphabet can be made by freely forming words out of an alphabet of 25.99852... (this is the root r_1 of (3) when $A = 26$ and $L = 3$) letters, so to speak (!!).

Remarks. (1) Some of the ideas of this column were worked out by Dr. Albert Nijenhuis and myself. When these words appear in print, he will have retired. I would like to take this opportunity to wish him well.

(2) Another chapter in this story will be told in one of these columns in the near future. There we will find that $f(n)$ can be expressed in a form that doesn't have a thing to do with the roots of the equation (3). The equality of these two forms will give interesting identities. We'll also have an update on recent results about the MAGMA-like situation where the hypotheses of theorem 1 are not fulfilled.

(3) A number α is called a P-V number (for Pisot-Vijayaraghavan, see [**4**]) if it satisfies a polynomial equation whose coefficients are integers, whose highest coefficient is 1, and if all of the roots of that equation other than α have absolute values less than 1. The number r_1 that occurred above, is a P-V number.

These numbers have many important properties. For example a beautiful theorem of Salem and Zygmund [**5**] shows their relationship with uniqueness of Fourier series. A set S is a set of uniqueness if every Fourier series that converges to zero except possibly at the points of S converges to zero identically. Suppose we manufacture a Cantor set by using, instead of the customary dissection ratio of $1/3$, a ratio of ξ. This means that we will subdivide in the ratios $\xi : 1 - 2\xi : \xi$. The question is, for exactly which ξ do we obtain a set of uniqueness for Fourier series? The answer is, precisely when $1/\xi$ is a P-V number.

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NOTES

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The Bisection Method: Which Root?

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Suppose that a continuous real-valued function f has $2N-1$ roots, $r_1, r_2, \ldots, r_{2N-1}$, in an interval (a, b) whose endpoints are not roots. If all the roots are simple, then $f(a)f(b) < 0$ and the bisection method can be used to find a root of f. (Notice that when there are an even number of simple roots, $f(a)f(b) > 0$, so that the bisection method does not apply over (a, b).) It is the purpose of this note to show that the probability that the bisection method locates the ith smallest root is zero when i is even, assuming any continuous joint distribution for the roots. Furthermore, assuming the roots are independent and uniformly distributed on (a, b), the probability that it finds the ith smallest root is $1/N$, when $i = 1, 3, \ldots, 2N-1$.

Recall that the bisection method locates a root in (a, b) by first examining the sign of $f(c)$, where $c = (a+b)/2$. If $f(c) = 0$, a zero-probability event, then a root has been found. If $f(a)f(c)$ is negative, the process begins anew with b replaced by c. Otherwise the process recommences with a replaced by c. Ultimately, this procedure will find a root of f to any desired accuracy by finding an interval of arbitrarily small length containing a root.

Let (a, b) contain roots $r_1, r_2, \ldots, r_{2N-1}$ where $a < r_1 < r_2 < \cdots < r_{2N-1} < b$. Let $P(i)$ denote the probability that the bisection method finds root r_i, for $i = 1, 2, \ldots, 2N-1$. Let X denote the number of roots in (a, c).

We first prove, by induction on N, that $P(i) = 0$ when i is even. When $N = 1$, the assertion is immediate. Assume the assertion is true for any interval (α, β) which contains $2j-1$ roots, $j = 1, 2, \ldots, N$. Suppose (a, b) has $2N+1$ roots. There are four cases to consider:

(i) $X = 2N+1$ (ii) $X = 0$ (iii) $X = 2j-1$ for some $j \in \{1, \ldots, N\}$

(iv) $X = 2j$

for some $j \in \{1, \ldots, N\}$.

Notice that the bisection method will always search next the half of the interval that contains an odd number of roots. In case (i) or (ii), the bisection method will search the interval that contains all the roots and will eventually reach case (iii) or (iv). In case (iii), the bisection method will replace b with c and begin again. Here, $P(i) = 0$ for $i > 2j-1$, and by our induction hypothesis $P(i) = 0$ when $i = 2, 4, \ldots, 2j-2$. Thus $P(i) = 0$ for $i = 2, 4, \ldots, 2N$. In case (iv), a will be replaced by c and the process begins anew. Here $P(i) = 0$ for $i \leqslant 2j$ and by our induction hypothesis, $P(i) = 0$ for $i = 2j + k$ for all even k. Thus $P(i) = 0$ for $i = 2, 4, \ldots, 2N$, and the induction is complete.

Thus, (1) becomes

$$(1 - 4^{-N})P(i) =$$

$$\left(\frac{1}{2}\right)^{2N+1}\left[\sum_{\substack{k=i\\ k \text{ odd}}}^{2N-1}\binom{2N+1}{k}\left(\frac{2}{k+1}\right) + \sum_{\substack{k=2\\ k \text{ even}}}^{i-1}\binom{2N+1}{k}\left(\frac{2}{2N+2-k}\right)\right].$$

Substituting j for $k - 1$ gives us:

$$(1 - 4^{-N})P(i) =$$

$$\left(\frac{1}{2}\right)^{2N}\left[\sum_{\substack{k=i\\ k \text{ odd}}}^{2N-1}\frac{(2N+1)!}{(k+1)!(2N+1-k)!} + \sum_{\substack{j=1\\ j \text{ odd}}}^{i-2}\frac{(2N+1)!}{(j+1)!(2N+1-j)!}\right]$$

$$= 4^{-N}\sum_{\substack{j=1\\ j \text{ odd}}}^{2N-1}\frac{(2N+1)!}{(j+1)!(2N+1-j)!},$$

which no longer depends on i. Therefore, $P(i) = 1/(N+1)$. The induction is complete.

More generally, through the probability integral transformation, we obtain the same results when the roots are independent and identically distributed over (a, b) according to an arbitrary positive continuous distribution f, and when the point c for which the sign of $f(c)$ is tested is the median of f restricted to the "current" subinterval (α, β).

Spaces with Locally Compact Completions are Compact

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Department of Mathematics, University of Catania, Catania, Italy

In this note we characterize compactness in metrizable spaces in a way that has apparently not been pointed out elsewhere. We refer the reader to [**4**] for terminology. In particular, given a metrizable space X, we say that a metric d on X is *compatible* provided that d generates the topology of X; also, we say that a metric space $(\hat{X}, \hat{d})$ is *a metric completion of* X provided that $(\hat{X}, \hat{d})$ is the metric completion of some metric space (X, d), with d a compatible metric on X.

THEOREM 1. *A metrizable space X is compact if and only if every metric completion of X is locally compact.*

The key, in the proof of Theorem 1, is the following well-known result on extending metrics, due to Hausdorff [**2**], and rediscovered by Bing [**1**]; see also [**4**], Problem 22.E.

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d extends to a compatible metric on X, which we also denote by d. Let $(\hat{X}, \hat{d})$ be the completion of the metric space (X, d). Then, since the metric spaces H and C are isometrically homeomorphic, the same thing can be said about their metric completions $\mathrm{Cl}_{\hat{X}}(H)$ and $\bar{C} = \mathrm{Cl}_Y(C)$ respectively (see [4], Theorems 24.4 and 24.10). Hence, $\mathrm{Cl}_{\hat{X}}(H)$ is not locally compact, because $\bar{C}$ is not. By Lemma 3 (necessity) $\hat{X}$ is not locally compact. This ends the proof.

Remark. Another well-known characterization of compactness for metrizable spaces, which may be established by means of Hausdorff extension theorem, is that *a metrizable space is compact if and only if it is complete in every compatible metric* (see [4], Problem 24.C). This condition (originally due to Niemytzki and Tychonoff [3]) and that of Theorem 1 do not seem to be immediately deducible from each other as may be seen by considering two examples: a bounded open interval of $\mathbb{R}$, with the usual metric, which is not complete, while its completion is compact, and an infinite dimensional Banach space, with the norm metric, which is complete but not locally compact. However, a slight modification of the above proof of Theorem 1 shows that the following result holds. The details are left to the reader.

THEOREM 2. *A metrizable space X is compact if and only if, for every metric completion $\hat{X}$ of X, either $X = \hat{X}$ or $\hat{X}$ is locally compact.*

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Disjoint Covering Systems of Congruences

R. J. SIMPSON
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A *disjoint covering system* (henceforth, DCS) is a finite set of ordered pairs of integers

$$\{\langle a_i, d_i\rangle : i = 1, 2, \ldots, t\}$$

with the property that every integer n satisfies exactly one congruence

$$n \equiv a_i (\mathrm{mod}\ d_i).$$

The integers d_i will be called the *moduli* of the DCS. In recent years DCS's have attracted considerable interest; the reader is referred to section F14 of [1] and to [3].

In this note I consider the following conjecture made by Stefan Znam in [3].

d extends to a compatible metric on X, which we also denote by d. Let $(\hat{X}, \hat{d})$ be the completion of the metric space (X, d). Then, since the metric spaces H and C are isometrically homeomorphic, the same thing can be said about their metric completions $\mathrm{Cl}_{\hat{X}}(H)$ and $\bar{C} = \mathrm{Cl}_{Y}(C)$ respectively (see [4], Theorems 24.4 and 24.10). Hence, $\mathrm{Cl}_{\hat{X}}(H)$ is not locally compact, because $\bar{C}$ is not. By Lemma 3 (necessity) $\hat{X}$ is not locally compact. This ends the proof.

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In this note I consider the following conjecture made by Stefan Znam in [3].

CONJECTURE 1. *Let* $\mathscr{D} = \{d_1, d_2, \dots, d_t\}$ *be a sequence of positive integers with the property that*

$$\sum_{i=1}^{t} 1/d_i = 1. \tag{1}$$

Then there exists a DCS whose sequence of moduli is $\mathscr{D}$ *if and only if for each prime* p *there exists a partition of* $\mathscr{D}$,

$$\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_p$$

with the properties

(i) $d_i \in \mathscr{D}_0$ *if and only if* $(d_i, p) = 1$,

(ii) *the sum*

$$\sum_{d_i \in \mathscr{D}_j} 1/d_i$$

is constant for $j = 1, 2, \dots, p$.

In [3] Znam proved that the conditions of this conjecture are necessary for $\mathscr{D}$ to be the sequence of moduli of a DCS.

In the following theorem I show that the moduli of any DCS must satisfy a set of conditions which are stronger than those in conjecture 1. I then present a counterexample to this conjecture. This is a sequence of integers which does not satisfy the conditions of the theorem and is, therefore, not the sequence of moduli of any DCS, but which does satisfy the conditions of conjecture 1. Finally I will use the ideas of the theorem to make a new conjecture.

THEOREM. *Let* $\mathscr{A} = \{\langle a_i, d_i \rangle : i = 1, \dots, t\}$ *be a DCS. Then for each prime* p *and each positive integer* α *the sequence* $\mathscr{D} = \{d_1, d_2, \dots, d_t\}$ *may be partitioned.*

$$\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_p$$

such that

(i) $d_i \in \mathscr{D}_0$ *if and only if* $(d_i, p^\alpha) < p^\alpha$,

(ii) *The sum*

$$\sum_{d_i \in \mathscr{D}_j} 1/d_i$$

is constant for $j = 1, 2, \dots, p$.

Proof. The proof uses induction on α. When $\alpha = 1$, (i) and (ii) reduce to the conditions of Conjecture 1 which Znam has shown to be necessary if $\mathscr{A}$ is a DCS. We assume the theorem holds when $\alpha = \alpha_0 - 1$ for some $\alpha_0 \geqslant 2$ and show that it holds when $\alpha = \alpha_0$.

We first fix an arbitrary prime p and make the partition

$$\mathscr{D} = \mathscr{D}^{(0)} \cup \mathscr{D}^{(1)} \cup \cdots \cup \mathscr{D}^{(p)}$$

where $d_i \in \mathscr{D}^{(0)}$ if and only if $(p, d_i) = 1$ and if p divides d_i then $d_i \in \mathscr{D}^{(k)}$ implies

$$a_i \equiv k \pmod{p}.$$

For arbitrary k, $1 \leqslant k \leqslant p$, let $\mathscr{A}^{(k)}$ be the sequence of ordered pairs in $\mathscr{A}$ whose second elements belong to $\mathscr{D}^{(0)} \cup \mathscr{D}^{(k)}$. We now form a new set of ordered pairs $\langle a_i^*, d_i^* \rangle$ from $\mathscr{A}^{(k)}$ using the mapping

$$\langle a_i, d_i \rangle \to \langle a_i^*, d_i^* \rangle, \tag{2}$$

where

$$d_i^* = d_i/(d_i, p)$$

and a_i^* is a solution of

$$a_i^* p/(d_i, p) \equiv (a_i - k)/(d_i, p) (\operatorname{mod} d_i^*).$$

Further let $\mathscr{D}^{*(k)}$ be the sequence of second elements of these ordered pairs.

In [**2**] I called this new set of ordered pairs the *reduction of $\mathscr{A}$ via k modulo p* and showed that it is also a DCS. We now apply the induction hypothesis to this DCS so that $\mathscr{D}^{*(k)}$ may be partitioned

$$\mathscr{D}^{*(k)} = \mathscr{D}_0^{*(k)} \cup \mathscr{D}_1^{*(k)} \cup \cdots \cup \mathscr{D}_p^{*(k)}$$

where a modulus d_i^* belongs to $\mathscr{D}_0^{*(k)}$ if and only if $(d_i^*, p^{\alpha_0 - 1}) < p^{\alpha_0 - 1}$ and the sum

$$\sum_{d_i^* \in \mathscr{D}_j^{*(k)}} 1/d_i^*$$

is constant for $j = 1, 2, \ldots, p$. By considering the mapping (2) we see that the sequence of all d_i^*'s appearing in these p sums consists precisely of those integers d_i/p for which $d_i \in \mathscr{D}^{(k)}$ and d_i is divisible by p^{α_0}.

These manipulations may be performed with each of $\mathscr{D}^{(1)}, \mathscr{D}^{(2)}, \ldots, \mathscr{D}^{(p)}$. Summing over the index k we find that

$$\sum_{k=1}^{p} \sum_{d_i/p \in \mathscr{D}_j^{*(k)}} p/d_i$$

is constant for $j = 1, 2, \ldots, p$. We may write this sum as

$$p \sum_{d_i \in \mathscr{D}_j} 1/d_i$$

where

$$\mathscr{D}_j = \left\{ d_i \colon d_i/p \in \bigcup_{k=1}^{p} \mathscr{D}_j^{*(k)} \right\}.$$

With $\mathscr{D}_0$ defined by (i) the sets $\mathscr{D}_0, \mathscr{D}_1, \ldots, \mathscr{D}_p$ fulfil the requirements of the theorem. This holds for every prime p, so the result is proved by induction. □

Counterexample to Conjecture 1. It is easily checked that the sequence of integers

$$\underbrace{30, \ldots, 30}_{\text{29 times}}, 180, 300, \underbrace{450, \ldots, 450}_{\text{11 times}}$$

satisfies the conditions of conjecture 1. However, with $p = 2$ and $\alpha = 2$, it fails to satisfy conditions (i) and (ii) of the theorem. It is therefore not the sequence of moduli of a DCS.

CONJECTURE 2. *I conjecture that if $\mathscr{D}$ is a sequence of integers satisfying* (1) *and which for any prime p and any positive integer α admits a partition*

$$\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_p$$

satisfying conditions (i) and (ii) of the theorem, then there exists a DCS whose sequence of moduli is $\mathscr{D}$.

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EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

A Method of Obtaining Pythagorean Triples

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Let x, y, z be three positive integers. When

$$x^2 + y^2 = z^2, \tag{1}$$

x, y, z is called a Pythagorean triple. If x, y, z satisfy (1), then x and/or y must be even since otherwise $z^2 = x^2 + y^2 \equiv 2 \pmod 4$ which is impossible. A Pythagorean triple x, y, z is called *primitive* if x, y, z are relatively prime. We will solve the next problem and then consider a well-known theorem that determines all of the primitive Pythagorean triples.

Problem. Let $4a^2$ be an even square. Find all of the positive integers y and z that satisfy

$$4a^2 + y^2 = z^2. \tag{2}$$

To solve this problem we use an elementary fact, the proof of which is immediate, concerning a quadratic equation with integer coefficients: If a rational number p/q, $(p, q) = 1$, is a solution of an equation $X^2 + cX + d = 0$, then $q = 1$ and $p|d$.

Solution. Let $f_b(X) = X^2 - bX + a^2$ where a and b are positive integers. (Think of a as being fixed and b as a parameter.) If u and v are integral solutions of $f_b(X) = 0$ such that $u > v > 0$ then, from the factor theorem, $a^2 = uv$ and $b = u + v$. Suppose that $a = p_0^{e_0} p_1^{e_1} \cdots p_n^{e_n}$ where $2 = p_0 < p_1 < \cdots < p_n$ are primes and $e_0 \geqq 0$, $e_1 \geqq 1, \ldots, e_n \geqq 1$. Using the fact stated above we see that

$$u = p_0^{h_0} \cdots p_n^{h_n}, \quad v = p_0^{2e_0 - h_0} \cdots p_n^{2e_n - h_n} \quad \text{where } 0 \leqq h_i \leqq 2e_i. \tag{3}$$

The discriminant of $f_b(X)$ is the square of an integer and it follows that $b^2 - 4a^2 = y^2$, say. Since $b = u + v$ and $a^2 = uv$, $y = u - v$. It follows that $y = u - v$ and $z = b = u + v$ is a positive integral solution of (2). Conversely, suppose that y, z is a positive integral solution of (2). Then y and z have the same parity and if we let $u = (z + y)/2$ and $v = (z - y)/2$ then $u > v > 0$ where u and v are integers. Also, $uv = (z^2 - y^2)/4 = a^2$, and if $b = u + v$ then u and v are solutions of $X^2 - bX + a^2 = 0$. Thus, there is a one-to-one correspondence between the positive integral solutions of (2) and the integral solutions u, v of $f_b(X) = 0$ such that $u > v > 0$.

It is easy to see that the number of pairs u, v in (3) such that $u > v$ is given by $\{(2e_0 + 1) \cdots (2e_n + 1) - 1\}/2 = N$, say. Thus, the equation (2) has exactly N solutions for a given value of a.

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The Missing Fields

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The presentation of the Galois Theory usually culminates in one grand theorem: the general equation of degree greater than or equal to 5 cannot be solved in terms of radicals. Although the importance of this result can hardly be overstated, it is somewhat unsatisfying to have just one application of the machinery and lemmas that have been developed. The purpose of this note is to give another (similarly suprising) result that uses the Galois Theory.

Corresponding to the Cauchy theorem for groups,[1] one might conjecture the "theorem" for field extensions: If L is a finite extension of the field F having degree n and the prime p divides n, then there is a field K, $F \subseteq K \subseteq L$, of degree of p over F. (Of course, with this result, one could show that if d is any divisor of n, then there is an extension K of F, $K \subseteq L$, of degree d over F.) However, the proposed theorem is false, and not merely for certain exotic fields, but for F equal to the rationals, and for all n.

THEOREM. *For any positive integer n, there exists an extension K of the rationals such that $[K:\mathbf{Q}] = n$ and for no proper divisor d of n is there a field F, $\mathbf{Q} \subseteq F \subseteq K$ with $[F:\mathbf{Q}] = d$. (Alternatively: there is no field F properly between $\mathbf{Q}$ and K.)*

The theorem is clearly true if $n = 1$ or 2, so we assume $n \geq 3$. The proof is based upon the following lemma.

LEMMA. *For any $n \geq 1$, there is no proper subgroup H of S_n such that H properly contains a subgroup isomorphic to S_{n-1}.*

Proof. Although this is fairly easy to prove by brute force, it seems preferable to use this as an opportunity to review some of the results and techniques previously developed and employed.

Assume $S_{n-1} \subsetneqq H \subsetneqq S_n$ and let $H_1 = H \cap A_n$. It follows that $A_{n-1} \subsetneqq H_1 \subsetneqq A_n$ and thus the index of H_1 in A_n is less than n. By the generalization of Cayley's Theorem [**1**, Thm. 2.9.2, p. 73], there is a homomorphism of A_n into the group of permutations of the right cosets of H_1 in A_n and the kernel of this homomorphism is in H_1. If $n \neq 4$, A_n is simple, which implies that the kernel of this homomorphism is $\{e\}$, the homomorphism is an isomorphism, and the index of H_1 in A_n is at least n. This contradicts the observation made above and takes care of all n

[1] If G is a finite group of order n and the prime p divides n, then G has a subgroup of order p.

except $n = 4$. However, since A_4 has no subgroup of order 6, there is no such H_1 if $n = 4$, and this completes the proof of the lemma.

We now turn to proof of the theorem.

Proof of the theorem. With $n \geq 3$, let $f(x) \in \mathbf{Q}[x]$ be a polynomial whose Galois group[2] is S_n and let L be the splitting field of $f(x)$ over $\mathbf{Q}$. If K is the fixed field of the group S_{n-1}, then $[K:\mathbf{Q}] = |S_n|/|S_{n-1}| = n$. (Of course $K = \mathbf{Q}(\alpha)$ where α is a root of $f(x) = 0$.) By the Fundamental Theorem of Galois Theory, for any field F with $\mathbf{Q} \subseteq F \subseteq K$, the group of automorphisms of L that leave every element of F fixed lies between $G(L, \mathbf{Q}) = S_n$ and $G(L, K) = S_{n-1}$, and the correspondence is one-to-one (since L is a normal extension of $\mathbf{Q}$). We know that there is no group between S_{n-1} and S_n and thus there is no field (properly) between $\mathbf{Q}$ and K. This completes the proof of the theorem.

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Addition Theorems in Ordinary Differential Equations

ABRAHAM UNGAR
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Students in undergraduate, ordinary differential equations (ODE) courses, already familiar with the binomial theorem from algebra, the sin/cos addition formulas from trigonometry, and the exponential addition law from calculus, are likely to be impressed by a link between them provided by methods of ODE. Accordingly, the aim of this note is to place the binomial theorem and the addition theorems for exponential, trigonometric, and hyperbolic functions in the context of a single addition theorem generated by an initial value problem (IVP).

Consider the $(n+1)$th order ODE, $n = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{n+1} a_k f^{(k)}(x) = 0, \tag{1}$$

where a_k are constants. Let a solution $f(x)$ of (1) generate the function $F(c, x)$ of two variables c and x, given by the equation

$$F(c, x) = \sum_{m=0}^{n} a_{m+1} \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(c - x). \tag{2}$$

[2]Although it is generally an unsolved problem whether there is a polynomial over $\mathbf{Q}$ having as its Galois group an arbitrary subgroup of S_n, there are in fact simple constructions for such polynomials with the group S_n. See, for example, [2, §61].

except $n = 4$. However, since A_4 has no subgroup of order 6, there is no such H_1 if $n = 4$, and this completes the proof of the lemma.

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Since

$$\begin{aligned}
\frac{\partial F(c,x)}{\partial x} &= \sum_{m=0}^{n} a_{m+1} \frac{\partial}{\partial x} \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(c-x) \\
&= \sum_{m=0}^{n} a_{m+1} \left[\sum_{k=0}^{m} f^{(k+1)}(x) f^{(m-k)}(c-x) \right. \\
&\qquad \left. - \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k+1)}(c-x) \right] \\
&= \sum_{m=0}^{n} a_{m+1} \left[\sum_{k=1}^{m+1} f^{(k)}(x) f^{(m-k+1)}(c-x) \right. \\
&\qquad \left. - \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k+1)}(c-x) \right] \\
&= \sum_{m=0}^{n} a_{m+1} \left[f^{(m+1)}(x) f(c-x) - f(x) f^{(m+1)}(c-x) \right] \\
&= f(c-x) \sum_{m=0}^{n} a_{m+1} f^{(m+1)}(x) - f(x) \sum_{m=0}^{n} a_{m+1} f^{(m+1)}(c-x) \\
&= -f(c-x) a_0 f(x) + f(x) a_0 f(c-x) \\
&= 0,
\end{aligned}$$

the expression

$$F(c,x) = \phi(c)$$

is *independent* of x. Replacing c by $x + y$, equation (2) thus takes the form

$$\sum_{m=0}^{n} a_{m+1} \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(y) = \phi(x+y)$$

in $-\infty < x, y < \infty$, for some analytic function ϕ. In particular, we have the equality

$$\sum_{m=0}^{n} a_{m+1} \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(y) = \sum_{m=0}^{n} a_{m+1} \sum_{k=0}^{m} f^{(k)}(0) f^{(m-k)}(x+y), \tag{3}$$

which is an *addition theorem* [**1**]. To simplify it we normalize (1) by selecting

$$a_{n+1} = 1,$$

and single out a unique solution $f(x)$ of (1) by imposing the initial conditions

$$\begin{aligned}
&f^{(k)}(0) = 0, \qquad k = 0, 1, 2, \ldots, n-1 \\
&f^{(n)}(0) = 1.
\end{aligned} \tag{4}$$

These reduce equation (3) to

$$f(x+y) = \sum_{m=0}^{n} a_{m+1} \sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(y), \tag{5}$$

thus obtaining an elegant *addition theorem* for the solution of the IVP formed by equations (1) and (4).

The existence of a relationship between addition theorems and some linear homogeneous constant coefficient ODE is well known in the literature. Thus, for example, the addition formula for the exponential function is deduced in [**2**] from the ODE $f'(x) = f(x)$ and the trigonometric addition formulas are deduced in [**3**] from the ODE $f''(x) = -f(x)$.

When $a_m = 0$, $m = 0, 1, 2, \cdots n$, the function $f(x)$ satisfying the IVP of equations (1) and (4) is $f(x) = x^n/n!$, and the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{6}$$

is recovered from equation (5). The binomial theorem (6) can thus be viewed as an addition theorem generated by an IVP. It is obviously linked to other well-known addition theorems that can be recovered from equation (5):

(i) When $n = 0$, $a_0 = -1$, and $a_1 = 1$ in equation (1), $f(x)$ of equations (1) and (4) specializes to the exponential function e^x, and the addition theorem (5) specializes to

$$e^{x+y} = e^x e^y;$$

(ii) when $n = 1$, $a_0 = 1$, $a_1 = 0$ and $a_2 = 1$ in equation (1), $f(x)$ of equations (1) and (4) specializes to the trigonometric function $\sin x$ and the addition theorem (5) specializes to

$$\sin(x+y) = \sin x \cos y + \cos x \sin y;$$

and

(iii) when $n = 1$, $a_0 = -1$, $a_1 = 0$ and $a_2 = 1$ in equation (1), $f(x)$ of equations (1) and (4) specializes to the hyperbolic function $\sinh x$ and the addition theorem (5) specializes to

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

For an example with general $n > 1$ one may consider the *higher-order hyperbolic sine function* $f(x)$ given by the equation

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{rk}}{(rk)!}, \quad r = n+1,$$

and satisfying the IVP formed by equations (1) and (4) with $a_0 = -1$, $a_{n+1} = 1$, and $a_k = 0$ for $k = 1, 2, \ldots, n$. Higher-order sine functions, satisfying the ODE $f^{(r)}(x) \pm f(x) = 0 (r > 2)$, have a long, checkered history. In spite of their attractive features, these functions did not find their way into the accessible literature and

as a result they have been repeatedly rediscovered since they first appeared in a 1757 paper by Vincenzo Riccati [**4**]. Further details may be found in [**5**] [**6**].

In the field of ODE the binomial theorem is commonly treated only as a mathematical tool; e.g. [**7**]. The binomial theorem has a long history [**8**] and is found throughout the mathematical literature, mainly in treatises of combinatorial analysis, algebra, statistics, and number theory. It is therefore of interest to students of ODE to realize that the binomial theorem and its proof belong naturally to their study.

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PROBLEMS AND SOLUTIONS

EDITED BY PAUL T. BATEMAN, HAROLD G. DIAMOND, KENNETH B. STOLARSKY (ADVANCED PROBLEMS), AND DOUGLAS B. WEST (ELEMENTARY PROBLEMS)

*For instructions about submitting **proposed** problems for publication in this department see the inside front cover. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the address given on the inside front cover. Two copies suffice.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed by March 31, 1988, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3231. *Proposed by Herbert Guelicher, Muenster, West Germany.*

In a triangle $P_1P_2P_3$ let p_i be the side opposite vertex P_i and let s_i be a line parallel to p_i (but different from p_i). Suppose that s_i divides P_iP_{i+1} in the (signed) ratio λ_i, so that if s_i meets p_{i-1} in Q_i, then $\lambda_i = P_iQ_i/Q_iP_{i+1}$. (Subscripts are taken modulo 3.) Prove that the lines s_1, s_2, s_3 are concurrent if and only if

$$\lambda_1\lambda_2\lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3) = 2.$$

E 3232. *Proposed by Jordi Dou, Barcelona, Spain.*

Given lines l_1, l_2, l_3, l_4, l_5 and points Q_1, Q_2, Q_3, Q_4, Q_5 in the plane such that Q_i does not lie on l_i $(i = 1, 2, 3, 4, 5)$, prove that there exist points P_i, R_i on line l_i such that the angle $P_iQ_iR_i$ is a right angle $(i = 1, 2, 3, 4, 5)$ and such that the ten points P_1, P_2, P_3, P_4, P_5, R_1, R_2, R_3, R_4, R_5 lie on a conic.

(a) Show that any chain of circles $C_1, C_2, \ldots$ such that C_i, C_{i+1} are companion incircles for every i consists of at most six distinct circles.

(b) Give a ruler and compass construction for the unique chain which has only three distinct circles.

(c) For such a chain of three circles show that the three subdividing lines are concurrent.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Integral Identity

E 3069 [1985, 57]. *Proposed by Zhang Zaimin, Yuxi Teachers' College, Yunan, China.*

The m by m determinant $I = |a_{rs}|$ has $a_{rs} = \int_0^1 x^{s-1} F(x)^{m-r+1}\,dx$, where F is nondecreasing on $[0,1]$. Prove that $I \geqslant 0$.

Solution by P. Y. Wu, National Chiao Tung University, Hsinchu, Taiwan. The assertion as stated does not hold. For example, let $m = 2$ and $F(x) = x$. Then $a_{rs} = \int_0^1 x^{s-r+2} = 1/(s-r+3)$, whence $I = -1/72 < 0$. Requiring F to be nonincreasing still won't work, as $m = 1$ and $F(x) = -x$ show. In this case, $I = -1/2 < 0$. We propose the following modification, which is slightly more general: If the $m \times m$ determinant $I = |a_{rs}|$ has $a_{rs} = \int_p^q G(x)^{s-1} F(x)^{m-r+1}\,dx$, where G is nondecreasing and $F \geqslant 0$ is nonincreasing on $[p,q]$, then $I \geqslant 0$.

For the proof, we use a result from G. Pólya and G. Szegő's *Problems and Theorems in Analysis*, Vol. I, Springer-Verlag, 1972. Since the functions G^{s-1} and F^{m-r+1} are integrable over $[p,q]$, it follows from Problem II.68 therein that we can "interchange" the determinant and integration to obtain

$$I = \frac{1}{m!}\int_p^q \cdots \int_p^q |b_{rs}||c_{rs}|\,dx_1 \cdots dx_m,$$

where $b_{rs} = G(x_s)^{r-1}$ and $c_{rs} = F(x_s)^{m-r+1}$. The integrand is a product of Vandermonde determinants (after factoring $F(x_s)$ from column s of $|c_{rs}|$), with values given by $|b_{rs}| = \prod_{i<j}(G(x_j) - G(x_i))$ and $|c_{rs}| = \prod_i F(x_i) \prod_{i<j}(F(x_i) - F(x_j))$. By the conditions on F and G, the integrand is nonnegative, and therefore so is I.

Editorial comment. With I defined as above, the proposer's original statement read "For $F(x)$ a positive, monotone decreasing function defined in $0 \leqslant x \leqslant 1$,

prove that $I \geqslant 0$." In the initial printing, the words "positive, monotone" were inadvertently replaced by "non." Both are necessary for the truth of the statement. Some solvers fixed the problem as intended. Others kept F nondecreasing but changed $m - r + 1$ to $r - 1$, which has essentially the same proof.

The theorem from Pólya and Szegő cited above is that if the functions $f_1(x), \dots, f_m(x), g_1(x), \dots, g_m(x)$ are bounded and integrable on $[p, q]$, then

$$|a_{rs}| = \frac{1}{m!}\int_p^q \cdots \int_p^q |b_{rs}||c_{rs}|\,dx_1 \cdots dx_m,$$

where

$$a_{rs} = \int_p^q f_r(x) g_s(x)\,dx, \qquad b_{rs} = f_r(x_s), \quad \text{and} \quad c_{rs} = g_r(x_s).$$

The basic idea behind this theorem is that the integrals are linear operators on the columns of $|a_{rs}|$, and can be taken out since the determinant is a multilinear function of its columns. I.e., letting $\bar{f}(x_i)$ be the column vector $(f_1(x_i), \dots, f_m(x_i))^T$ and $x_i = [ip + (n - i)q]/n$, the first column of integrals in $|a_{rs}|$ is the limit of the linear combination $\sum_{i=1}^{n} g_1(x_i)\bar{f}(x_i)(p - q)/n$ as $n \to \infty$. Hence pulling the integrals out is merely the interchange of this limit with the finite multiplications of the determinant.

Four other solvers, N. Elkies, R. Farwig (W. Germany), D. Schmidt, and G. Sylvester, noted along with Wu the generalization replacing x^{s-1} by $G(x)^{s-1}$, where G is an arbitrary increasing function of x. Elkies and Farwig noted further that the same conclusion holds when the powers $s - 1$ and $m - r + 1$ are replaced by arbitrary strictly increasing and strictly decreasing sequences of real numbers $\alpha(s)$ and $\beta(r)$. In the last step, one then obtains generalized Vandermonde matrices in place of ordinary ones, but the argument can still be carried through (see Gantmacher, *The Theory of Matrices*, 1959).

Two further solvers J.-C. Leccia (France) and W. A. Newcomb, noted that if F is positive and nondecreasing, then the sign of I is $(-1)^{\lfloor m/2 \rfloor} = (-1)^{m(m-1)/2}$.

Also solved by the proposer. Several other readers supplied counterexamples to the statement as printed.

Uncountable Sets with No Long Arithmetic Progressions

E 3119 [1985, 736]. *Proposed by Paul K. Stockmeyer, College of William and Mary, and the Editors.*

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$$|a_{rs}| = \frac{1}{m!} \int_p^q \cdots \int_p^q |b_{rs}||c_{rs}| \, dx_1 \cdots dx_m,$$

where

$$a_{rs} = \int_p^q f_r(x) g_s(x) \, dx, \qquad b_{rs} = f_r(x_s), \quad \text{and} \quad c_{rs} = g_r(x_s).$$

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Also solved by the proposer. Several other readers supplied counterexamples to the statement as printed.

Uncountable Sets with No Long Arithmetic Progressions

E 3119 [1985, 736]. *Proposed by Paul K. Stockmeyer, College of William and Mary, and the Editors.*

(a) Prove that for every integer $n \geqslant 3$ there exist sets $S_n \subseteq [0,1]$ such that S_n contains no arithmetic progression of length n and $S_n \cup \{x\}$ contains such a progression for every $x \in [0,1] - S_n$.

(b) For each $n \geqslant 3$ exhibit such a set S_n by explicit construction.

Solution (a) by Bjorn Poonen (student) Harvard College. Let $U = \{S \subseteq [0,1]$ such that S contains no arithmetic progression of length $n\}$, and order it by inclusion ($S \leqslant T$ iff $S \subseteq T$). We claim $(U, \leqslant)$ satisfies Zorn's condition: for any chain $C \subseteq U$ the set $T = \bigcup_{S \in C} S$ is an upper bound. To show this, it suffices to show that there do not exist n numbers $x_1, x_2, \ldots, x_n$ in T forming an arithmetic progression. If such numbers exist, then each x_i is in S_i for some S_i in C. But all the S_i are comparable, so one of them must contain all the others. This element of C contains an arithmetic progression of length n, a contradiction. Hence we can apply Zorn's Lemma to pick a maximal element $M \in U$. For this M, if $M \cup \{x\} \in U$ for some $x \in [0,1] - M$, then M would not be maximal. Thus M must have the desired property.

Note: This method can clearly be generalized to universes other than $[0,1]$, and the condition "contains an arithmetic progression of length n" can be replaced by "contains n elements satisfying (some relation)."

Part (a) only also solved by W. A. Newcomb and V. Pambuccian (Romania). There were no correct solutions to part (b).

An Inequality for Two of Thirteen

E 3121 [1985, 736]. *Proposed by Calvin T. Long, Washington State University.*

Given any thirteen distinct real numbers, show that there exist at least two which satisfy the inequality

$$0 < \frac{x-y}{1+xy} < \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}\,.$$

Solution by David J. Dixon, Southern Illinois University at Edwardsville, and James E. Kistner, Southwestern Bell Telephone. First note that

$$\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} = \tan\frac{\pi}{12}.$$

Let $a_1, a_2, \ldots, a_{13}$ be thirteen distinct real numbers. Then $a_i = \tan u_i$, for $i = 1, \ldots, 13$, where the u_i are distinct and in the interval $(-\pi/2, \pi/2)$. Partitioning this interval, of length π, into twelve equal subintervals of length $\pi/12$, we may conclude that among our thirteen u_i, there exist at least two of them, say u_j and u_k, such that $0 < u_j - u_k < \pi/12$ (Pigeonhole Principle). Renaming $\tan u_j = a_j = x$ and $\tan u_k = a_k = y$, and taking the tangent of each side of the above inequality, we obtain our solution. To wit:

$$0 < u_j - u_k < \pi/12,$$

$$\tan 0 < \tan(u_j - u_k) < \tan(\pi/12),$$

$$0 < \frac{\tan u_j - \tan u_k}{1 + \tan u_j \tan u_k} < \tan(\pi/12),$$

$$0 < \frac{x - y}{1 + xy} < \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}}\,.$$

This calculation can be generalized as follows:

Given any n distinct real numbers, $n \geqslant 4$, there exist at least two which satisfy the inequality:

$$0 < \frac{x - y}{1 + xy} < \tan\{\pi/(n-1)\},$$

Also solved by 70 other readers and the proposer.

Classification of 3 × 3 Matrices

E 3126 [1986, 60]. *Proposed by F. S. Cater, Portland State University.*

Let M be a 3 by 3 matrix with entries in a field F. Prove that M is similar over the field to precisely one of these three types:

$$\underset{\text{Type I}}{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}} \underset{\text{Type II}}{\begin{pmatrix} b & 0 & 0 \\ 1 & c & 0 \\ 0 & 0 & c \end{pmatrix}} \underset{\text{Type III}}{\begin{pmatrix} d & 1 & 0 \\ e & 0 & 1 \\ f & 0 & 0 \end{pmatrix}} \quad (a, b, c, d, e, f \in F).$$

Solution by Edmond D. Dixon, Tennessee Technological University, Cookeville. M is similar over F to a unique rational canonical form which depends on the invariant factors of M. If M has only one invariant factor it is similar to a companion matrix, type III. If it has three invariant factors they must be identical and the rational canonical form is type I. If it has two invariant factors they must be $(x - c)$ and $(x - b)(x - c)$, in which case the rational canonical form is

$$\begin{pmatrix} 0 & -bc & 0 \\ 1 & b+c & 0 \\ 0 & 0 & c \end{pmatrix},$$

the direct sum of companion matrices of the invariant factors. This matrix is similar to type II, using

$$P = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Comment by the proposer. When one of each type of matrix in the classical canonical forms

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \begin{pmatrix} 0 & e & 0 \\ 1 & d & 0 \\ 0 & 0 & f \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & g \\ 1 & 0 & h \\ 0 & 1 & i \end{pmatrix}$$

is written, 9 parameters are used altogether, though we needed only 6. Also these matrices could be similar to each other. On the other hand, the Jordan forms

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \begin{pmatrix} d & 0 & 0 \\ 1 & d & 0 \\ 0 & 0 & e \end{pmatrix}, \quad \begin{pmatrix} f & 0 & 0 \\ 1 & f & 0 \\ 0 & 1 & f \end{pmatrix}$$

require only 6 parameters, but this canonical form requires the field F to be algebraically closed.

Also solved by David Callan, P. M. Cohn (England), E. R. Gentile (Argentina), Robert Gilmer, Z. P. Lin (England), R. B. Richter and W. R. Wardlaw, P. Y. Wu (Taiwan), and the proposer. Several solvers noted that the problem is essentially exercise 11 on p. 203 of K. Hoffman and R. Kunze, *Linear Algebra* (Prentice-Hall, 1961), 5th printing.

to give

$$p_{n+1}^3 < \prod_{i=1}^{n} p_i \qquad (n \geqslant 5), \tag{3}$$

which implies (1). (See also Rademacher and Toeplitz, op. cit., Ch. 28.)

Also solved by the proposer and 39 other readers.

ADVANCED PROBLEMS

6557. *Proposed by Clark Kimberling, University of Evansville.*

Let Γ denote the circumcircle of a triangle ABC. Let A' be the point, other than A, where the A-median of ABC meets Γ. Let A'' be the point, other than A, where the A-altitude of ABC meets Γ. Similarly define B', C' and B'', C''. Let DEF be the tangential triangle of ABC (e.g., D is the point where the line tangent to Γ at B meets the line tangent to Γ at C). Define (as usual) the Euler line of ABC to be the line containing the orthocenter, the circumcenter, and the centroid of ABC. Prove that the lines DA', EB', FC' and the lines DA'', EB'', FC'' concur in points that lie on the Euler line of ABC.

6558. *Proposed by Barry Powell, Kirkland, Washington.*

Let $p > 3$ be prime. By a theorem of Faltings [*Invent. Math.*, 73(1983) 349–366] the Fermat equations

$$2x^p + y^p = z^p, \qquad 2^{p-1}x^p + y^p = z^p, \quad \text{and} \quad x^p + y^p = z^p$$

have only finitely many nontrivial integral solutions.

(a) Show there exists a positive integer N such that for q prime, $m \geqslant 1$, and $q^m > N$, the equation

$$q^m x^{2p} + y^{2p} = z^{2p}$$

has no positive integral solutions with x, y and z pairwise relatively prime.

(b) Show that for any positive integer $n \geqslant 2$, the Diophantine equation

$$x^6 - y^6 = z^n$$

has at most finitely many integral solutions with x and y relatively prime.

6559. *Proposed by Benjamin G. Klein, Davidson College, and John Layman, Virginia Polytechnic Institute and State University.*

Let $c_0, c_1, c_2, \ldots$ be a sequence of real numbers determined by the values of c_0, c_1, and c_2 together with the recursion

$$c_{n+1} = (c_n c_{n-1} + 1)/c_{n-2},$$

for $n \geqslant 2$. Further, suppose that $c_0 = c_2 = a$, where $a > 0$.

(i) For what values of c_1 does the sequence of ratios c_{2n+2}/c_{2n} converge? Determine the value of the limit for each such value of c_1.

(ii) Show that for each value of a there is exactly one value of c_1 such that the sequence of ratios c_{n+1}/c_n converges. Find this value of c_1 and then determine the associated value of the limit of the sequence of ratios.

SOLUTIONS OF ADVANCED PROBLEMS

6434 [1983, 403]. *Proposed by Allen J. Schwenk, University of Waterloo.*

The Petersen graph shown below has 15 edges. Its adjacency matrix P is formed by setting the (i, j) entry equal to 1 if vertex i is adjacent to vertex j and to 0 otherwise.

(a) Show that its characteristic polynomial is

$$(x-3)(x-1)^5(x+2)^4.$$

(b) Can the 45 edges of the complete graph K_{10} be partitioned into three copies of the Petersen graph?

Composite solution by O. P. Lossers, Eindhoven Univ. of Technology, Eindhoven, Netherlands, and the proposer (who is now at Western Michigan University).

(a) Consider the $3 + 3^2$ walks of length 1 or 2 that begin at a specified vertex v. Each other vertex is the end of one of these, and v is the end of three of them. Hence $P^2 + P = 2I + J$, where J is the matrix of all 1's. Factoring $P^2 + P - 2I$ and applying $P - 3I$ to both this and J yields $(P-3I)(P-I)(P+2I) = 0$. Hence $(\lambda - 3)(\lambda - 1)(\lambda + 2)$ is the minimum polynomial of P, and P has eigenvalues $3, 1, -2$. To determine the multiplicities a, b, c use the fact that for every $j \geqslant 0$, $a \cdot 3^j + b \cdot 1^j + c \cdot (-2)^j = \operatorname{trace} P^j$. For $j = 0, 1, 2$, $\operatorname{trace} P^j = 10, 0, 30$, and the resulting three equations in three unknowns yield $a, b, c = 1, 5, 4$.

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4. A. J. Hoffman and R. R. Singleton, On Moore graphs of diameters 2 and 3, *IBM J. Res. Dev.*, 4 (1960) 497–504.

Also solved by Juraj Bosák (Czechoslovakia), Emeric Deutsch (Polytech. Inst. of NY), Marietjie Frick (South Africa), A. A. Jagers, (Universiteit Twente, Netherlands), Peter Rowlinson (U. Stirling, Scotland), Robert Singleton (Portland, CT), John H. Smith (Boston College), Lajos Takács (Case Western), Rand Africaans Univ. Problem Solving Group (South Africa), and the proposer.

6515 [1986, 305]. *Proposed by Robert E. Shafer, Berkeley, CA.*

For complex μ and ν with $\operatorname{Re}(\mu), \operatorname{Re}(\nu) > 0$, prove that

$$\binom{\nu-1}{\mu-1}(\psi(\mu)-\psi(\nu)) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\binom{\nu-1}{\mu-1+m},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\binom{\alpha}{\beta} = \Gamma(\alpha+1)/\{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)\}$ (cf. this MONTHLY, 6480 [1984, 651]).

Combined Solution. Since for $\operatorname{Re}(\beta-\alpha) > 0$ we have

$$\psi(\beta) - \psi(\beta-\alpha) = \sum_{m=1}^{\infty} \frac{1}{m} \frac{\Gamma(\alpha+m)/\Gamma(\alpha)}{\Gamma(\beta+m)/\Gamma(\beta)}$$

(see Y. L. Luke, The Special Functions and Their Approximations, Academic Press, New York, 1969, p. 111 formula (42)) we have for $\operatorname{Re}\nu > 0$ that

$$\psi(\mu) - \psi(\nu) = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)} \sum_{m=1}^{\infty} \frac{\Gamma(\mu-\nu+m)}{m\Gamma(\mu+m)}.$$

By the gamma function complementation formula,

$$\begin{aligned}
\Gamma(\mu-\nu+m) &= \frac{\pi}{\sin\pi(\mu-\nu+m)\Gamma(1-\mu+\nu-m)} \\
&= \frac{\pi(-1)^m}{\sin\pi(\mu-\nu)\Gamma(1-\mu+\nu-m)} \\
&= (-1)^m \frac{\Gamma(\mu-\nu)\Gamma(1-\mu+\nu)}{\Gamma(1-\mu+\nu-m)}
\end{aligned}$$

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so

$$\psi(\mu) - \psi(\nu) = \Gamma(\mu)\Gamma(1-\mu+\nu)\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\frac{1}{\Gamma(\mu+m)\Gamma(1-\mu+\nu-m)}$$
$$= \frac{\Gamma(\mu)\Gamma(1-\mu+\nu)}{\Gamma(\nu)}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\binom{\nu-1}{\mu+m-1}.$$

Note that (as William A. Newcomb emphasized in his solution) no restriction on μ is needed. The solutions of O. P. Lossers (The Netherlands), S. K. Rangarajan (India), and especially N. Ortner (Austria) are essentially that given above. Newcomb and the proposer introduced integrals of the form

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\log(1-x)\,dx$$

and deduced the result from first principles rather than the above ψ difference formula.

It should be mentioned that Rangarajan's solution to Advanced Problem 6480 [1986, 220] also contained a solution to this problem. Moreover, Rangarajan's solution to Elementary Problem 3065 [1987, 378] contained a *generalization* of the above identity (to which it reduces for $k' = -\mu$ and $p = -\mu - 1$).

Reviews

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Numerical Recipes, The Art of Scientific Computing. By W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. Cambridge University Press, 1986. xi + 817 pp.

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Numerical analysis is concerned with the numerical solution of mathematical problems. This simple description already provides a reason why the subject is hard to teach. It is necessary to understand the mathematical task before taking up its numerical solution. Obviously, then, the mathematical training of the students limits the tasks that might be taken up in a course. It is usual in an introductory course to study a few mathematical problems that are sufficiently basic to be familiar to most students. Tasks that require more mathematical training or sophistication are left to topics courses. Unfortunately, students may have a need to solve mathematical problems that they understand poorly, if at all. Besides this difficulty with the diversity of tasks, there is a difficulty with the diversity of methods used to attack them. Although there are basic principles common to the numerical treatment of problems as different as the computation of eigenvalues and the numerical solution of ordinary differential equations (ODEs), the methods brought to bear on one task may have little in common with those required for another. As a consequence, an introductory course is likely to be perceived as a collection of unrelated problems and methods. To a considerable extent this perception is correct, for the student sees a body of theory only when treating a topic in depth.

The variety of reasons for studying numerical analysis complicates the teaching of the subject. Scientists and engineers often want to learn no more than how to prepare their problems for a library routine and how to interpret the results. Despite the readability of the documentation, the broad applicability of the codes, the reliability of the methods, and the protection afforded by the best mathematical software, this is simply not enough. The scientist is quite likely to encounter problems that can be solved with library codes only if he understands what the codes are doing, and why. The problems that arise in practice are often too complicated to be solved directly with library codes. To devise ways to solve them, an understanding of the principles of numerical analysis is essential. The tasks are mathematical, and in the main the methods for their solution are, too. Numerical analysis is, then, a branch of applied mathematics and so an appropriate subject for a mathematician to study. Computer scientists are often required to take a course in numerical computation as a cultural matter. Perhaps this sounds odd, but it is entirely appropriate. Numerical computation is an important part of computing, and it has surprisingly little in common with mainstream computer science.

Corresponding to the variety of potential students is a variety of backgrounds. Scientists and engineers may have a shallow understanding of some of the mathematics necessary for a survey course. The mathematician is likely to have an adequate training in this regard, although he may not be familiar with why one wants to solve such problems, but is not nearly so likely as a scientist or engineer to be familiar with programming and the use of computers. The computer scientist is well equipped for a study of numerical analysis in many respects, but may not be comfortable with the mathematics of the continuous processes that are the principal object of numerical analysis—the reason for the word "analysis" in the name.

Proper treatment of a topic in numerical analysis involves the reason for solving the problem, the mathematics of the problem and a way to approximate its solution, a programming language, and certain aspects of computation such as computer arithmetic, data manipulation, and documentation. Because the potential students have quite different backgrounds, and interests, in these matters, it is difficult to provide a well-rounded treatment of a topic that is palatable to all. Several ways of responding to the difficulties are seen. One is to skip over some aspects of a complete treatment. This is dangerous because it encourages "amateurism." If the complexity of sound numerical analysis is not made clear, the student is encouraged to think that it is an easy matter to produce code with the qualities of mathematical software enumerated earlier. The result is unsound and inferior work. Another way to reduce the material to be developed is to assume that all the students have the same background. A well-rounded treatment is possible, even for the full range of students, if the number of topics and methods taken up is kept quite small. The disadvantage of this approach is that the teacher must be very sure that the few tasks taken up and, especially, the few methods for their solution that are treated are the important ones.

An item of mathematical software is complex and involves art as well as science. There are few publications that seek to explain in detail even one kind of mathematical software, and they are appropriate only to topics courses. Some introductory texts direct students to "black boxes" in libraries and concentrate on how to use the software and interpret the results. Others include state-of-the-art codes, but discuss only the general principles of the codes; they work with "grey boxes." A third approach is to explain in detail an implementation that is of high quality but that does not include all of the art and the subtleties of the best codes. Each of these approaches involves substantial codes, so the author adopting one must deal with the issue of making the codes accessible to the students. Nowadays numerical analysis texts are expected to contain codes. Providing codes and explaining them is so difficult that some authors choose to provide only fragments of code, perhaps in a pseudo-language, that make specific points about how software is constructed. By providing mere translations of mathematical algorithms into some computer language, it is possible to satisfy the expectation of codes without having to face up to the difficulties.

These general observations explain why popular numerical analysis texts present the subject in very different ways. Let us now try to fit *Numerical Recipes* into the spectrum of possibilities. The book is encyclopedic in scope. Only a small fraction

of the volume could be covered in a single course. The breadth of topics requires a team, and this book has four authors. In only one instance did I note a reference in the book to a paper by one of the authors. From this, the affiliations of the authors, and the tenor of their remarks, I take them to be people who do a lot of numerical computing, as opposed to people interested in developing new methods or in developing mathematical software. A complete review of the book would also require a team. My own background includes training as a numerical analyst, two decades' experience in industry and universities, many papers published about the numerical solution of ODEs, and considerable experience with the writing and testing of mathematical software. Despite all this, I am not in a position to comment about large portions of this book in any way. Whether the quality of chapters I studied is representative, I do not know.

The chapter on the initial value problem for ODEs treats the most common methods. Little is said about the preparation of problems for their solution with the codes provided. The treatment of Runge-Kutta methods is limited to the classical four stage, fourth-order formula with local error estimate by doubling and local extrapolation. The authors seem pretty enthusiastic about the method. It is remarkable that they do not even mention the embedded formulas that are the basis of the codes found in all the well-known libraries. Most attention is devoted to extrapolation of the modified midpoint rule. The authors do not explain what is "modified," and so Gragg's important contribution to the method is never acknowledged. The scheme they describe is obsolete; current practice is described in the nice survey of Deuflhard [**1**]. The authors extol the virtues of rational extrapolation in this chapter, but they use polynomial extrapolation in the quadrature chapter. The authors have a light-hearted style that is very agreeable, but it sometimes falls flat. For example, in connection with predictor-corrector methods they say, "Let us here dispose of two silly ideas." The trouble is that their explanations are silly. Plainly they neither understand nor appreciate the powerful variable order, variable step multistep codes that are so widely used. This is odd in view of their treatment of stiff problems; the authors direct the reader to Gear's implementation of the backward differentiation formulas, which is just such a code. This chapter describes numerical methods for ODEs from the viewpoint of 1970. If the authors had consulted an expert in the subject or read one of the good survey articles available, I think they would have assessed the methods rather differently and presented more modern versions of the methods.

The quadrature chapter focuses on extrapolation methods. Examination of popular libraries, collections like QUADPACK, and the published advice of experts, such as the appropriate chapter in [**2**], shows the great importance of adaptive methods. No reference is made to these methods. The chapter on linear equations highlights Gauss-Jordan elimination with complete pivoting. Not many experts would think it a good idea to invert a matrix every time a linear system is solved. The authors do provide a Crout reduction code with partial pivoting. In connection with residual correction, the authors say just that it is a good idea to calculate the residual in double precision (if available); this is inadequate. Indeed, the TURBO Pascal version does use single precision without a warning of the implications. The

Jacobi method is highlighted as a way to compute the eigenvalues of symmetric matrices. This is despite a description of more modern methods and codes along the lines of EISPACK. The authors favor these less popular methods because they are robust and "simple." Presenting "simple" methods is of some pedagogical value, but that does not seem to be one of the aims of the book. I would rather present the methods that people really use rather than those that are easy to explain to the student.

The book is subtitled, *The Art of Scientific Computing*. This suggests interesting examples and illustrative computations. However, I noticed only one interesting example, the computation of spheroidal harmonics as solutions of boundary value problems for ODEs. I expected that a companion volume labeled "Example Book" would contain some substantial examples with a discussion of their significance, a discussion of how to prepare the problem for solution with the codes provided, and an interpretation of the results. The examples are of the most straightforward kind with practically no discussion at all. They do provide a partial check that the codes are performing properly. My guess is that few people would find an example book and its accompanying diskette to be worth the cost.

In order to present a great many topics, the authors devote little space to any one method. The treatment of each method is adequate to appreciate the basic idea, and references are provided to the literature. Generally the treatment is too superficial for a textbook. Further, the lack of examples, illustrative computations, and exercises make the book unsuitable for the classroom. The presentation of the codes is quite good in every respect. I have tried out a number of the codes on an IBM PC AT with two FORTRAN compilers and one Pascal compiler. There were no problems, with the single exception that one FORTRAN compiler complained, correctly, about the absence of EXTERNAL statements when the name of a subroutine was passed through several levels of subroutine calls. The codes themselves are of better than average quality for a survey book. However, they are far from being mathematical software. Even in the cases of a high quality code taken from the literature, the documentation expected of mathematical software is absent. Although the authors repeatedly express their distaste for "black boxes," they do refer the reader to such codes in a number of instances. With few exceptions, the reader would be well advised to turn to reputable sources of mathematical software rather than to the codes given in this book. The advice offered does not always correspond to the methods advocated by leading practitioners and implemented in leading libraries, c.f. **[2]**.

In my view, *Numerical Recipes* is best described as a reference book providing a quick introduction to the solution of a mathematical task and a lead-in to more thorough treatments.

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Mathematical Surveys and Monographs, Number 21, Bearnstein II, et al., ed. *The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof. Providence: American Mathematical Society*, 1986, pp. xvi + 218.

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The Bieberbach conjecture has challenged mathematicians since its inception in 1916. Attempts to prove, and even to disprove, this conjecture inspired researchers to develop elegant and useful techniques in complex analysis but no one resolved more than some special cases. In 1984, however, Louis de Branges of Purdue University focused the power of his ingenuity, combined with some tools of operator theory and special functions, to raise the conjecture to the status of a theorem.

DE BRANGES' THEOREM. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is holomorphic and univalent (one-to-one) in the unit disk $|z| < 1$, then $|a_n| \leqslant n$ for $n = 2, 3, \ldots$. Furthermore, if $|a_n| = n$ for some integer $n \geqslant 2$, then $f(z) = z/(1 - \omega z)^2$, where ω is a fixed complex number, $|\omega| = 1$.*

Prior to de Branges' result, the Bieberbach conjecture was known to be true, after sixty-eight years of effort, only for the integers $n \leqslant 6$.

Much like the proof of de Branges, the *Proceedings* of the Symposium, held in March of 1985 at Purdue University, contains a medley of topics in analysis, the historical development of the conjecture, and personal accounts of the individuals who played a role in the final stages of the proof. There is something in the book for every mathematician—well-written surveys, open problems, research articles, applications, advice, predictions, and even a most amusing poem befitting the event. The mathematical articles are based primarily on hour-long presentations by various mathematicians at the conference, although there were a number of interesting shorter talks that were not included in the edition. In particular, there is an article in the book by de Branges that generalizes his original theorem on the Bieberbach conjecture.

As stated above, de Branges' Theorem is, in fact, a corollary of a much stronger result. Indeed, a hierarchy of conjectures about holomorphic univalent functions has been formulated over the years. The most notable are (a) the Rogosinski conjecture for subordinate functions, (b) the Robertson conjecture for odd univalent functions, and (c) the Milin conjecture for the coefficients of $\log(f(z)/z)$. The most general of these conjectures (c) implies (b), which implies (a), and (a) implies the Bieberbach conjecture. This chain of implications is one way. What is fantastic is that de Branges proves (c) and, hence, all of these conjectures at once. In his article in these Proceedings, moreover, de Branges shows that his method provides considerably more information than is necessary to prove conjecture (c) but the approach cannot be directly used to establish the Robertson conjecture (b). Whereas this success story may encourage mathematicians to generalize other conjectures, anyone

into Russian the portion of de Branges book directly related to the proof of the conjecture. On the basis of this translation, which replaced the terminology of functional analysis with that of complex analysis, Milin developed a variant of the proof. It appears this version of the proof met a goal of Milin and sparked recognition by experts in univalent function theory regarding the correctness of de Branges' argument.

The Russian mathematicians helped de Branges polish his proof but one problem remained. The equality statement of de Branges' Theorem was yet to be established. As a consequence, some other mathematicians resolved this problem before de Branges could find time to establish the equality directly by using his original argument. Shortly thereafter, however, de Branges produced a more self-contained version of his proof that included, in a most natural way, the case of equality. The authors of other approaches delayed their publication until de Branges' paper appeared (Acta Math., 154 (1985) 137–152).

The proof itself is not simple. A considerable amount of preliminary work on the part of many mathematicians had to be done before such an argument could unveil de Branges' Theorem. The proof, and some historical details, is outlined by Jacob Korevaar [**1**] in this MONTHLY. A more complete historical account of the progress on the Bieberbach conjecture is concisely articulated in the preface of this book and amplified in a number of the articles. This development is also an important aspect of the story of the proof, since it links the accomplishments that eventually lead to the Milin conjecture and subsequently the work of de Branges.

Univalent function theory shall remain alive and well even after the status change of its most famous conjecture. It is likely, nonetheless, that in the field of univalence the aesthetic appeal of de Branges' Theorem shall never be surpassed.

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The Chip with the College Education: the HP-28C

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Five years ago in this MONTHLY [**15**], our present editor augured a future in which students would have pocket calculators that could do symbolic calculus. Exactly five years later, *The Wall Street Journal* [**1**] announced the calculus

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Five years ago in this MONTHLY [**15**], our present editor augured a future in which students would have pocket calculators that could do symbolic calculus. Exactly five years later, *The Wall Street Journal* [**1**] announced the calculus

calculator: the HP-28C. This hand-held machine deserves some attention—if it could walk into a standard lower-division mathematics course, it might well pass on its own. The following examples, which demonstrate the new capabilities of the HP-28C, provide a basis for the subsequent discussion of the potential of such supercalculators in the teaching of mathematics.

1. The power of the HP-28C

CALCULUS. First, watch how the HP-28C solves homework problems selected from various calculus texts.

Problem 1. Let $f(x, y) := x \ln xy$; find $\dfrac{\partial b}{\partial x}$ (Fleming [**4**, p. 79, #1]).

To find $\partial f/\partial x$, enter the formula for f in the form `'X*LN(X*Y)'`, specify the variable by entering `'X'`, and press the differentiation key. The calculator answers

`'LN(X*Y)+X*(Y/(X*Y))'`.

To simplify this expression, select the `ALGEBRA` menu and then the `FORM` submenu; move the cursor onto the second `*` and execute the `COLCT` command. The machine collects similar terms and displays

`'LN(X*Y)+1'`.

Problem 2. Find the Maclaurin polynomial of degree 3 for $\sqrt{1+x}$ (Stein [**13**, p. 547, #10]).

To determine this Taylor polynomial, enter the formula `'√(1+X)'`, specify the variable with `'X'` and the degree with 3, and select the `TAYLR` command from the `ALGEBRA` menu. The HP-28C responds: `'1+.5*X-.125*X^+.0625*X^3'`.

Problem 3. Calculate $\int(ax^2 + bx + c)\,dx$ (Leithold [**10**, p. 376, #20]).

To calculate this indefinite integral, enter the integrand, `'A*X^2+B*X+C'`, the variable of integration, `'X'`, and the degree of the integrand, 2; then press the integration key. Now add your favorite constant to the display,

`'C*X+B/2*X^2+A*2/2/3*X^3'`.

If desired, the `COLCT` command can simplify the redundant form `A*2/2/3`. This redundancy arises from the Maclaurin polynomial of the integrand, which the HP-28C integrates term by term. Although this procedure may seem unwieldy for mere polynomials, it also enables the calculator to tackle harder problems.

Problem 4. Find the Maclaurin series for $(2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt$ (Hurley [**6**, p. 618, #31]).

As in problem 3, enter the integrand, `'2/√π*e^-T^2'`, the variable, `'T'`, and the degree of the desired Taylor polynomial, for example **3** (with a higher degree, the calculator runs out of memory space); then press the integration key. The HP-28C replies:

`'2/√π*T+2/√π*(LN(e)*(-2)/2/3*T^3)'`.

To end this calculus quiz, let the calculator try a curve-sketching problem:

Problem 5. Graph the function $f(x) = e^{\sin x}$ (Spivak [**12**, p. 326, #4b]).

To sketch this curve, store the formula `'e^SIN(X)'`, or `'EXP(SIN(X))'`, into the `PLOT` menu, and execute the `DRAW` command. Within thirty seconds, the HP-28C traces the graph in exhibit 1a. Since the curve does not quite fit into the display, translate the center of the screen upward by 1.4; this will produce the graph in exhibit 1b. For a hard copy, enter the command `CLLCD DRAW PRLCD` and point the calculator toward its printer (with which it communicates by infrared beam).

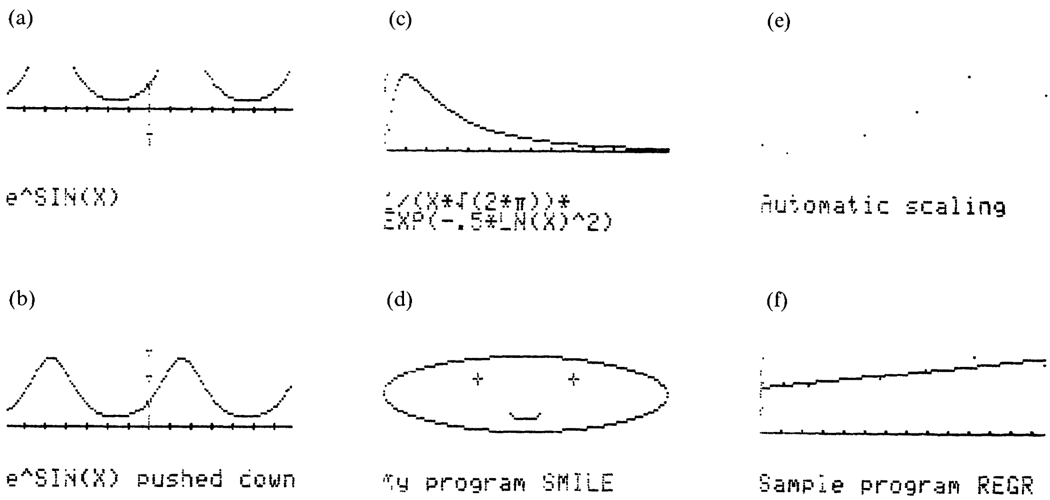

EXHIBIT 1. These slightly enhanced, actual-sized copies from the HP-82240A printer are identical to the HP-28C displays, in both size and resolution. (a) and (b) graphs from Problem 5. (c) and (d) other examples of graphs. (e) and (f) scatter plots from problem 6.

STATISTICS. In addition to computing means, variances, correlations, and regressions, the HP-28C also distinguishes itself with two other novelties. First, it draws scatter plots.

Problem 6. Fit a least-squares line to the data (Freund [**5**, p. 352, #11.1]):

$$(5,16), (1,15), (7,19), (9,23), (2,14), (12,21).$$

"Always plot the data" [**5**, p. 367]. Therefore, enter the data with the `STAT` menu, revert to `PLOT` and execute the `DRWΣ` command. At first the screen shows the axes but no data, because the points lie outside its range. To correct this mismatch, press the `SLCΣ` key, which automatically fits the display onto the data set (but sends the axes away), as in exhibit 1e. To superimpose the least-squares line and bring the axes back into the picture, run the following sample program, which produces exhibit 1f:

```
« SCLΣ (0,0) PMIN LR « X PREDV » STEQ CLLCD DRAW DRWΣ » .
```

Besides drawing scatter plots, the HP-28C computes upper-tail probabilities, $upt(x) := \int_x^{\infty} f(t)\,dt$, for normal, chi-square, t, and F random variables. For example, to compute the probability that a χ^2 random variable with 357 degrees of freedom takes a value bigger than 401.9, enter 357 and 401.9 and execute the `UTPC` command. The calculator gives `.050599`..., meaning that $P(\chi^2_{357} > 401.9) \approx 0.0506$.

Combined with the HP-28C equation solver, upper-tail probabilities also give an easy solution to the following "percentile problem."

Problem 7. Determine the 99th percentile of the distribution χ^2_{357}.

To determine the value of x such that $P(\chi^2_{357} > x) = 1 - 0.99$, program this equation in the form `« .01 357 X UTPC - »` and invoke `SOLVR`, the equation solver. After about a minute, the HP-28C displays `X: 422.08`... To appreciate this prowess, recall that this amounts to solving for x the equation

$$\int_x^{\infty} \frac{1}{\Gamma(357/2)2^{357/2}} t^{355/2} e^{-t/2}\,dt = 0.01.$$

NUMERICAL ANALYSIS AND LINEAR ALGEBRA. In addition to its equation solver, the HP-28C offers further numerical routines that evaluate definite integrals, solve linear systems, and compute dot and cross products, determinants, operator norms, and inverses of real or complex matrices. Since similar features have been available for over five years on a predecessor, the HP-15C, a few illustrations will suffice to describe the speed and accuracy of the HP-28C.

Problem 8. Solve the following moderately ill-conditioned system, with condition number $\|A\|_1 \|A^{-1}\|_1 \approx 10^6$ (Burden and Faires, [**2**, p. 331, #5c]):

$$\begin{aligned}
v + \frac{1}{2}w + \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z &= 1,\\
\frac{1}{2}v + \frac{1}{3}w + \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z &= 1,\\
\frac{1}{3}v + \frac{1}{4}w + \frac{1}{5}x + \frac{1}{6}y + \frac{1}{7}z &= 1,\\
\frac{1}{4}v + \frac{1}{5}w + \frac{1}{6}x + \frac{1}{7}y + \frac{1}{8}z &= 1,\\
\frac{1}{5}v + \frac{1}{6}w + \frac{1}{7}x + \frac{1}{8}y + \frac{1}{9}z &= 1.
\end{aligned}$$

After entering the right-hand vector, B, and the matrix of coefficients, A, simply press the division key, $\div$. The calculator thinks for three seconds and displays its solution:

```
[5.00000076461  -120.000014446  630.000062793
                        -1120.00009538  630.0000468863]
```

This result compares favorably to that of a large mainframe CDC CYBER 180/855 running IMSL (International Mathematical and Statistical Library), which blinked for just 0.01 second and printed:

```
5.000000002094  -120.000000038932  630.000000167638
                             -1120.000000253036  630.000000123768
```

Problem 9. Evaluate $P(X) = 8118X^4 - 11482X^3 + X^2 + 5741X - 2030$ for $X = 0.707107$ (Kulisch and Miranker [**9**, p. 12, #5]).

According to Kulisch and Miranker, $P(0.70107) = -1.91527325270\ldots \times 10^{-11}$. Using double precision (28 digits) the CYBER found $-1.91527325270819 \times 10^{-11}$. Working with 16 digits only, the HP-28C returned the single digit 0.

However, the HP-28C and the CYBER agreed on the answer to this last problem:

Problem 10. Find the eigenvectors of the following matrix (Johnson and Riess [**8**, p. 104, #7]).

$$A = \begin{pmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{pmatrix}.$$

One possible solution (which takes advantage of the HP-28C built-in matrix multiplication and transposition) consists of programming Jacobi's method according to the algorithm in [**14**, pp. 341–342]. Three sweeps of Jacobi's method take only a minute and yield the following eigenvalues and eigenvectors: $\lambda_1 = -15$, $\lambda_2 = -1$, and $\lambda_3 = 5 = \lambda_4$, with

$$v_1 = \begin{pmatrix} .5 \\ .5 \\ .5 \\ .5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -.5 \\ .5 \\ .5 \\ -.5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -.155356107285 \\ .689829312165 \\ .689829312166 \\ .155356107285 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} -.689829312165 \\ .155356107285 \\ -.155356107285 \\ .689829312166 \end{pmatrix}.$$

COMPUTER SCIENCE. Besides its symbolic and numerical capabilities, the HP-28C also provides bit-by-bit logical operators (`AND`, `OR`, `XOR`, `NOT`), register shifts, and hexadecimal, octal, and binary arithmetic, all on 64-bit words. This relatively large word-length makes the HP-28C well suited to one of computer scientists' favorite homework assignments: the simulation of one machine on another.

Exercise. The CDC CYBER mainframe computer operates with 60-bit words, in which it represents integers by "complement to 1." (Thus, the CYBER stores a positive integer as its binary expansion, but it represents a negative integer as the

bit-by-bit logical complement of its absolute value.) Simulate the integer arithmetic of the CYBER on the HP-28C.

2. Supercalculators in the mathematics classroom.

FIRST ACADEMIC REACTIONS. Left to their own devices, four freshmen familiarized themselves with the HP-28C in just two hours, with the help of the well-written *Getting Started Manual.* However, they felt that they needed a better understanding of the mathematics involved in order to use the calculator more intelligently. "Anyway," sighed one, "teachers won't allow it on tests, or will they?"

Nobody knows; in a recent informal poll, faculty showed mixed reactions: "What would I ask on tests now?" wondered one professor, obviously feeling threatened, while at another school, a colleague exclaimed: "It will save hours of calculations!" Because of such a divergence of opinions, the HP-28C and its successors will probably influence individual mathematics curricula in different ways, as does the use of different textbooks now.

Possibly, the HP-28C might enable students instantly to punch, read, and speak calculus. In extremely cook-book courses, students might do nothing but scan the HP-28C UNITS menu from "`Å`" to "`tsp`" to find that one teaspoon equals `'4.92892159375E-6 m^3'`.

The HP-28C may also allow users to leave the calculations to the machine, and to focus on ideas and strategies. For thinkers, including non-mathematicians, the availability of supercalculators may increase the practical importance of theory. Indeed, this conjecture seems supported by the following informal survey.

THE EDUCATION OF THE UNDERGRADUATE MATHEMATICAL PRACTITIONER. What do employers look for in the mathematical education of a new graduate?

Peter Eriksen, who holds a bachelor's degree in mathematics and works for Boeing Military Airplane Co. near Seattle, also has a degree in *philosophy*, which he finds more helpful than mathematics. He considers most useful "the training from a particular philosophy professor, who insisted that we analyze problems logically, that we arrive at some answer, and that we write up our argument in a flawless style."

"Proficiency in undergraduate mathematics, experience in utilizing the mathematics library, attention to detail, and written communication skills," says Dr. Stephen P. Keeler, who hires and supervises mathematicians at Boeing Computer Services Co. He illustrates the need for these intellectual abilities with the following example: "Suppose that you have to code a two-dimensional integration routine. Then you must understand something about the Riemann integral, be able to review the literature on your own, code your algorithm correctly, and document your work in a manner understandable to your colleagues." Unfortunately, Dr. Keeler has found that he cannot assume such an intellectual maturity from students with only a bachelor's degree in mathematics. He suggests one way in which supercalculators might help in education: "To emphasize the importance of details, give students a

mathematical programming assignment [for instance as above] and insist that they get it absolutely right, be it on a LISP machine or on an HP-28C."

Aside from the aerospace industry, indications about the potentials of supercalculators in education may also come from elsewhere in the corporate world.

THE MATHEMATICAL EDUCATION OF THE EXECUTIVE. The Executive Master of Business Administration (EMBA) Program of the University of Washington offers a propitious environment for testing new ideas in the teaching of business calculus, including the use of fancy calculators. Immediately before entering the program, the participating senior executives attend a "business calculus" course designed to meet their needs on the job and in such EMBA courses as finance, microeconomics, and statistics. For this mathematics course, every executive must bring a powerful financial HP-12C (or a scientific HP-15C), which allows for more substantial case-studies, as in the following example.

Example 1. Consider a thirty-year Treasury bond purchased on 15 May 1984 for \$9933.90 with \$662.50 interest coupons every six months. The "yield rate" of this bond, r, is *by definition* given by the solution $v = 1/(1 + r/2)$ of the equation

$$10{,}000v^{60} + 662.50(v^{60} + v^{59} + \cdots + v^2 + v) - 9933.90 = 0.$$

Calculus shows that this equation has exactly one positive solution. While the calculators were computing the yield rate, one banker remarked that "the equation implies that you reinvest every coupon into a similar bond." Freed from the computations, the executive realized what the yield rate *means* and how to interpret it in business. Then the calculators gave the yield rate in the form of rationals on either side of a Dedekind cut or the starts of equivalent Cauchy sequences. The calculators also left time to explain those concepts.

Nevertheless, executives do not feel that supercalculators free them from mastering the basics. "I still need to understand my algebra thoroughly," says a company vice-president, "so that I can explain to myself what a formula means for my business." A health-services director adds that "we need much more graphical analysis, including the concepts of slope and area." Even a supercalculator would not help in the following assignment.

Example 2. Imagine that you sit on the board of directors of your local utility company. Discuss the advantages and disadvantages of setting the price equal to the marginal cost, instead of the average cost.

CONCLUSIONS. The HP-28C introduces one new element into the teaching of mathematics, namely awesome computing power at both a modest price and size, with admirable user-friendliness (all three characteristics compared to those of a CYBER, for instance). Students may thus purchase, carry, and utilize a power close to that of a main-frame as easily as they do textbooks. Still, in spite of the availability of this hand-held power, proficiency in certain basic skills remains essential to the students' ability to *apply* mathematics. Indeed, it appears that a new trend toward the use of the HP-28C and its successors would require that students

understand the underlying concepts even better than before in order to decide what computations to perform, to interpret the results with lucidity [**11**, pp. 40–42], or even first to recognize that no calculator can address the issue at hand. In practice, the need for a deeper understanding of theory grows dramatically, as seen in two excerpts from *The Wall Street Journal:*

> software defects have killed sailors, maimed patients, wounded corporations and threatened to cause the government-securities market to collapse [**3**].

> Morton Thiokol Inc., admitting that it never fully understood the working of the booster rocket blamed for the explosion of the space shuttle Challenger, said it made major changes [**7**].

Shortly after the Challenger disaster, a junior mathematics major at a university expressed the desire to work on the shuttle program but could not cope with the evaluation of $\int_{-1}^{1}|x|\,dx$. The faculty nevertheless decided to graduate the student. With pressure on the faculty to pass students weighed against the need to train students to detect software defects in supercalculators, mathematics instructors face a difficult choice. We may refrain from feeling partly responsible for mistested drugs and shuttle crashes, or we may insist that students (even students with supercalculators) be able to solve unfamiliar problems, detect errors in proofs and programs, and verify the validity of mathematical algorithms, models, and theories.

Acknowledgement. I thank Joyce D. Kehoe, Seattle writer, for her professional help in editing this review.

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TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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T: Textbook	P: Professional Reading	1–4: Semesters
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S: Supplementary Reading	13: Grade Level	??: Questionable

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General, S, L*. *Sliding Piece Puzzles.* Edward Hordern. Rec. in Math., V. 4. Oxford U Pr, 1986, xiii + 249 pp, $26.95. [ISBN: 0-19-853204-0] A delightful collection of more than 250 sliding piece puzzles, graded according to difficulty (from one to six), complete with optimal (or near optimal) solutions. Necessary "push-out" pieces are included. The main purpose of the book is to provide entertainment, and to get you hooked; the author graciously includes a few instructive examples for mastering basic principles. LCL

General, S, L*.** *NYSML-ARML Contests, 1973-1985, Second Edition.* Ed: Harry D. Ruderman. NCTM, 1987, vi + 193 pp, $8 (P). [ISBN: 0-87353-247-3] Problems and answers, by year, of two prestigious high school mathematics competitions: The New York State Mathematics League, and The Atlantic Region Mathematics League. An invaluable resource for teachers of mathematical problem solving (even at the college level), and for students competing in mathematics contests. There are three types of questions: Short Answer Questions (time limits given), Power Questions (requiring discovery and explanation), and Relay Questions (for team competitions)—all designed to generate interest and encourage ingenuity. LCL

General, P, L. *Symmetry: Unifying Human Understanding.* Ed: István Hargittai. Intern. Ser. in Mod. Appl. Math. & Comput. Sci., V. 10. Pergamon Pr, 1986, xiii + 1045 pp, $125. [ISBN: 0-08-033986-7] An interdisciplinary issue consisting of articles by over 70 authors from a wide range of disciplines and nations. JNC

General, S*, P*. *CRC Handbook of Mathematical Sciences, 6th Edition.* Ed: William H. Beyer. CRC Pr, 1987, 860 pp, $64.95. [ISBN: 0-8493-0656-6] Same general format as previous editions. Additions include material on solutions to nonlinear equations, statistical tests of hypothesis, statistical confidence intervals, and analysis of variance tables. Omissions include some power and root tables, logarithms of binomial coefficients, and financial tables. Still contains page after page of integral tables, special functions, probability and statistics, differential equations, analytic geometry, and many other things we always want to look up. MR

General, L. *The High School Mathematics Library.* William L. Schaaf. NCTM, 1987, x + 83 pp, $7.80 (P). [ISBN: 0-87353-238-4] A list of about 1000 books recommended for high school libraries. About 200 titles have been added (and an equal number dropped) since the 1982 edition (TR, October 1982). Includes exposition, biography, history, mathematical topics, computer science, and professional books for teachers. LAS

General, P, L. *Science, Computers, and People: From the Tree of Mathematics.* Stanislaw M. Ulam. Birkhauser Boston, 1986, xxi + 264 pp, $39. [ISBN: 0-8176-3276-X] A sampler of Ulam's non-mathematical essays—reviews, editorials, exposition, commentary, personal reflections. Includes a personal account of Ulam's style by Françoise Ulam, a memorial statement by Gian-Carlo Rota, and an introduction by Martin Gardner. LAS

Elementary, T(13: 1). *Mathematics for Electronics, Second Edition.* Donald P. Leach. Prentice-Hall, 1987, xiii + 433 pp, $34.95. [ISBN: 0-13-562455-X] "How-to-apply-it" approach to topics from intermediate algebra and elementary trigonometry, using a variety of problems from electronics. Includes ma-

trices (2 x 2 and 3 x 3) and determinants; complex numbers; logarithms. (*First Edition,* TR, November 1979.) LCL

Mathematics Appreciation, S(13-15), L. *Mathematics and the Search for Knowledge.* Morris Kline. Oxford U Pr, 1985, vii + 257 pp, $19.95. [ISBN: 0-19-503533-X] A non-technical account of those aspects of physical reality that are known chiefly through the medium of mathematics, from Greek astronomy through nineteenth century electromagnetism to contemporary quantum physics. Vintage Kline, playing new variations on his standard theme: readers of his earlier volumes will find these ideas very familiar. New readers will learn much about the sources of mathematical power. LAS

Mathematics Appreciation, S*(13-14), L. *Numbers At Work and At Play.* Stephen P. Richards (Box 501, New Providence, NJ 07974), 1987, 213 pp, $8.95 (P). [ISBN: 0-9608224-2-9] An elementary exposition of a wide variety of contemporary mathematics (e.g., NP-problems, fractals, monster groups) using only examples based on numbers. Typewritten manuscript; too few illustrations. Inventive, imaginative, intriguing. A superb sketch of the mathematical way of thought in down-to-earth terms. LAS

Precalculus, T(13: 1). *Precalculus Mathematics.* John M. Thomason. Wiley, 1987, xvii + 811 pp, $40.50. [ISBN: 0-471-01153-3] The author assumes a background of two years of algebra and one of geometry but still needs 650 pages of text to develop the elementary concepts of fractions, exponentials and logarithms, trigonometry and analytic geometry. Some computer exercises. JS

Precalculus, S(13). *Calculator Programs for Classical Algebra.* C.W. Young. C&R Pr, 1986, 337 pp, $29.95 (P). [ISBN: 0-9617321-0-5] A collection of algorithms from college algebra, laid out in the form of detailed flow charts convenient for programming an electronic calculator. LCL

Precalculus, T(13: 1). *Trigonometry.* John R. Durbin. Wiley, 1987, ix + 402 pp, $31.50. [ISBN: 0-471-03366-9] Trigonometry presented in the general context of functions and their graphs. Text begins with a review of functions and ends with complex numbers, the exponential and logarithmic functions, and analytic geometry. Lots of examples, exercises, and chapter review. LC

Precalculus, T(13: 1). *Algebra and Trigonometry: A Pre-Calculus Approach, Third Edition.* Max A. Sobel, Norbert Lerner. Prentice-Hall, 1987, xvii + 826 pp. [ISBN: 0-13-021502-3-01] This edition includes more material on trigonometry, an introduction to complex numbers, and chapters on linear systems, matrices and determinants and permutations, combinations and probability; a set of sample test questions has also been added to each chapter. (*First Edition,* TR, October 1979; *Second Edition,* TR, August-September 1983.) JNC

Precalculus, T(13). *College Algebra and Trigonometry, Second Edition.* Bernard Kolman, Arnold Shapiro. Academic Pr, 1986, xiii + 648 pp, $24. [ISBN: 0-12-417905-3] A basic college algebra and trigonometry text. Complex numbers are introduced in the first chapter to allow for the solution of any quadratic equation. Writing style is serious and not condescending to the reader. Answers are given to odd problems and chapter reviews. (*First Edition,* TR, December 1981.) SM

Precalculus, T(13: 2). *College Algebra, Second Edition.* Ralph C. Steinlage. West, 1987, xii + 608 pp, $28.36. [ISBN: 0-314-29531-3] Major changes from *First Edition* focus on lowering the difficulty level for the student. Harder exercises have been reworked or replaced and the more difficult ones have been numbered in red. Chapter overviews and historical perspectives have been added. MR

Education, P. *Teaching and Learning: A Problem-solving Focus.* Ed: Frances R. Curcio. NCTM, 1987, viii + 116 pp, $12 (P). [ISBN: 0-87353-240-6] A collection of seven essays beginning with Pólya's "On Learning, Teaching, and Learning Teaching," and ending with a summary of "indications" from research on problem solving by Suydam. Two essays are especially recommended: a brief but lively history of problem solving by Schoenfeld, and a provocative piece by Kilpatrick whose two-sentence title begins "Is Teaching Teachable?" Worthwhile reading for teachers of mathematics and "teachers" of teachers of mathematics. SG

Education, P. *Neue Ideen zur Stochastik.* Wolfgang Riemer. Lehrbucher und Mono. zur Didaktik der Math., Band 3. Bibliographisches Institut, 1985, 156 pp, (P). [ISBN: 3-411-03119-0] On how to teach probability and statistics at the secondary level. Central, in the author's opinion, should be subjective probability, Bayes' theory, and the early weighing of alternative hypotheses. JD-B

Education, P, L. *Providing Opportunities for the Mathematically Gifted, K-12.* Ed: Peggy A. House. NCTM, 1987, iv + 100 pp, $9 (P). [ISBN: 0-87353-239-2] A survey of gifted education with guidelines and examples of programs in mathematics. Primarily procedural; little discussion of the mathematics appropriate for gifted children. Includes an extensive bibliography. LAS

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be taught. Little new in general ideas expressed, but much of interest in illustrating what they mean in particular situations. JD-B

Education, P. *Numerische Mathematik im Rahmen der Schulmathematik: Ansäte zu einer Didaktik.* Jürgen Blankenagel. Lehrbucher und Mono. zur Didaktik der Math., Band 2. Bibliographisches Institut, 1985, 192 pp, (P). [ISBN: 3-411-03118-2] On how numerical calculation should be taught in the schools, where hand calculators or computers are now generally available. Many interesting examples. JD-B

Education, P. *Different Ways Children Learn to Add and Subtract.* Thomas A. Romberg, Kevin F. Collis. Journ. for Res. in Math. Educ., No. 2. NCTM, 1987, xiii + 178 pp, $7.50 (P). A report on five studies carried out in Australia in 1979-80 examining how children in grades 1-3 learn addition and subtraction. Several interesting conclusions are drawn including the suggestion that more curricular emphasis be placed on counting strategies, verbal problems, and writing. SG

Education, S, L. *Teaching Thinking Skills: Mathematics.* Marcia Heiman, *et al.* NEA, 1987, 48 pp, $6.95 (P). [ISBN: 0-8106-0203-2] A superficial survey of the research and policy literature concerning how children learn to think about mathematics. Focuses on word problems and Piaget's research; gives only lip service to heuristics (no mention of Pólya) and computers (just one page). LAS

Education, P, L. *Learning and Teaching Geometry, K-12: 1987 Yearbook.* Mary Montgomery Lindquist, Albert P. Shulte. NCTM, 1987, vi + 250 pp, $16. [ISBN: 0-87353-235-X] Twenty chapters by individual authors in five groupings: perspectives; problem solving and applications; activities; geometry related to other mathematics; teacher preparation. Ideas related to curriculum, approaches to teaching, problem-solving at all levels are discussed throughout. Chapter references. RJA

Education, S*, P, L*. *Stories of Excellence: Ten Case Studies from a Study of Exemplary Mathematics Programs.* Mark Driscoll. NCTM, 1987, 106 pp, $9 (P). [ISBN: 0-87353-236-8] Profiles of excellence in school mathematics, leading to one overwhelming indicator of excellence: teacher collegiality. Teachers who talk with each other, who share ideas, who learn from each other weave a network of support for the risk-taking so necessary to learning. LAS

History, P, L. *Papers of John von Neumann on Computing and Computer Theory.* Ed: William Aspray, Arthur Burks. Charles Babbage Inst. Repr. Ser. for the Hist. of Comput., V. 12. MIT Pr, 1986, xviii + 624 pp, $65. [ISBN: 0-262-22030-X] One dozen of von Neumann's major computer papers, grouped into four categories (architecture, coding, high speed computing, automata), with various overviews and modern supplements (e.g., Knuth's paper on von Neumann's first computer program). Most papers are also in von Neumann's *Collected Works* (Macmillan, 1963), but that multi-volume work is out-of-print. LAS

History, S(13).** *To Infinity and Beyond: A Cultural History of the Infinite.* Eli Maor. Birkhauser Boston, 1986, xvi + 275 pp, $49.50. [ISBN: 0-8176-3325-1] Interesting story of the role of infinity in our culture. Touches on subjects ranging from mathematics, cosmology, art, and music. Very readable. Occasionally a topic is not covered in sufficient detail (at least for a mathematician). Good but not exhaustive bibliography for those who wish to read further. Work of Escher is discussed quite thoroughly. A well-done book for both the lay person and the scientist. MR

History, L*. *Hugo Steinhaus Selected Papers.* Hugo Steinhaus. PWN, 1985, 899 pp. [ISBN: 83-01-00825-3] Approximately one-third of Steinhaus' 250+ publications, introduced by a reprinted biographical essay by E. Marczewski and a complete Steinhaus biography. Papers were selected to represent fairly the very broad scope of Steinhaus' work and thought. Includes many expository and reflective essays. Articles originally published in Polish have been translated into English. LAS

History, P, L. *Invariantetheorie.* Paul Gordan. Chelsea, 1987, xi + 360 pp, $49.95. [ISBN: 0-8284-0328-7] Reprint, with only minor changes, of the *First Edition,* published in two volumes in 1885 and 1887. JD-B

History, P. *Selected Papers on Algebra and Topology by Garrett Birkhoff.* Ed: Gian-Carlo Rota, Joseph S. Oliveira. Birkhauser Boston, 1987, xvii + 608 pp, $78. [ISBN: 0-8176-3114-3] Reprints of 41 papers published from 1933 to 1976, chosen by the author as his most interesting and influential in the two fields. Also six brief essays giving the historical background of groups of related papers. JD-B

History, P, L. *Leonardo Pisano Fibonacci: The Book of Squares.* L.E. Sigler. Academic Pr, 1987, xx + 124 pp, $19.95. [ISBN: 0-12-643130-2] The first complete English translation of Leonardo Pisano's 1225 *Liber quadratorum,* the high point of number theory from the *Arithmetica* of Diophantus to Fermat. The subject is square numbers and how they "arise out of the increasing sequence of odd numbers;" the methods are a virtuso display of geometrical algebra à la Euclid. Sigler's extensive commentary on each section restates each argument in random form to help the reader grasp the original reasoning. LAS

History, L*. *Alfred Tarski, Collected Papers, V. 1-4.* Ed: Steven R. Givant, Ralph N. McKenzie. Birkhauser Boston, 1986, $630 set [ISBN: 0-8176-3284-0]. *V. 1, 1921-1934,* 659 pp; *V. 2, 1935-1944,*

699 pp; *V. 3, 1945-1957,* 682 pp; *V. 4, 1958-1979,* 757 pp. A complete collection of all Tarski papers, abstracts, and reviews (but not books) forming the "bedrock" of modern logic both within mathematics and in philosophy: theory of truth, undecidable theories, algebraic logic, universal algebra, foundations of geometry, and set theory. Concludes with a comprehensive scholarly Tarski bibliography giving explicit references to the numerous translations of Tarski's papers. LAS

Logic, P, L*.** *Ω-Bibliography of Mathematical Logic.* Ed: Gert H. Müller, Wolfgang Lenski. Perspect. in Math. Logic. Springer-Verlag, 1987. *Volume I: Classical Logic.* Ed: Wolfgang Rautenberg, xxxix + 485 pp, $170 [ISBN: 0-387-17321-8]; *Volume II: Non-Classical Logics.* Ed: Wolfgang Rautenberg, xxxvii + 469 pp, $170 [ISBN: 0-387-15521-X]; *Volume III: Model Theory.* Ed: Heinz-Dieter Ebbinghaus, xlv + 617 pp, $200 [ISBN: 0-387-15522-8]; *Volume IV: Recursion Theory.* Ed: Peter G. Hinman, xlv + 697 pp, $200 [ISBN: 0-387-15523-6]; *Volume V: Set Theory.* Ed: Andreas R. Blass, li + 791 pp, $230 [ISBN: 0-387-15525-2]; *Volume VI: Proof Theory, Constructive Mathematics.* Ed: Jane E. Kister, Dirk van Dalen, Anne S. Troelstra, xli + 405 pp, $155. [ISBN: 0-387-15524-4] A comprehensive bibliography of all works in mathematical logic published between 1879 and 1985, classified according to the *Mathematical Reviews* categories in Section 03, expanded to account for older specialties. Except for *Volumes I* and *II* which together make up Section 03B, other volumes match *MR* Sections 03C, 03D, 03E, and 03F. Each volume contains complete introductory sections on the organization of the work; a Subject Index giving within each subtopic a chronological list of papers (author *and* title); an Author Index giving bibliographic details (including references to reviews); a Source Index giving full details on cited journals, collected works, proceedings and publishers; and miscellaneous lists with cross reference to other volumes in the set. A definitive reference work that is easy to use and attractive for browsing. Of particular value is the historical view of each topic that emerges from scanning the chronologically organized Subject Index. LAS

Logic, T(17-18: 2), P, L*. *Theory of Recursive Functions and Effective Computability.* Hartley Rogers, Jr. MIT Pr, 1987, xxi + 482 pp, $15 (P). [ISBN: 0-262-68052-1] Corrected paperback reprinting of a pioneering 1967 McGraw-Hill monograph—the first comprehensive treatment of the theory of algorithms, partial functions, recursion, and unsolvability. Subsequent developments in computer science have confirmed the continuing significance of this seminal work. LAS

Logic, P. *Logic and Combinatorics.* Ed: Stephen G. Simpson. Contemp. Math., V. 65. AMS, 1987, xi + 394 pp, $37 (P). [ISBN: 0-8218-5052-0] Proceedings of the AMS-IMS-SIAM joint summer research conference on applications of mathematical logic to finite combinatorics held in August 1985 at Humboldt State University. LC

Logic, P. *Mathematical Logic and Theoretical Computer Science.* Ed: David W. Kueker, Edgar G.K. Lopez-Escobar, Carl H. Smith. Lect. Notes. in Pure & Appl. Math., V. 106. Dekker, 1987, xviii + 383 pp, $74.75 (P). [ISBN: 0-8247-7746-8] Collection of articles by a subset of the participants in the University of Maryland's XVI Special Year in Mathematics. Topics include denotational semantics, recursion theoretic aspects of computer science, logic, artificial intelligence, model theory and algebra, Automath and automated reasoning, stability theory, toposes. Participant list. List of talks by session. Index. RJA

Foundations, T(17-18), P. *A Theory of Sets, Second Edition.* Anthony P. Morse. Pure & Appl. Math., V. 108. Academic Pr, 1986, xxxii + 179 pp, $59. [ISBN: 0-12-507952-4] A formal, unified treatment of logic and set theory, that "can be used to build just about any mathematical structure on some suitable foundation of definitions and axioms." (*First Edition,* TR, March 1967.) LCL

Foundations, S*(16-18), P*, L*. *Kurt Gödel: Collected Works, Volume I: Publications 1929-1936.* Ed: Solomon Feferman, *et al.* Oxford U Pr, 1986, xvi + 474 pp, $35. [ISBN: 0-19-503964-5] The first of two volumes of published works to be succeeded by volumes of unpublished manuscripts and lectures. Gödel, the most outstanding logician of the twentieth century, "succeeded in establishing the subject of mathematical logic as one that could be pursued with results as decisive and significant as those in the more traditional branches of mathematics." Each article is preceded by an introductory note which elucidates and places it in historical context. Those articles originally written in German are reproduced with English translations on facing pages. Includes a biographical essay of Gödel's life and work. LCL

Graph Theory, T(14-16: 1), S, P, L. *Graph Theory 1736-1936.* Norman L. Biggs, E. Keith Lloyd, Robin J. Wilson. Oxford U Pr, 1986, xi + 239 pp, $26.95 (P). [ISBN: 0-19-853916-9] Paperback issue of the corrected 1976 edition (TR, March 1977), giving self-contained introduction to graph theory using historical (hence, problem-oriented) approach, allowing reader to learn graph theory the way it was discovered. Includes 37 extracts taken from the original sources, integrated with the textual development. RM

Graph Theory, T(16-17: 1), S, L. *Graph Theory: A Development from the 4-Color Problem.* Martin Aigner. Transl: L. Boron, C. Christenson, B. Smith. BCS Associates, 1987, vii + 226 pp, $28 (P). [ISBN: 0-914351-03-6] Readable introduction to graph theory and the role the 4-color problem played

in its development. Presents origins and early attempts at solving the 4-color problem, plus standard topics of planarity, coloring factorization, Hamiltonian circuits, and matroids. Ends with discussion of final solution to the 4-color problem and questions arising from the solution. LC

Graph Theory, T(17-18: 2), S, P. *Some Topics in Graph Theory.* H.P. Yap. London Math. Soc. Lect. Note Ser., V. 108. Cambridge U Pr, 1986, 230 pp, $24.95 (P). [ISBN: 0-521-33944-8] A very personal selection of topics used as a textbook for a first course in graph theory. Much of the material is taken directly from recent research papers. Topics include: edge colorings, symmetries, packing, and computational complexity. SS

Combinatorics, S(16-18), P, L.** *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds.* G. Pólya, R.C. Read. Springer-Verlag, 1987, vii + 148 pp, $33. [ISBN: 0-387-96413-4] An English translation of Pólya's classic paper on combinatorial enumeration. The range of applications of Pólya's theorem is enormous, as shown even in this paper; here we find a comprehensive and unified treatment of problems which had previously been solved, if at all, only by *ad hoc* methods. Read has included a useful and informative survey of the many different kinds of research that have stemmed from this paper during the past fifty years. Its profound influence is still growing. LCL

Combinatorics, P. *Lecture Notes in Mathematics-1238: Injective Choice Functions.* Michael Holz, Klaus-Peter Podewski, Karsten Steffens. Springer-Verlag, 1987, vi + 183 pp, $15.80 (P). [ISBN: 0-387-17221-1] The marriage problem is the following: given a set of boys, each of whom know several girls, what conditions are necessary so that we can marry off the boys in a way so that each boy is paired with a girl he knows? These notes generalize this problem: find necessary and sufficient conditions which decide if a family of sets has an injective choice function. LC

Combinatorics, T(17-18: 2), P. *Design Theory.* Thomas Beth, Dieter Jungnickel, Hanfried Lenz. Cambridge U Pr, 1986, 688 pp, $79.50. [ISBN: 0-521-33334-2] Comprehensive treatment of combinatorial design theory including analysis of incidence structures, interplay of groups and designs, constructive aspects (such as constructions based on Abelian groups), and characterizations of the classical designs. May be used as a text or as a reference for researchers. Includes exercises and over 500 references. (1985 Bibliographisches Institut edition, TR, October 1986.) LC

Combinatorics, P. *Surveys in Combinatorics 1987.* Ed: C. Whitehead. London Math. Soc. Lect. Note Ser., V. 123. Cambridge U Pr, 1987, 226 pp, $27.95 (P). [ISBN: 0-521-34805-6] Papers of the nine invited speakers at the eleventh British combinatorial conference held at the University of London Goldsmith's College, July 1987. Includes papers by R.L. Graham and P. Erdös. LC

Discrete Mathematics, T(14: 1). *Mathematical Structures for Computer Science, Second Editon.* Judith L. Gersting. Books in Math. Sci. WH Freeman, 1987, xv + 618 pp, $35.95. [ISBN: 0-7167-1802-2] Revisions include expanded coverage of logic, induction, combinatorics, and graphs; addition of topics such as recursion and recurrence relations, matrices and their operations; plus more examples and exercises illustrating computer science applications. (*First Edition,* TR, March 1983; Extended Review, June-July 1984.) LC

Discrete Mathematics, T(13-14: 1), L. *Discrete Mathematical Structures for Computer Science, Second Edition.* Bernard Kolman, Robert C. Busby. Prentice-Hall, 1987, xv + 464 pp. [ISBN: 0-13-216003-X] Revisions of the *First Edition* (TR, August-September 1984) include new material on combinations, permutation, new section on pigeonhole principle, more exercises and examples, and use of second color. LC

Discrete Mathematics, T(13: 1). *Discrete Mathematics: A Bridge to Computer Science and Advanced Mathematics.* Olympia Nicodemi. West, 1987, xiv + 491 pp, $30.56 [ISBN: 0-314-28503-2]; *Solutions Manual to Accompany,* iii + 121 pp, (P). [ISBN: 0-314-59201-6] Logic, set theory, functions, elementary combinatorics, graph theory, relations, and an introduction to algebra (groups, rings, fields). Optional section at each chapter end gives an application to computer science. Assumes "mathematical maturity usually provided by one semester of calculus and a high-level computer language." LC

Discrete Mathematics, T(14: 1), L. *Discrete Mathematics and Algebraic Structures.* Larry J. Gerstein. Math. Sci. WH Freeman, 1987, xv + 413 pp, $32.95. [ISBN: 0-7167-1804-9] Introduction to concepts and techniques in discrete mathematics and abstract algebra, including familiarity with symbols (quantifiers, set notation), proof techniques (induction, contradiction), and abstract ideas such as countability, congruence, and groups. Also covers topics such as language and finite automata and elementary combinatorics. LC

Discrete Mathematics, T(13-14: 1). *Introductory Discrete Structures with Applications.* Bernard Kolman, Robert C. Busby. Prentice-Hall, 1987, xiv + 364 pp. [ISBN: 0-13-500794-1-01] Organized into two parts. Part I is elementary theory of sets, counting, relations, and functions. Part II is dedicated to applications, with chapters on Boolean algebra, trees, languages, finite state machines and probability. Exercises, answers. JS

Number Theory, P. *Arithmetic Duality Theorems.* J.S. Milne. Perspect. in Math., V. 1. Academic

Pr, 1986, x + 421 pp, \$38. [ISBN: 0-12-498040-6] An organized account, with proofs, of some important duality theorems concerning the Galois cohomology of finite modules and Abelian varieties over local and global fields, with generalizations to cohomology groups. LCL

Number Theory, S(16-18), L*. *Essais Historiques sur la Théorie des Nombres.* André Weil. Mono. No. 22. L'Enseignement Mathematique, 1975, 55 pp, (P). Three high-level expository essays—a long one in English and two shorter ones in French—stressing the continuity of number theory research from Fermat to the present. LAS

Number Theory, T(17-18: 1, 2), S, P*, L*. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* Jonathan M. and Peter B. Borwein. Canadian Math. Soc. Ser. of Mono. & Adv. Texts. Wiley, 1987, xv + 414 pp, \$49.95. [ISBN: 0-471-83138-7] How fast can you get five thousand digits of π? The familiar arctangent formulas lose to the arithmetic-geometric mean (AGM). The how and why necessitate a study of elliptic integrals and theta functions, with an eye on computational complexity. Numerous exercises. Suitable for sharp undergraduates. BC

Number Theory, T(17), S, P. *Diophantine Inequalities.* R.C. Baker. London Math. Soc. Mono. New Ser., V. 1. Clarendon Pr, 1986, xii + 275 pp, \$65. [ISBN: 0-19-853545-7] An overview of recent developments in the study of fractional parts of polynomials. Most of the principal results are analogous for higher-degree polynomials of Dirichlet's theorem on integer multiples of an arbitrary point in Euclidean space. Proofs are given in detail. Excellent list of references. CEC

Number Theory, P. *Number Theory.* Ed: H. Kisilevsky, J. Labute. CBMS Conf. Proc., V. 7. AMS, 1985, x + 466 pp, \$54 (P). [ISBN: 0-8218-6012-7] Proceedings of the Montreal conference held June 17-29, 1985. 21 papers on a variety of topics. Main papers by Gross ("Heights and the special values of L-series"), and Stark ("Modular forms and related objects"). MR

Number Theory, T(18: 4), S, P. *Introduction to Complex Hyperbolic Spaces.* Serge Lang. Springer-Verlag, 1987, viii + 271 pp, \$58. [ISBN: 0-387-96447-9] A complex space is hyperbolic if the distance between two points is defined by summing over the hyperbolic length of local geodesics. Lang's point of view stems from the relationship between the subject and diophantine geometry; in particular, he conjectures that a projective variety is hyperbolic if and only if it has only a finite number of rational points in every finitely generated field over the rationals. MR

Number Theory, T(18), S. P*. *Quadratic Forms and Hecke Operators.* Anatolij N. Andrianov. Grund. der math. Wissenschaften, V. 286. Springer-Verlag, 1987, xii + 374 pp, \$102. [ISBN: 0-387-15294-6] For twenty years Andrianov and collaborators have elaborated the theory of theta-series and the Hecke algebra in the setting of Siegel modular forms. This excellent book gives a complete, self-contained account of the subject, eminently suitable as an introduction and valuable as a reference. Includes numerous exercises. BC

Number Theory, P. *Lecture Notes in Mathematics-1240: Number Theory.* Ed: D.V. Chudnovsky, *et al.* Springer-Verlag, 1987, 324 pp, \$27.80 (P). [ISBN: 0-387-17669-1] Sixteen papers from the CUNY number theory seminar. Includes excellent survey papers by D.V. and G.V. Chudnovsky on computer-assisted number theory, and by S. Wagstaff and J. Smith on methods of factoring large numbers. BC

Number Theory, T*(18: 4), S, P.** *Algebraic Number Theory.* Serge Lang. Grad. Texts in Math., V. 110. Springer-Verlag, 1986, xiii + 354 pp, \$29.80. [ISBN: 0-387-96375-8] A reissue of the 1970 edition (TR, February 1971). Divided into three parts: Basic Theory, including algebraic integers, completions, ideles and adeles, and elementary properties of Zeta functions and L-series; Class Field Theory, including the Artin symbol, existence theorem for class fields, and L-series; and Analytic Theory, including both Hecke and Tate's proof of the functional equation for the Zeta function, and for the Brauer-Siegal theorem which gives an asymptotic relationship between the class number, regulator, and discriminant of a number field. Excellent modern introduction to subject. MR

Number Theory, P. *The Theory of Riemann Zeta-Function, Second Edition.* E.C. Titchmarsh. Oxford U Pr, 1986, x + 412 pp, \$49.95 (P). [ISBN: 0-19-853369-1] A reprinting of this classic treatise with corrections and extensive end-of-chapter notes by D.R. Heath-Brown. SG

Number Theory, P. *Geometry of Numbers, Second Edition.* P.M. Gruber, C.G. Lekkerkerker. Math. Lib., V. 37. Elsevier Science, 1987, xv + 732 pp, \$95.50. [ISBN: 0-444-70152-4] Essentially a reprinting of the *First Edition* with supplemental notes added to each chapter. Topics covered in depth include convex bodies, covering constant, star bodies, homogeneous and inhomogeneous forms. Extensive bibliography. A basic reference on the subject. SG

Linear Algebra, T(15: 1), S, L*. *Linear Algebra, Third Edition.* Serge Lang. Undergrad. Texts in Math. Springer-Verlag, 1987, ix + 285 pp, \$34. [ISBN: 0-387-96412-6] For upper-level undergraduates. Covers vector spaces, matrices, linear maps, scalar products, structure theorems for eigenvalues, eigenvectors, quadratic and Hermitian forms, spectral theorems for finite dimensional spaces, Jordan canonical form, and the Krein-Milman theorem. Many parts of the text have been rewritten and re-

organized, and a substantial number of new exercises have been added in this edition. (*First Edition*, TR, December 1967; *Second Edition*, TR, December 1971.) CEC

Linear Algebra, T(17-18: 1). *Matrix Analysis for Applied Sciences, Volume 2.* Ivo Marek, Karel Žitný. Teub-Texte zur Math., B. 84. BG Teubner, 1986, 152 pp, 15M (P). Functional analysis on finite-dimensional spaces, treating matrices as representations of linear operators. Particular emphasis on pseudoinverse operators. The translation into English is at times rather awkward. BC

Linear Algebra, T*(14: 1), S, L. *Linear Algebra: A Comprehensive Introduction.* Donald H. Pelletier. Prentice-Hall, 1986, xii + 548 pp, $29.95. [ISBN: 0-8359-4064-0] A linear algebra book for second-year students which is simultaneously introductory and comprehensive. Well-written, but its thoroughness will force the instructor to be selective. Additional routine exercises are needed in a few sections. Some of the illustrations are outstanding. Does not include applications. Very suitable for an honors class. CEC

Linear Algebra, S(15-16). *Some Eclectic Matrix Theory.* Kenneth S. Miller. Robert E Krieger, 1987, ix + 130 pp, $14.50. [ISBN: 0-89874-895-X] A collection of special topics from linear algebra, e.g., positive definite matrices, inversion of matrices having special forms, applications of symmetric functions to eigenvalue problems. Interesting, clearly written, and accessible to undergraduates with a good course in linear algebra. AM

Linear Algebra, T(16-17: 1). *Applied Linear Algebra.* Riaz A. Usmani. Pure & Appl. Math., V. 106. Dekker, 1987, vi + 258 pp, $39.75. [ISBN: 0-8247-7622-4] Text for a second course in linear algebra. Reviews vector spaces and matrix theory in first two chapters, then covers topics such as generalized inverses of matrices and their applications to least squares and minimum norm solutions, eigenvalues of band, companion and circulant matrices, solving sets of first order differential and difference equations, and irreducible and monotone matrices. Includes examples and exercises. LC

Linear Algebra, T*(14: 1), S. *Elementary Linear Algebra.* D.J. Hartfiel, Arthur M. Hobbs. Prindle, Weber & Schmidt, 1987, xii + 435 pp. [ISBN: 0-87150-038-8] Includes the standard material covered in second-year linear algebra texts along with sections on numerical methods and linear programming. Most sections contain routine exercises, exercises involving complex numbers, theoretical exercises, applications, and computer exercises. The bulk of the text consists of worked examples and proofs of theorems. CEC

Linear Algebra, T(14: 1), S.** *Elementary Linear Algebra.* Robert S. Johnson, Thomas O. Vinson, Jr. Harcourt Brace Jovanovich, 1987, viii + 342 pp, $30.95. [ISBN: 0-15-521082-3] A highly readable and useable text for students at the freshman or sophomore level. Includes standard sections on systems of equations, matrices, determinants, vectors, vector spaces, linear maps, inner products, and eigenvalues. Also includes a section on complex spaces and linear differential equations. Good collection of exercises and examples. CEC

Group Theory, S(18), P. *Skew Linear Groups.* M. Shirvani, B.A.F. Wehrfritz. Math. Soc. Lect. Note Ser., V. 118. Cambridge U Pr, 1986, 253 pp, $22.95 (P). [ISBN: 0-521-33925-1] Presents a coherent survey of the current state of development in the theory of matrix groups over division rings. Assumes considerable background in group theory and some ring theory. Treatment includes locally finite groups, absolutely irreducible groups, applications to group rings. Bibliography, indexes. JS

Group Theory, P. *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume II.* Charles W. Curtis, Irving Reiner. Wiley, 1987, xv + 951 pp, $95. [ISBN: 0-471-88871-0] The completion of this most remarkable, thorough, self-contained treatment of representation theory. Contains an extensive introduction to algebraic K-theory, a nearly complete survey of results on class groups, chapters on block theory, finite groups of Lie type, rationality questions, indecomposable modules, and representation rings. Volumes *I* and *II* replace the old Curtis-Reiner as the bible of representation theory. SG

Algebra, T(17: 2). *Algebras, Lattices, Varieties, Volume I.* Ralph N. McKenzie, George F. McNulty, Walter F. Taylor. Math. Ser. Wadsworth, 1987, xii + 361 pp, $44.95. [ISBN: 0-534-07651-3] General theory of algebras and its connections to lattices and varieties. Assumes familiarity with basic set theory and classical algebraic structures (e.g., groups, rings). Includes exercises. First of four volumes. LC

Algebra, P. *Algebra and Order.* Ed: S. Wolfenstein. Res. & Expos. in Math., V. 14. Heldermann Verlag, 1986, xi + 385 pp, DM 72 (P). [ISBN: 3-88538-214-8] Proceedings of the First International Symposium on Ordered Algebraic Structures, held in Luminy-Marseilles in 1984. The conference was unique in bringing together researchers from the entire spectrum of ordered algebraic structures: lattice ordered groups, ordered fields and real algebraic geometry, f-algebras and function spaces, as well as ordered semigroups, semirings, and related structures. LCL

Algebra, T(18: 2), S, P. *The Algebraic Structure of Group Rings.* Donald S. Passman. Robert E Krieger, 1985, xiv + 734 pp, $59.95. [ISBN: 0-89874-789-9] Written to be accessible to second-year graduate students, this book offers a readable, largely self-contained analysis of a substantial portion of the modern theory of group rings for infinite groups.

Part 1 is basic concepts, Part 2 studies the role of the center of the group ring and related concepts, and Part 3 treats finiteness conditions and Noetherian group rings. Exercises, references, and index, all quite thorough and helpful. Reprint of the 1977 Wiley edition (TR, April 1978). JS

Algebra, S(18), P. *The Jacobson Radical of Group Algebras.* Gregory Karpilovsky. Math. Stud., V. 135. Elsevier Science, 1987, x + 532 pp, $86.75 (P). [ISBN: 0-444-70190-7] Assuming a background of standard algebraic concepts, the author presents a fairly self-contained and comprehensive look at the current theory of the Jacobson radical of a group algebra. Several introductory chapters are followed by chapters on group algebras for characteristic p, induced modules, Loewy length, nilpotency index, and radicals of blocks. Bibliography, survey of further results. JS

Algebra, S(18), P. *Malcev-Admissible Algebras.* Hyo Chul Myung. Prog. in Math., V. 64. Birkhauser Boston, 1986, xvi + 353 pp, $55. [ISBN: 0-8176-3345-6] Assumes familiarity with the standard theory of Lie and Malcev algebras to develop a systematic discussion of Malcev-admissible algebras. Topics include flexible and power-associative algebras, the case where the associated algebra is simple, and that where the radical is not zero. Concludes with a classification theory for algebras up to dimension eight over an algebraically closed field. Index, bibliography. JS

Algebra, T(17-18: 1), S, P. *Commutative Ring Theory.* Hideyuki Matsumura. Transl: M. Reid. Cambridge U Pr, 1986, xiii + 320 pp, $49.50. [ISBN: 0-521-25916-9] An English version of the original 1980 Japanese book, this is a fairly self-contained introduction to the central concepts of commutative rings, giving special prominence to work of Krull, I.S. Cohen, and Serre. Some applications, exercises (with hints and solutions), appendices, extensive bibliography. JS

Algebra, P. *Buchsbaum Rings and Applications: An Interaction Between Algebra, Geometry and Topology.* Jürgen Stückrad, Wolfgang Vogel. Springer-Verlag, 1986, 286 pp, $65.50. [ISBN: 0-387-16844-3] An exposition of the work of the authors and others on Buchsbaum rings. Connections are exhibited between commutative algebra, algebraic geometry, combinatorics, and topology. A thorough reference for researchers. SG

Algebra, P. *Representations of Algebras: Proceedings of the Durham Symposium 1985.* Ed: P. Webb. London Math. Soc. Lect. Note Ser., V. 116. Cambridge U Pr, 1986, 199 pp, $29.95 (P). [ISBN: 0-521-31288-4] Seven expository lectures given at a 1985 Durham symposium on representations of algebras. Diagrammatic or "quiver" representation theory, including almost split sequences, theory of tubes, tiling functors, hammocks, triangulated categories; representation type of local rings of singularities; modular representation theory of finite groups. Designed to be self-contained surveys of cutting edge research. RB

Calculus, T*(13-14: 2, 3), S, L. *Calculus and Analytic Geometry, Fourth Edition.* Sherman K. Stein. McGraw-Hill, 1987, xx + 1061 pp, $45.95. [ISBN: 0-07-061159-9] Covers the standard syllabus, in unusually intuitive, student-directed, "user-friendly" style—e.g., margins contain countless notes to student readers. Many useful exercises bridge the gap between routine drill and rigorous proofs—e.g., series of exercises on issue of integrability in closed form. Many graphs are in sections, fewer in exercise sets. Slightly shortened from earlier editions. (*First Edition,* TR, November 1973; Extended Review, February 1976; *Second Edition,* TR, May 1977; *Third Edition,* TR, October 1982.) PZ

Calculus, T(13: 4). *Calculus.* James Stewart. Brooks/Cole, 1986, xxx + 1084 pp, $50. [ISBN: 0-534-06690-9] Good, solid mainstream text with emphasis on problem solving techniques. Thoroughly worked examples with some attempts at explaining heuristics. Rigorous proofs of most theorems, but could be omitted at instructors discretion. MR

Calculus, T(13: 1, 2), S, L. *Top-down Calculus: A Concise Course.* S. Gill Williamson. Comput. & Math. Ser., V. 11. Computer Science Pr, 1987, xv + 429 pp, $25.95. [ISBN: 0-88175-072-7] A short, intuitive, very informal treatment of calculus of algebraic and transcendental functions. "Top-down" (a phrase from computer programming) describes expository style—topics appear first intuitively and graphically, later in more detail. Graphical viewpoint is stressed throughout, e.g., in motivation for chain rule, fundamental theorem. Much standard material is omitted, e.g., most proofs, mean value theorem, partial derivatives, any review of trigonometric functions. BASIC programs are used (e.g., to tabulate function values), but numerical methods (e.g., trapezoid rule, Newton's method) are hardly mentioned. PZ

Real Analysis, T(14-15: 2). *Introductory Analysis: The Theory of Calculus.* J.A. Fridy. Harcourt Brace Jovanovich, 1987, xiii + 354 pp, $36.95. [ISBN: 0-15-501845-0] "Theoretical development of the calculus concepts that are presented intuitively in the typical freshman-sophomore calculus course." Covers sequences, series, continuity, differentiation, Riemann integration, power series, some multivariable calculus, and introduces metric spaces. Somewhat tersely written. BH

Complex Analysis, T(18: 1), S, P. *Lectures on Quasiconformal Mappings.* Lars V. Ahlfors. Math. Ser. Wadsworth, 1987, 146 pp, $24.95 (P). [ISBN: 0-534-08118-5] A reprint of the 1966 original edition (TR, August-September 1967). A brief,

brisk introduction to theory—especially geometric theory—of quasiconformal mappings. (Quasiconformality involves a positive parameter—the complex dilatation—which measures deviation from conformality.) Advanced level; intuitive style. No exercises, few references. PZ

Complex Analysis, P. *Complex Analytic Singularities.* Ed: T. Suwa, P. Wagreich. Adv. Stud. in Pure. Math., V. 8. Elsevier Science, 1987, ii + 697 pp, Dfl. 330. [ISBN: 0-444-70200-8] Papers from a seminar held at Tsukuba University, July 16-20, 1984. 30 papers on the topic. MR

Complex Analysis, T(15-16: 1), L.** *Invitation to Complex Analysis.* Ralph Philip Boas. Math. Ser. Birkhauser Boston, 1987, xii + 348 pp, $26. [ISBN: 0-394-35076-6] A "primer of complex analysis" in the clear, elementary Boas style. Cauchy's theorem comes quickly in simple form. Later chapters treat analytic continuation, harmonic functions, conformal mappings, Riemann surfaces, and univalent functions. Extensive solutions to exercises occupy 20% of the volume. LAS

Differential Equations, T(14-15: 1, 2), S, L. *Differential Equations: A Modeling Approach.* Robert L. Borrelli, Courtney S. Coleman. Prentice-Hall, 1987, xvii + 684 pp, $40. [ISBN: 0-13-211533-6-01] Solid treatment of ordinary differential equations. Prerequisite: mostly single variable calculus. Linear algebra ideas developed as needed. Setting book apart is emphasis on modeling, especially on growth processes, motion of mechanical systems, and electrical circuits. Suitable for year or semester course. KK

Differential Equations, S(17), P, L. *Third Order Linear Differential Equations.* Michal Greguš. Math. & Its Applic. D Reidel, 1986, xv + 270 pp, $59. [ISBN: 90-277-2193-9] Concerns mainly the theory of solutions of third order equations in normal form: fundamental properties, oscillatory behavior, asymptotic properties, boundary value problems. Also surveys results about such equations when linear, homogeneous, and with continuous coefficients. A short chapter on applications of the linear theory. Intended for a general audience. DFA

Differential Equations, P. *Differential Equations: Qualitative Theory.* Ed: B. Sz.-Nagy, L. Hatvani. Elsevier Science, 1987, $160 set [ISBN: 0-444-70093-5]. *Volume I,* 589 pp; *Volume II,* 579 pp. Expanded versions (in English) of papers and lectures given at an August 1984 colloquium in Szeged, Hungary. Most address asymptotic properties and stability problems of solutions of ordinary and functional differential equations; others concern bifurcation of solutions; boundary value problems; periodic solutions; applications in mechanics, physics, and biology. DFA

Differential Equations, P. *From Local Times to Global Geometry, Control and Physics.* Ed: K.D. Elworthy. Pitman Res. Notes in Math., V. 150. Longman Scientific & Technical (US Distr: Wiley), 1986, 344 pp, $49.95. [ISBN: 0-470-20785-X] A collection of articles reporting work carried out during the 1984/85 symposium on Stochastic Differential Equations and Applications held at Warwick University. Topics covered include classical and quantum probability theory, differential analysis in infinite dimensions, potential theory, stochastic control, index theorems, stochastic mechanics, analysis on manifolds, and Lorentz geometry. AM

Differential Equations, T(14: 1). *Elementary Differential Equations.* David L. Powers. Prindle, Weber & Schmidt, 1987, vii + 592 pp. [ISBN: 0-87150-093-0] For engineering and science students. First-order equations, second-order linear equations, power series methods, Laplace transform, numerical methods, linear systems, boundary value problems. An appendix contains the necessary matrix algebra. Emphasis throughout is on motivation, methods, and examples. Many exercises, some computer oriented. DFA

Differential Equations, P. *Elements of Superintegrable Systems: Basic Techniques and Results.* B.A. Kupershmidt. Math. & Its Applic. Kluwer Academic, 1987, xvi + 187 pp, $49.50. [ISBN: 90-277-2434-2] Supermathematics refers to Z_2-graded versions of algebra, analysis, and geometry where the even part reduces to ordinary algebra, analysis, and geometry. In theoretical physics, these ideas relate (Fermi) matter to (Bose) force. The first half of the text is devoted to construction and analysis of the super Lax equations, and description of the Lagrangian and Hamiltonian formalism in the presence of super variables. The second half discusses relations between Lie algebras and integrable systems, both super and even. AM

Partial Differential Equations, P. *Soliton Mathematics.* Alan C. Newell, *et al.* Pr U Montreal, 1986, 116 pp, $18 (P). [ISBN: 2-7606-0782-8] Lectures given at the University of Montreal in 1985 on recent work in soliton theory. Four basic topics: soliton equations and Kac-Moody-Lie algebras; the Hirota method; gauge and Bäcklund transformations; and the Painlevé method for partial differential equations. Primarily for people already in the field, but well-written for its audience. BC

Partial Differential Equations, P. *Infinitesimal Symmetries: A Computational Approach.* P.H.M. Kersten. CWI Tract, V. 34. Math Centrum, 1987, iii + 155 pp, Dfl. 24.20 (P). [ISBN: 90-6196-314-1] Infinitesimal symmetries are employed in the study of nonlinear differential equations. Following a short survey of the mathematical background of this subject, this book presents software used to calculate infinitesimal symmetries using the symbolic language REDUCE. This software is used in the latter part of the book in the computation of infinitesimal symmetries of the partial differential equations of math-

ematical physics. Contains listings of source code in LISP. AM

Partial Differential Equations, T(16-17). *Introduction to Partial Differential Equations with Applications.* E.C. Zachmanoglou, Dale W. Thoe. Dover, 1986, x + 405 pp, $9.95 (P). [ISBN: 0-486-65251-3] An unabridged, corrected republication of a work first published in 1976 (TR, February 1977). AM

Partial Differential Equations, P*. *Pseudodifferential Operators and Spectral Theory.* M.A. Shubin. Transl: Stig I. Andersson. Springer-Verlag, 1987, x + 278 pp, $55. [ISBN: 0-387-13621-5] Pseudodifferential operators (PDO's) arise as tools in the study of differential operators. For example, the inverse of an elliptic differential operator will not be another differential operator, but will fall within the class of PDO's. This book presents an introduction to PDO's and applies them to the spectral theory of elliptic operators (one of the topics studied is the asymptotic behavior of eigenvalues). The author has tried to make the book accessible to students familiar with the theory of distributions. It also includes exercises and a good bibliography. AM

Partial Differential Equations, P. *A Unified Theory of Nonlinear Operator and Evolution Equations with Applications: A New Approach to Nonlinear Partial Differential Equations.* Mieczyslaw Altman. Pure & Appl. Math., V. 103. Dekker, 1986, xiv + 292 pp, $69.75. [ISBN: 0-8247-7613-5] Presents an extension of the Nash-Moser technique for solving nonlinear operator equations. Also presents a new theory of solving nonlinear evolution equations via the study of linearized evolution equations (which are better understood). The book contains no background material, but the author states that a first course in functional analysis is a sufficient prerequisite for understanding the main thrust of the book. AM

Partial Differential Equations, P. *Singular Integral Operators.* Solomon G. Mikhlin, Siegfried Prössdorf. Springer-Verlag, 1986, 528 pp, $49. [ISBN: 0-387-15967-3] A presentation of the fundamentals of singular integral operators and their applications using the methods of functional analysis. Topics covered include one-dimensional equations with discontinuous coefficients, singular integro-differential equations, a generalization of the Zygmund inequality to higher dimensions, the exact correlation of the differentiability properties between symbol and characteristic, etc. Many of the results presented are known only from journals; some are published for the first time. Assumes familiarity with analysis and functional analysis. AM

Numerical Analysis, P. *Lecture Notes in Mathematics-1230: Numerical Analysis.* Ed: J.P. Hennart. Springer-Verlag, 1986, x + 234 pp, $19.40 (P). [ISBN: 0-387-17200-9] Eighteen papers (in English) from the fourth workshop on numerical analysis held by the Institute for Research in Applied Mathematics and Systems of the National University of Mexico in July 1984. Numerical aspects of optimization; linear algebra; and differential equations, both ordinary and partial. DFA

Numerical Analysis, T(15), L. *Numerical Methods in Engineering and Science.* Carl E. Pearson. Van Nostrand Reinhold, 1986, x + 214 pp, $29.50. [ISBN: 0-442-27344-4] A well-written introduction to numerical analysis appropriate for undergraduates in physics or engineering. Emphasis is on applications. Details of derivations are often not explicitly given. Prerequisites include ordinary differential equations, linear algebra, and some complex variables. Algorithms in Fortran 77. Price is right. SM

Numerical Analysis, P. *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods.* J.C. Butcher. Wiley, 1987, xv + 512 pp, $74.95. [ISBN: 0-471-91046-5] Theoretical introduction to numerical methods for ordinary differential equations, dealing in particular with Runge-Kutta and general linear methods. Nearly 100 pages of bibliography. LC

Numerical Analysis, P. *Discrete Fourier Transforms and Their Applications.* Václav Čížek. Adam Hilger, 1986, 141 pp, $31. [ISBN: 0-85274-800-0] Basic information on discrete Fourier transforms for engineers and natural scientists. LC

Numerical Analysis, P. *Numerical Quadrature.* Ed: M. Mori, R. Piessens. Elsevier Science, 1987, 236 pp, $55.50. [ISBN: 0-444-701826] A collection of papers reprinted from the *Journal of Computational and Applied Mathematics,* Vol. 17, Nos. 1 and 2, including a translation of the Japanese original paper on the "IMT" rule, a successful schema for dealing with endpoint singularities. LAS

Numerical Analysis, P. *Approximation Theory V.* Ed: C.K. Chui, L.L. Schumaker, J.D. Ward. Academic Pr, 1986, xviii + 654 pp, $65. [ISBN: 0-12-174581-3] Proceedings of the Fifth International Symposium on Approximation Theory held at Texas A&M University on January 13-17, 1986. Nine survey papers (on complex approximation, Padé approximation, multivariate splines, signal processing, bases of Banach spaces, Haar systems, constrained spline interpolation), 98 short research articles, a 42-page bibliography of recent research on Bernstein polynomials. DFA

Numerical Analysis, T(16-17: 1). *Computational Methods in Engineering and Science, With Applications to Fluid Dynamics and Nuclear Systems.* Shoichiro Nakamura. Robert E Krieger, 1986, xii + 457 pp, $48.95. [ISBN: 0-89874-867-4] Reprint of the 1977 Wiley edition (TR, June-July 1978), with corrections. Methods include finite difference, finite

element, statistical (Monte Carlo) types. Studies numerical solution of eigenvalue problems of ordinary differential equations and ellitic, parabolic, and hyperbolic partial differential equations. DFA

Numerical Analysis, T(15-16: 1). *Numerical Methods with Fortran IV Case Studies.* William S. Dorn, Daniel D. McCracken. Robert E Krieger, 1987, xii + 447 pp, $45. [ISBN: 0-89874-982-4] Reprint of the 1972 Wiley edition (TR, February 1973; Extended Review, April 1977). DFA

Functional Analysis, P. *Functional Calculus of Pseudo-Differential Boundary Problems.* Gerd Grubb. Progress in Math., V. 65. Birkhauser Boston, 1986, vii + 511 pp, $49. [ISBN: 0-8176-3349-9] Pseudo-differential operators include differential operators, their solution operators, and certain integro-differential operators. Boundary-value problems arise in the reduction of multi-order systems, in implicit eigenvalue problems, and in singular perturbation problems. The resolvent is used as the basic tool in studying the functional calculus. MR

Analysis, T(18: 1), S, P*. *An Introduction to Tauberian Theory: From Tauber to Wiener.* J. van de Lune. CWI Syllabus, V. 12. Math Centrum, 1986, iii + 102 pp, Dfl. 16.50 (P). [ISBN: 90-6196-309-5] A readable, well-motivated introduction, at the advanced graduate student level, to the history of Tauberian theory, beginning with connections to Abel's theorem, and proceeding through the general Tauberian theorems of Wiener and Pitt. A convincing object lesson in interrelatedness of apparently diverse topics: integration, Fourier and Laplace transforms, Banach algebras, analytic number theory. PZ

Analysis, T*(16-17: 1, 2), S, L. *Power Series from a Computational Point of View.* K.T. Smith. Universitext. Springer-Verlag, 1987, viii + 132 pp, $23 (P). [ISBN: 0-387-96516-5] An unusual, effective, very concrete introduction to theory and applications of power series, and hence to basic analytic function theory. Well motivated—theory arises directly from easily stated numerical problems, e.g., to compute a given real integral with stated accuracy. Much complex analysis, through the monodromy theorem, is shown convincingly in action. PZ

Analysis, P. *Foundations of Analysis Over Surreal Number Fields.* Norman L. Alling. Math. Stud., V. 141. Elsevier Science, 1987, xvi + 373 pp, $66.75 (P). [ISBN: 0-444-70226-1] Lays the basis for analysis in subfields of Conway's field of surreal numbers (so christened by Knuth). The author investigates set theoretic, topological, algebraic, and analytic properties of these fields. A nicely-written, part expository, part research treatise. SG

Analysis, P. *Algebraic D-Modules.* A. Borel, *et al.* Perspect. in Math., V. 2. Academic Pr, 1987, xii + 355 pp, $29.95. [ISBN: 0-12-117740-8] "D-module is shorthand for a sheaf of modules over the sheaf of rings of germs of differential operators on a manifold, on algebraisation of the notion of a system of linear partial differential equations." This theory, developed extensively during the last decade, has important ramifications not only in partial differential equations but also in group representations. This volume provides a coherent survey of one part of the theory—for algebraic D-modules—in a series of related chapters by Borel, Malgrange, and others. LAS

Analysis, T(17), S.** *The Functions of Mathematical Physics.* Harry Hochstadt. Dover, 1986, xi + 322 pp, $8.95 (P). [ISBN: 0-486-65214-9] Wonderful coverage of historical interplay between pure and applied mathematics of late 18th and 19th century (a subject almost unmentioned in current graduate education). Includes orthogonal polynomials, regular and confluent hypergeometric functions, Bessel's function, and (the oft-neglected) Hill's equation. Excellent illustrations of applications in all areas. Unabridged, corrected reissue of 1971 work (TR, November 1971). Valuable source for both mathematics and physics courses. MR

Analysis, P. *A. Haar, Memorial Conference.* Ed: J. Szabados, K. Tandori. Elsevier Science, 1987, $155.50 set [ISBN: 0-444-70095-1]. *Volume I,* 474 pp; *Volume II,* 541 pp. Proceedings of a 1985 Budapest conference on the birth centenary of A. Haar. Contains 75 lectures, mostly on approximation theory, orthogonal series, Haar measure, and applications. Three overview papers assess Haar's mathematical contributions. PZ

Analysis, P. *Topics in Fourier Analysis and Function Spaces.* Hans-Jürgen Schmeisser, Hans Triebel. Wiley, 1987, 300 pp, $44.95. [ISBN: 0-471-90895-9] Studies several classes of Besov-Hardy-Sobolev function spaces on the Euclidean n-space and n-torus. LC

Analysis, P. *Lecture Notes in Mathematics-1249: Nonstandard Asymptotic Analysis.* Imme van den Berg. Springer-Verlag, 1987, ix + 187 pp, $15.80 (P). [ISBN: 0-387-17767-1] Presents nonstandard methods of asymptotic reasoning, some new results, and some nonstandard alternatives to the classical theory of asymptotic expansions. LCL

Algebraic Geometry, P. *Lecture Notes in Mathematics-1246: Hodge Theory.* Ed: E. Cattani, *et al.* Springer-Verlag, 1987, vii + 175 pp, $15.80 (P). [ISBN: 0-387-17743-4] Proceedings of a U.S.-Spain workshop held in Barcelona, June 1985. Fourteen papers on developments during the past 20 years concerning singular and open varieties, families, rational homotopy of varieties; variation of Hodge structure; mixed Hodge theory via cubical hyperresolutions and iterated integrals; L^2-realization of intersection cohomology, associated pure Hodge structures. RB

Differential Geometry, P. *Symplectic Geometry and Analytical Mechanics.* Paulette Libermann, Charles-Michel Marle. Transl: Bertram Eugene

Schwarzbach. Math. & Its Applic. D Reidel, 1987, xvi + 526 pp, $89. [ISBN: 90-277-2438-5] The invariant formulation of analytical mechanics may be discussed within the framework of symplectic geometry. This book covers symplectic vector spaces and vector bundles, symplectic structures on manifolds and their generalizations (e.g., Poisson manifolds), actions of Lie groups on symplectic manifolds, momentum maps, Lie-Poisson structures, and contact manifolds. AM

Geometry, T(13-14: 1), S. *Microcomputers in Geometry.* Adrian Oldknow. Math. & Its Applic. Halsted Pr, 1987, 211 pp, $69.95. [ISBN: 0-470-20805-8] Tries to show how geometric ideas can be investigated using computer programs on micros. Text is a mixture of Basic code and the geometry it is used to illustrate. Geometric topics: circle, line, curves, transformations, splines, approximations, surfaces, solids. Bibliography, index. RJA

Geometry, S(13). *Taxicab Geometry: An Adventure in Non-Euclidean Geometry.* Eugene F. Krause. Dover, 1986, viii + 88 pp, $3.95 (P). [ISBN: 0-486-25202-7] An easy to understand presentation which would be a valuable supplement for advanced high school or undergraduate students; an unabridged and corrected republication of the 1975 Addison-Wesley book (TR, November 1975). JNC

Geometry, T(16: 1), P, L. *Geometry.* Marcel Berger. Universitext. Springer-Verlag, 1987, $39 (P) each. *I,* xiii + 428 pp [ISBN: 0-387-11658-3]; *II,* x + 406 pp. [ISBN: 0-387-17015-4] A group theoretic treatment emphasizing the visual aspects of Euclidean, affine, and projective geometry. Translated from French. Based in part on the author's experience in preparing the geometry part of an exam to select the best high school teachers in France. JNC

Geometry, S(17-18), P. *Lecture Notes in Mathematics-1181: Buildings and the Geometry of Diagrams.* Ed: L.A. Rosati. Springer-Verlag, 1986, vii + 277 pp, $21.30 (P). [ISBN: 0-387-16466-9] Eight lengthy papers based on notes from courses given at the 1984 CIME session on "Buildings and the Geometry of Diagrams." The courses dealt with the principal aspects of this branch of geometry including its applications to group theory. SS

Geometry, T(13: 1), S, P. *The Non-Euclidean Revolution.* Richard J. Trudeau. Birkhauser Boston, 1987, xiii + 269 pp, $39. [ISBN: 0-8176-3311-1] A presentation of both Euclid's original work and non-Euclidean geometry interwoven with a non-technical description of the revolution in mathematics which resulted from non-Euclidean geometry. The pleasant conversational style is marred by the statement-reason format of the proofs. JNC

Geometry, T(15-16). *Orthogonality and Spacetime Geometry.* Robert Goldblatt. Universitext. Springer-Verlag, 1987, ix + 189 pp, $26 (P). [ISBN: 0-387-96519-X] A study of the geometric notion of orthogonality and its use in establishing a metric structure in affine geometry. Focuses on geometries having lines that are self orthogonal—as in the Minkowski space of special relativity; uses many of the constructions of projective geometry. Intended to be self-contained, requiring only an elementary knowledge of linear and abstract algebra. It should be accessible to undergraduates, however, there are no exercises. AM

Algebraic Topology, P. *Characteristic Classes and the Cohomology of Finite Groups.* C.B. Thomas. Stud. in Adv. Math., V. 9. Cambridge U Pr, 1986, xii + 129 pp, $29.95. [ISBN: 0-521-25661-5] An exposition of the work of the author and others on the cohomology, representations, and characteristic classes of discrete groups. Applications are given to linear groups over rings of algebraic integers and over finite fields, the symmetric group, and groups of p-rank at most 2. SG

Topology, T(16-17: 1, 2), S, L. *Topological and Uniform Spaces.* I.M. James. Undergrad. Texts in Math. Springer-Verlag, 1987, ix + 163 pp, $36. [ISBN: 0-387-96466-5] A brisk but readable text for beginning graduate or very well-prepared undergraduate students. In three main sections: basics of general topology; theory of uniform spaces; miscellaneous topics. A preliminary chapter covers rudiments of set theory, especially filters. Notable features: early, careful treatment of compactness; emphasis on continuous functions throughout. Relatively few exercises. PZ

Topology, P. *Papers on Group Theory and Topology.* Max Dehn. Transl: John Stillwell. Springer-Verlag, 1987, 396 pp, $34. [ISBN: 0-387-96416-9] Max Dehn, quiet giant of early twentieth-century topology and combinatorial group theory, contributed ideas which remain central in contemporary work, notably Thurston's. Stillwell has translated five major papers and three important unpublished works, supplied summary and historical introductions of each, and provided an appendix on Dehn's prior proof of Nielsen's theorem. RB

Topology, P. *Geometry and Topology: Manifolds, Varieties, and Knots.* Ed: Clint McCrory, Theodore Shifrin. Lect. Notes in Pure & Appl. Math., V. 105. Dekker, 1987, viii + 349 pp, $69.75 (P). [ISBN: 0-8247-7621-6] 24 papers submitted for the 1985 Georgia topology conference held at the University of Georgia, August 5-16, 1985. Includes papers on gauge theory and smooth structures on four-manifolds (S.K. Donaldson), knot theory, three-manifolds, group actions, algebraic varieties. AM

Topology, T(16-17: 1), S. *Topologie I: Topologische Räume.* Horst Herrlich. Heldermann Verlag, 1986, vii + 314 pp, $30 (P). [ISBN: 3-88538-102-8] Compactly written, modern text with many prob-

lems. Coverage fairly standard. Second volume will treat uniform spaces. JD-B

Topology, S*(15-17), P, L*. *A Topological Picturebook.* George K. Francis. Springer-Verlag, 1987, xv + 194 pp, $33. [ISBN: 0-387-96426-6] A fascinating collection of hand-drawn pictures supported by "topological stories" gathered from "expert friends" whose work the author has helped illustrate. Saddles, knots, catastrophe surfaces, braids, spheres, fibrations and bottles illustrate diverse aspects of low dimensional topology. An excellent resource to strengthen the faculty of spatial imagination. LAS

Topology, T(17-18: 1), P. *Decompositions of Manifolds.* Robert J. Daverman. Pure & Appl. Math. Academic Pr, 1986, xi + 317 pp, $55. [ISBN: 0-12-204220-4] A decomposition of a manifold M is a partition of that manifold into a collection G of disjoint sets. A fundamental problem in the theory of manifolds is to determine sufficient conditions which guarantee that M will be topologically equivalent to the quotient space, M/G. This book is intended as an introduction for students interested in geometric topology, but should also serve more mature readers seeking background in this area. Contains a proof of R.G. Edwards' Cell-like Approximation Theorem, a central result of the subject whose complete proof has never been published. AM

Operations Research, T(15-17: 1), S, L. *Introduction to Mathematical Programming.* N.K. Kwak, Marc J. Schniederjans. Robert E Krieger, 1987, xii + 356 pp, $36.50. [ISBN: 0-89874-710-4] Optimization techniques for business and MBA students. Linear programming (non-linear, dynamic, integer, goal); transportation and assignment problems; duality and sensitivity analysis; network models. Theory and general principles developed by carefully prepared examples, illustrations, and exercises. College algebra is the only mathematical prerequisite. LCL

Operations Research, T(16-17: 1), P, L. *Multiple Criteria Optimization: Theory, Computation, and Application.* Ralph E. Steuer. Wiley, 1986, xx + 546 pp, $39.95. [ISBN: 0-471-88846-X] A text suitable for a one-term course in multiple objective linear programming. Given a feasible set defined by a family of linear constraints and several linear objective functions, the problem is to find all feasible points where no objective function can be improved without detracting from some other. Extensive bibliography, many exercises. Simplex and parametric programming briefly reviewed. SM

Operations Research, T(14-16: 1). *Decision-Making Models in Production & Operations Management.* Michael Ballot. Robert E Krieger, 1986, xi + 295 pp, $28.50. [ISBN: 0-89874-825-9] Descriptions and case-study explanations for the major mathematical and statistical models used in management (e.g., linear programming, regression, scheduling, and inventory models). Minimal theory; no exercises. LCL

Optimization, P. *Lecture Notes in Control and Information Sciences-81: Stochastic Optimization.* Ed: V.I. Arkin, A. Shiraev, R. Wets. Springer-Verlag, 1986, x + 754 pp, $70.50 (P). [ISBN: 0-387-16659-9] Proceedings of a 1984 international conference held in Kiev: 78 papers on controlled stochastic processes, stochastic extremal problems, and stochastic optimization with incomplete information. LAS

Optimization, T(15-16), S, C, P, L. *Optimization Using Personal Computers With Applications to Electrical Networks.* IBM PC. Thomas R. Cuthbert, Jr. Wiley, 1987, xvi + 474 pp, $44.95. [ISBN: 0-471-81863-1] An introduction to algorithms for linear and nonlinear optimization featuring complete programs provided (on IBM-PC disk and in an appendix) in BASICA. Builds on Strang's LDU approach to linear algebra; includes Gauss-Newton and quasi-Newton methods. Concludes with a capstone chapter applying these methods to optimize ladder networks. LAS

Dynamical Systems, T(17), S, P. *Probabilistic Properties of Deterministic Systems.* Andrzej Lasota, Michael C. Mackey. Cambridge U Pr, 1985, x + 358 pp, $49.50. [ISBN: 0-521-30248-X] The state of a deterministic system, after many state transitions, can be represented by a probability density. The "uncertainty" arises from not knowing the precise number of transitions or the precise initial conditions of the system. This very readable book describes the construction and analysis of these densities. Knowledge of advanced calculus and differential equations is assumed. No exercises. SM

Dynamical Systems, T(17-18). *Stability Theory: An Introduction to the Stability of Dynamic Systems and Rigid Bodies, Second Edition.* Horst Leipholz. Wiley, 1987, ix + 359 pp, $62.95. [ISBN: 0-471-91181-X] *Second Edition* updates and revises work originally published in 1972 (TR, August-September 1972). In particular, the material on elastomechanics has been completely rewritten, and exercises have been added at the end of each chapter. AM

Control Theory, P. *Fourth IMA International Conference on Control Theory.* Ed: P.A. Cook. Academic Pr, 1985, xvi + 447 pp, $69. [ISBN: 0-12-187260-2] Contains 34 theoretical papers presented at the Fourth Institute of Mathematics and Its Applications Conference on Control Theory held at Robinson College, Cambridge, September 11-13, 1984. Major topics include linear systems, time delays and distributive parameters, optimal control, nonlinear systems, and nine papers on uncertainty and robustness. SM

Control Theory, T(16-17: 1, 2), S, L. *Foundations of Optimal Control Theory.* E.B. Lee, L. Markus. Robert E Krieger, 1986, x + 576 pp, $64.95. [ISBN: 0-89874-807-0] A slightly revised reprinting of

the 1967 Wiley text (TR, April 1968). Basic concepts of control theory, via the qualitative theory of deterministic differential systems. Definition-theorem-proof format, interspersed with examples, and exercises at the end of sections. Assumes advanced calculus and differential equations. BC

Systems Theory, P. *Almost Invariant Subspaces and High Gain Feedback.* H.L. Trentelman. CWI Tract, V. 29. Math Centrum, 1986, iv + 239 pp, Dfl. 51.70 (P). [ISBN: 90-6196-308-7] The theory presented here falls within an area of research commonly known as "the geometric approach" to linear systems theory (e.g., *Linear Multivariable Control: A Geometric Approach* by W.M. Wonham, Springer-Verlag, 1979). This work constitutes much of the content of the author's doctoral dissertation at the University of Groningen, The Netherlands, 1985. LCL

Systems Theory, P. *Lecture Notes in Control and Information Sciences-83: Analysis and Optimization of Systems.* Ed: A. Bensoussan, J.L. Lions. Springer-Verlag, 1986, xiv + 901 pp, $82.80 (P). [ISBN: 0-387-16729-3] Proceedings of an international conference, the seventh in a series, held June 25-27, 1986 in Antibes. 76 papers in French and English on optimization, adaptive control, nonlinear systems, stochastic systems, linear systems, singular perturbations, control theory, and filtering. LAS

Probability, P. *Wahrscheinlichkeitsrechnung, Statistik und mathematische Grundlagen: Begriffe, Definitionen und Formeln.* Erich Härtter. Vandenhoeck & Ruprecht, 1987, xxv + 675 pp, DM 98,00. [ISBN: 3-525-40731-9] A handbook containing an enormous amount of information about probability, statistics, and the mathematical foundations of the two. Definitions, theorems, tables, but no proofs. JD-B

Probability, T(13-15: 1), S. *Probability, An Introduction.* Samuel Goldberg. Dover, 1987, xiv + 322 pp, $7.95 (P). [ISBN: 0-486-65252-1] An unaltered republication of the 1960 Prentice-Hall edition—a pioneering introduction to the theory of probability, often imitated but still distinctive. Features extensive introductions to many applications, often in special lengthy problem sets. LAS

Statistics, S(16-17). *Applied Statistics: A Handbook of BMDP Analyses.* E.J. Snell. Chapman & Hall, 1987, ix + 171 pp, (P). [ISBN: 0-412-28410-3] Complement to Cox and Snell's 1981 book *Applied Statistics: Principles and Examples* (TR, April 1982). Gives instructions for using BMDP programs (1985 release) to analyze the 24 examples from that book, together with sample output. RSK

Statistics, T*(13-17), S, C*. *Understanding Statistics.* IBM PC. Bruce J. Chalmer. Dekker, 1987, x + 432 pp, $39.75. [ISBN: 0-8247-7322-5] An inviting introductory text for students with minimal algebraic skills featuring careful exposition that is verbal and conceptual rather than computational. Intended to be used with the statistical package *Statpal* (TR, April 1987), the text comes with a *Statpal* (Version 5.0) IBM PC disk in the front cover, and concludes with a 100-page appendix containing the entire *Statpal* manual. Topics are standard: first two-thirds on univariate methods including hypothesis testing; final third covers correlation and analysis of variance. LAS

Statistics, P. *Cyclic Designs.* J.A. John. Mono. on Stat. & Appl. Prob. Chapman & Hall, 1987, x + 232 pp, $32.50. [ISBN: 0-412-28240-2] Studies families of block and row-column designs which are constructed using cyclical methods. Aimed at post-graduate level students. Assumes familiarity with basic principles in experimentation, with factorial experimentation and with some standard designs. LC

Statistics, P. *Proceedings of the Thirty-Second Conference on the Design of Experiments.* US Army Research Office (PO Box 12211, Research Triangle Park, NC), 1986, xiv + 408 pp, (P). Proceedings of a conference held at Fort Ord in October 1986. Features papers from a special session on "Field Experimentation: The Analysis of Messy Data." LAS

Statistics, P*. *Multivariate Statistical Simulation.* Mark E. Johnson. Wiley, 1987, ix + 230 pp, $34.95. [ISBN: 0-471-82290-6] In the Wiley Series in Probability and Mathematical Statistics. Deals with computer generation of continuous multivariate probability distributions. Generation algorithms are presented together with many interesting three-dimensional and contour plots which illustrate distributional properties. Good set of references. RSK

Elementary Computer Science, T(13: 1). *Algorithms, Programming, Pascal.* Barbara LiSanti, Lydia Mann, Fred Zlotnick. Wadsworth, 1987, xix + 716 pp, $23.50 (P). [ISBN: 0-534-06678-X] Covers the curriculum recommended by the ACM Committee Task Force for CS1, as well as some material recommended for CS2. Provides a disciplined approach to problem solving and algorithm development, followed by the details of programming in Pascal language. LCL

Elementary Computer Science, T(13: 1, 2). *Introduction to Computers and Programming: Pascal.* Peter P. Smith. Wadsworth, 1987, xviii + 765 pp, $23.50 (P). [ISBN: 0-534-07194-5] Intended as a text for the CS1/CS2 sequence; may not have quite enough material. Data structures (arrays, linked lists, queues, stacks, binary trees, heaps) are studied only in the second half. Consistent top-down approach. Early introduction of procedures and functions. Full chapters on program development, debugging, recursion. Many examples, exercises, problems. Very readable. DFA

Elementary Computer Science, T(13-14: 1, 2), S, L.** *Computer Science, An Overview.* J.

Glenn Brookshear. Ser. in Comput. Sci. Benjamin/Cummings, 1985, xiii + 448 pp, $24.95 (P). [ISBN: 0-8053-0900-4] A textbook presenting a broad overview of computer science: machine architecture, operating systems, algorithms, programming languages, software engineering, data structures, file structures, databases, artificial intelligence, theory of computation. Suitable for second course (Pascal supplement fulfilling CS2 is available) or sophisticated literacy course; considerable supplement for two-term introductory sequence. RB

Elementary Computer Science, T*(13). *Building Pascal Programs: An Introduction to Computer Science.* Stuart Reges. Little Brown, 1987, xx + 667 pp, $30.75 (P). [ISBN: 0-316-73854-9] An introductory text on structured computer programming and the language Pascal intended for the CS1 course outlined by the ACM. Emphasizes a software-engineering approach to programming as opposed to a problem-solving approach. To this end, procedures are introduced in the first chapter to encourage the student to write programs whose parts are flexible enough to be used again for other problems. Includes discussions of pointers, recursion, binary trees, searching, and sorting. Chapter exercises. AM

Elementary Computer Science, T(13-14: 1, 2). *PASCAL: Programming and Problem Solving, Second Edition.* Sanford Leestma, Larry Nyhoff. Macmillan, 1987, xiii + 682 pp, $22.50 (P). [ISBN: 0-02-369690-7] Revisions of *First Edition* (TR, January 1985) include use of a second color for highlighting, more examples and exercises from non-mathematical applications, introducing procedures before functions, and expanding discussion of topics such as recursion, data structure design, algorithm analysis, and structured data types. LC

Programming, T?(14-17), S, P. *Software Components with Ada: Structures, Tools, and Subsystems.* Grady Booch. Ser. in Ada & Software Engin. Benjamin/Cummings, 1987, xx + 635 pp, $35.95. [ISBN: 0-8053-0610-2] Nice catalogue of reuseable software components in Ada, built using careful software engineering and object-oriented design principles. Much complete code, discussion of design and development, issues of structuring large systems, complexity analyses. Includes structures (lists, trees, graphs, many more), tools and utilities, design principles for large systems. RM

Programming, T*(13-18: 1, 2), S, L. *Programming in Prolog, Third, Revised and Extended Edition.* W.F. Clocksin, C.S. Mellish. Springer-Verlag, 1987, xiv + 281 pp, $19.95 (P). [ISBN: 0-387-17539-3] This *Third Edition* adds new material on use of accumulators and difference structures, on syntax errors, and operator precedences. (*Second Edition*, TR, February 1986.) RJA

Programming, S, P*. *68000 Assembly Language Programming, Second Edition.* Lance A. Leventhal, *et al.* Osborne McGraw-Hill, 1986, xi + 484 pp, (P). [ISBN: 0-07-881232-1] An introduction to assembly language programming using MC68000 microprocessors family (68000 to 68020; Motorola assembler syntax). Presumes no assembler experience, but sidesteps system calls (how do you print?). Focuses on examples, exercises; clearly indicates 68020 features; 106-page section on proper software development principles; amplified reference for each instruction. Well done. RB

Programming, T(13: 1). *Essentials of Structured BASIC.* Roy Ageloff, Richard Mojena. Ser. in Comput. Inform. Sys. Wadsworth, 1987, xiv + 210 pp, $20.25 (P). [ISBN: 0-534-06810-3] Text for a first, short course in BASIC, written in a friendly manner. Extensive use of color shading and margin notes to highlight points, ample exercises and sample programs. LC

Programming, T(13: 1), S. *BASIC on the IBM PC.* Jeffrey Bonar. Harcourt Brace Jovanovich, 1987, viii + 264 pp, $11.50 (P). [ISBN: 0-15-504918-6] Introduction to Basic on the IBM PC. Contains discussions of if-then statements, loops, functions, subroutines, and IBM Basic Graphics commands. Suitable for self-study. AM

Programming, T(13: 1). *Structured Programming in COBOL.* Robert R. Boettcher. Holt, Rinehart & Winston, 1987, xvi + 558 pp, (P). [ISBN: 0-03-070559-2] For the first course. Compatible with 1974 and 1985 standard versions of COBOL. Gets to programming quickly. Many programming examples and exercises. Language reference section. DFA

Programming, P. *VAX Architecture Reference Manual.* Ed: Timothy E. Leonard. Digital, 1987, xii + 417 pp. [ISBN: 0-932376-86-X] The VAX computer family, produced by Digital Equipment Corporation, includes over a dozen computers ranging from powerful stand-alone workstations to multiprocessor super-minicomputers supporting hundreds of timesharing users. All share the same (machine language level) architecture or design incorporating virtual memory principles. This work is the ultimate, authoritative reference for VAX architecture. RB

Programming, P. *Prolog Multiprocessors.* Michael J. Wise. Prentice-Hall, 1987, xii + 168 pp, $26.67. [ISBN: 0-13-730755-1] A monograph on the development of a variant (EPILOG) of Prolog suitable for parallel architectures related to the data-flow model of computation (as opposed to traditional von Neumann architectures). Surveys of data-flow model, Prolog concepts; problems in data-flow model solved by Prolog-like software; EPILOG description, specifications; proposed architectures; experimental results. RB

Programming, T?(13: 1), S. *Modula-2 Made Easy.* Herbert Schildt. Osborne McGraw-Hill, 1986,

xvii + 377 pp, (P). [ISBN: 0-07-881241-0] Simple introduction to Modula-2. Describes standard procedures; discusses differences with Pascal. Readable introduction, with many simple code examples; suitable for self-study; too few exercises. RM

Programming, S(14-17), P, L. *A Little Smalltalk.* Timothy Budd. Addison-Wesley, 1987, xv + 280 pp, (P). [ISBN: 0-201-10698-1] The Smalltalk-80 object-oriented programming system requires special hardware, e.g., mouse, graphics. The author's Little Smalltalk interpreter (1984, in C) is a version of Smalltalk-80 language for Unix systems, conventional terminals (sacrificing interface). This handbook presents Little Smalltalk for general computer science students, details implementation for more advanced readers. Software is public domain. RB

Languages, P. *Lecture Notes in Computer Science-249 & 250: TAPSOFT '87.* Ed: Hartmut Ehrig, *et al.* Springer-Verlag, 1987, $25 each (P). *249: Volume 1,* xiv + 289 pp [ISBN: 0-387-17660-8]; *250: Volume 2,* xiv + 336 pp. [ISBN: 0-387-17611-X] Contains papers from the Advanced Seminar on Foundations of Innovative Software Development, from the Colloquium on Trees in Algebra and Programming, and from the Colloquium on Functional and Logic Programming all held in Pisa, Italy from March 23-27, 1987. Author index. RJA

Languages, T(13-14: 1), S. *Logic Tools for Programming.* Philip and Larry Pace. Delmar, 1987, xi + 349 pp, (P). [ISBN: 0-8273-2582-7] Text built around seven logic tools used in computer industry: flowcharts, pseudocode, hierarchy charts, IPO charts, Nassi-Shneiderman charts, Warnier-Orr diagrams, structure charts. Six nonstandard application problems. Chapter summaries, review questions, exercises. Appendices on arrays, files, sorting; index. RJA

Languages, P. *Object-Oriented Concurrent Programming.* Ed: Akinori Yonezawa, Mario Tokoro. Ser. in Comput. Syst. MIT Pr, 1987, vii + 282 pp, $25. [ISBN: 0-262-24026-2] A collection of nine research papers on issues in object-oriented concurrent programming, in the contexts of specific languages (e.g., Actor, ABCL/1, modified Smalltalk 80), together with an introductory outline defining the field and describing relevant issues. Philosophy of Actor model; object-oriented problem solving schemes; distributed computing; knowledge representation; a music composition application. RB

Languages, T(16-17), S, P*, L. *A Practical Introduction to Denotational Semantics.* Lloyd Allison. Computer Science Texts, V. 23. Cambridge U Pr, 1987, xii + 132 pp, $10.95 (P), $34.50. [ISBN: 0-521-31423-2; 0-521-30689-2] Denotational semantics is a formal method for defining meaning in programming languages based on functions, i.e., sets of ordered pairs. (Compare λ-calculus, functional programming, Lisp; contrast predicate/axiomatic approaches, logic programming, Prolog.) This text stresses practical work. Requires programming experience (e.g., Pascal); compiler design, functional programming helpful. Attention: mathematicians investigating computer science. RB

Languages, T?, P, L. *Object-Oriented Programming: An Evolutionary Approach.* Brad J. Cox. Addison-Wesley, 1986, xiii + 274 pp. [ISBN: 0-201-10393-1] An introduction to object-oriented "programming in the large" (or system-building programming) through the Objective-C language (Productivity Products International). Modularity, reusability needs in software system building; survey of object-oriented programming (Smalltalk-80, Ada, C++); Objective-C features; Objective-C libraries; application to building iconic user interfaces; further applications. Valuable, but tastes somewhat of sales pitch. RB

Algorithms, P. *Lecture Notes in Computer Science-251: Unobstructed Shortest Paths in Polyhedral Environments.* Varol Akman. Springer-Verlag, 1987, 103 pp, $15 (P). [ISBN: 0-387-17629-2] Presents algebraic and geometric algorithms to solve a shortest path problem on polyhedra in Euclidean three-space. Has significance for artificial intelligence and robotics. References. RJA

Algorithms, T(14-18: 1, 2), S. *Applied Data Structures Using Pascal.* Guy J. Hale, Richard J. Easton. DC Heath, 1987, xiv + 588 pp, $29.50. [ISBN: 0-669-07579-5] Emphasizes applications and "real-life" examples. Many Pascal programs, figures, diagrams, tables. Extensive use of pointers. Contains all standard data structures with all implementation details in Pascal. Chapter summaries and exercises. Appendices. Answers to selected exercises. Index. RJA

Algorithms, T(14-15: 1), S, L. *File Structures Using Pascal.* Nancy E. Miller. Ser. in Comput. Sci. Benjamin/ Cummings, 1987, xiv + 487 pp, $29.95. [ISBN: 0-8053-7082-X] Textbook for course in file organization and processing (ACM CS5) using Pascal. Rationale: students learn more about involved data structures from Pascal than from a language (e.g., COBOL, PL/I) with more built-in support for file organizations (including direct files, ISAM). Prerequisites: two terms of Pascal, e.g., CS1/2. Case studies, exercises, sizeable glossary. RB

Algorithms, T(15-16: 1). *File Structures: A Conceptual Toolkit.* Michael J. Folk, Bill Zoellick. Addison-Wesley, 1987, xxi + 538 pp, $32.95. [ISBN: 0-201-12003-8] Covers concepts and constraints of designing file structures (material outlined as Course CS-5 in the ACM Curriculum '78). Features programs written in pseudocode, Pascal, and C. LC

Computer Systems, T(14: 1), S, P. *UNIX System Administration.* Frank Burke. Harcourt Brace

Jovanovich, 1987, xvi + 175 pp, (P). [ISBN: 0-15-593025-7] Textbook for a Unix system administration course: a source of practical procedures supplemented by basic concepts concerning login administration, RS232 communications, file systems, process management. Collected administrative procedures: general operations; security; resource monitoring; system configuration planning; system generation; *uucp* networking; upgrading software, hardware. Exercises. RB

Computer Systems, P. *Lecture Notes in Computer Science-244: Advanced Programming Environments.* Ed: Reidar Conradi, Tor M. Didriksen, Dag H. Wanvik. Springer-Verlag, 1986, vii + 604 pp, $49 (P). [ISBN: 0-387-17189-4] Proceedings, in a strong sense, of an international workshop held at Trondheim, Norway, June 1986: papers supplemented by transcripts of the ensuing discussions. 33 papers: specialized editors (e.g., context-sensitive); system level environment architectures; version control; tool integration; software engineering databases; program reuse; knowledge-based environments. RB

Computer Systems, P. *Parallel Processing: The CM* Experience.* Edward F. Gehringer, Daniel P. Siewiorek, Zary Segall. Digital, 1987, xiii + 454 pp. [ISBN: 0-932376-91-6] Final report of the ten-year CM* project (1975-85) conducted at Carnegie-Mellon University, which combined 50 DEC LSI-11 microcomputers and 100 man-years labor to explore dozens of significant questions in parallel computing research. Description, analysis of hardware architecture, two operating systems, several programming environments developed for project. Overview, specific reports on experiments performed. RB

Computer Systems, T(17: 1, 2), P. *Decision Support Systems: Tools and Techniques.* Sitansu S. Mittra. Wiley, 1986, xviii + 433 pp, $35.95. [ISBN: 0-471-81641-8] This book serves computer science students interested in decision support systems (DSS's) and system professionals who want to build one. It "is a hard-core technical book" requiring at least a working knowledge of linear algebra and elementary statistics. SM

Computer Systems, T(14-15: 1), L. *Logical Introduction to Databases.* John Grant. Harcourt Brace Jovanovich, 1987, xi + 466 pp, $28. [ISBN: 0-15-551175-0] A textbook on databases with an eye towards new developments: applications of logic, Fifth Generation project, microcomputer-oriented systems. Standard database models (network, hierarchical, relational), design, components and implementation; application of first-order logic (and extensions), Prolog to databases; some specific systems. Languages, e.g., SQL, expressed in Backus-Naur Form. Exercises; pointers to the literature. RB

Computer Systems, P*. *RSX, A Guide for Users.* John F. Pieper. Digital, 1987, 366 pp, (P). [ISBN: 0-932376-90-8] An introduction/supplement to official RSX documentation offered as a complete reference for casual users, developed from an in-house training manual (engineering research environment) written by a (formerly) casual user. Technicalities, complexities, advanced features omitted in favor of clear, readable prose explaining basic capabilities users may need in straightforward applications. RB

Computer Systems, S*(14-17), P*, L*. *Professional Software.* Henry Ledgard, John Tauer. Addison-Wesley, 1987, (P). *Volume I: Software Engineering Concepts,* xv + 218 pp [ISBN: 0-201-12231-6]; *Volume II: Programming Practice,* xvii + 219 pp. [ISBN: 0-201-12232-4] An established author/consultant's viewpoints on the process of writing large programs in teams (*Volume I*) and the craftsmanship of quality programs (*Volume II*). Software lifecycle (vs. prototyping), programming teams and team dynamics, human factors considerations, software decomposition, empirical methods. Programming in modular packages, global variables, comments, naming, layout, programs which run right the third (not 300th) time. Case study developed in *Volume I,* reviewed in *Volume II.* Practical issues, separating professional programmers from amateurs and less. RB

Computer Graphics, P. *Three-Dimensional Computer Vision.* Yoshiaki Shirai. Ser. in Symbolic Comput. Springer-Verlag, 1987, xii + 297 pp, $98. [ISBN: 0-387-15119-2] A sizeable monograph reporting on some current techniques in three-dimensional computer vision, by an experienced Japanese researcher. Basic methods of computer vision; interpreting two-dimensional line drawings as 3-D scenes; stereo computer vision; shape interpretations of monocular images; techniques for processing, describing 3-D scenes, including knowledge representation. Hundreds of enriching illustrations. RB

Theory of Computation, S(16-17), P. *The Semantics of Destructive Lisp.* Ian A. Mason. Lect. Notes, No. 5. Center for the Study of Language & Information (Stanford U), 1986, ii + 282 pp, $14.95 (P); $29.95. [ISBN: 0-937073-06-7; 0-937073-05-9] Uses notion of memory structure as model to produce formal semantics (neatly separating control from data) for variant of LISP with destructive functions (e.g., Rplaca, Rplacd). Major theme is duality between program verification and program derivation/transformation; new inferential programming paradigm of Verification = Transformation + Induction. Many verification examples, simple and complex. RM

Artificial Intelligence, P. *Proceedings: Sixth Canadian Conference on Artificial Intelligence.* Pr U Quebec (US Distr: Morgan Kaufmann), 1986, 268 pp, (P). [ISBN: 2-7605-0409-3] Topics: learning, natural language understanding, formal reasoning, logic programming, recognition and perception, computer

vision, expert systems, knowledge representation, applications. Author index. RJA

Artificial Intelligence, P. *Model-Based Image Matching Using Location.* Henry S. Baird. ACM Dist. Dissert., 1984. MIT Pr, 1985, xix + 105 pp, $25. [ISBN: 0-262-02220-6] Humans recognize rigid visual patterns even when distorted by translation, rotation, scaling. Computer vision researchers have studied planar pattern recognition under such distortions. This award-winning dissertation presents an efficient pruned tree-search algorithm for recognizing patterns with small errors in location, recasting problem into linear inequalities. Method extends to higher dimensions, attributes other than location. RB

Artificial Intelligence, P. *Human and Machine Vision II.* Ed: Azriel Rosenfeld. Perspect. in Comput., V. 13. Academic Pr, 1986, x + 364 pp, $35. [ISBN: 0-12-597345-4] Fourteen papers, most presented at the Second Workshop on Human and Machine Vision, Montreal, August 1984, concerning visual perception and computer vision. Perception of transparency, spatial layout, surface orientation, organization; description of surfaces; preattentive processing in human vision; impact of parallelism. RB

Computer Science, T(17-18: 1), S*, P. *Microprocessor Logic Design: The Flowchart Method.* Nick Tredennick. Digital, 1987, ix + 369 pp. [ISBN: 0-932376-92-4] Knowing logic design principles, how would you actually design a microprocessor? This graduate-level textbook presents "flowchart method" of microprocessor design used by the author, a member of Motorola MC68000, IBM Micro/370 design teams. Micro/370 design process is detailed illustrative example; balance struck between academic methods, practitioner's solutions. Delightful tell-it-like-it-is style; memorable introductory section bears general reading. RB

Computer Science, T(17-18: 1), S, P. *Temporal Logic of Programs.* Fred Kröger. EATCS Mono. on Theor. Comput. Sci., V. 8. Springer-Verlag, 1987, viii + 148 pp, $39. [ISBN: 0-387-17030-8] Temporal logic, "a logic of propositions whose truth and falsity may depend on time," provides a basis for mathematical study of program execution sequences via formal logic mechanisms. This monograph could serve as graduate text, after prerequisite mathematical (classical propositional, first order) logic. "Linear time" temporal logic; temporal semantics of parallel programs; applications. RB

Computer Science, P. *The Papers of the Eighteenth SIGCSE Technical Symposium on Computer Science Education.* Ed: A.K. Rigler, D.C. St. Clair. ACM (ACM Order Dept., PO Box 64145, Baltimore, MD 21264), 1987, xvii + 541 pp, (P). [ISBN: 0-89791-217-9] Proceedings of the 1987 annual technical symposium of the ACM special interest group in computer science education held February 19-20 in St. Louis. LAS

Computer Science, T(13-16: 1, 2), L. *Computer Organization and Assembly Language Programming for the VAX.* G. Michael Schneider, Ronald Davis, Thomas Mertz. Wiley, 1987, xviii + 684 pp, $34.95. [ISBN: 0-471-83850-0] Split about equally between general principles of architecture and system software on the one hand, and the VMS MACRO assembler for the VAX family. The book is designed to be the text for a variety of courses with different emphasis. Well done, readable, and worthy of serious consideration for any course in this area that wants to avoid digital logic or extensive work on operating systems and I/O. Especially solid on data representation, von Neumann architecture, and VAX assembler. Reasonable index, no bibliography. JAS

Computer Science, P. *Lecture Notes in Computer Science-247: STACS 87.* Ed: F.J. Brandenburg, G. Vidal-Naquet, M. Wirsing. Springer-Verlag, 1987, x + 484 pp, $33.60 (P). [ISBN: 0-387-17219-X] Papers submitted to the Fourth Annual Symposium on Theoretical Aspects of computer science. Subject groupings: algorithms; complexity; formal languages; abstract data types; rewriting systems; denotational semantics; semantics of parallelism; net theory; fairness; distributed algorithms; system demonstrations. Author index. RJA

Computer Science, T, L. *Local Networks, Second Edition.* William Stallings. Macmillan, 1987, xiv + 434 pp. [ISBN: 0-02-415520-9] A text for an advanced undergraduate course or a reference for the professional, this expository volume treats the computer science material lying between electrical engineering and the applications of local networks. This edition gives additional information on various now complete IEEE standards and includes new material on fiber optics. The emphasis is on hardware and software protocols. However, except for basic computer science knowledge (through about CS4 of the ACM 1978 curriculum), the book is self-contained. Problems and an extensive bibliography. JAS

Applications, P. *Foundations and Applications of Montague Grammar, Part 2: Applications to Natural Language.* T.M.V. Janssen. CWI Tract, No. 28. Math Centrum, 1986, v + 237 pp, Dfl. 36.80 (P). [ISBN: 90-6196-306-0] Work draws on several fields: mathematics, philosophy, computer science, logic, and linguistics. Topics in present volume include the *PTQ*-fragment, variants, derivations, partial rules, constituent structures, relative clause formation, ambiguities. Appendices; index; references. RJA

Applications, P, L. *Real-Time Control of Walking.* Marc D. Donner. Progress in Comput. Sci., V. 7. Birkhauser Boston, 1987, xv + 160 pp, $29. [ISBN: 0-8176-3332-4] How can robots walk? The author studied animal walking, proposed a decompo-

sition of the walking task based on insect walking, designed a concurrent programming language (OWL) for real-time walking algorithms, wrote a program causing the SSA six-legged robot to walk, and conducted evaluative experiments, for this, his dissertation. Program listing included. RB

Applications (Communication Theory), P. *Error-Control Techniques for Digital Communication.* Arnold M. Michelson, Allen H. Levesque. Wiley, 1985, xix + 463 pp, $38.95. [ISBN: 0-471-88074-4] This work is primarily a reference book for a communications systems engineer knowledgeable in modern communications techniques. Using a minimal amount of sophisticated mathematics, the author seeks to provide guidance as to when and which error control techniques should be used. SM

Applications (Engineering), T(16). *Linear Systems Analysis.* A.N. Tripathi. Wiley, 1987, xii + 320 pp, $19.95. [ISBN: 0-470-20354-4] Covers basic principles and techniques for modelling, analysis, and simulation of linear dynamic systems. First two chapters introduce a variety of models (e.g., automobile suspension, biomedical system) and their classification. Later chapters develop techniques needed for analysis. Many in-depth exercises. MR

Applications (Engineering), P*. *Image Recovery: Theory and Application.* Ed: Henry Stark. Academic Pr, 1987, xix + 543 pp, $75. [ISBN: 0-12-663940-X] From the preface: "Is this a permanent defect of humans—to want more knowledge than it is possible to get? ... Image recovery is a mathematically intensive activity; it requires more than basic calculus and Fourier transforms—the staple of engineers educated not many years ago. It is deeply rooted in functional analysis, linear algebra, and analytic function theory." BC

Applications (Engineering), P. *Mathematics of Random Phenomena: Random Vibrations of Mechanical Structures.* Paul Krée, Christian Soize. Math. & Its Applic. D Reidel, 1986, xv + 438 pp, $98. [ISBN: 90-277-2355-9] Deterministic models (e.g., differential equations) are often totally inadequate for yielding accurate results for certain types of forces (e.g., the action of wind on structures, or earthquakes, sea waves, or random loading). These are better modeled by random functions. This work, in three parts, presents (a) the theoretical underpinnings and essential probabilistic ideas; (b) some applications; (c) further theoretical considerations. LCL

Applications (Fluid Mechanics), T*(15-17: 1), L.** *An Informal Introduction to Theoretical Fluid Mechanics.* James Lighthill. IMA Mono. Ser., V. 2. Clarendon Pr, 1986, xi + 260 pp, $35. [ISBN: 0-19-853631-3] Discursive yet properly mathematical, this slim volume offers a lucid introduction to fluid mechanics, and to the "efficient cooperation between theory and experiment" required for progress in this increasingly important field. Builds understanding with well-chosen, well-explained, and well-illustrated examples. Exercises at end of volume make it useful as an introductory text. LAS

Applications (Physics), P*. *Quantum Physics: A Functional Integral Point of View, Second Edition.* James Glimm, Arthur Jaffe. Springer-Verlag, 1987, xxii + 535 pp, $57. [ISBN: 0-387-96476-2] A mathematical structure of quantum theory and statistical mechanics whose central theme is the quantization of nonlinear partial differential equations and the physics of systems with an infinite number of degrees of freedom. This *Second Edition* includes new chapters on correlation inequalities, the cluster expansion, and nonabelian gauge theories. AM

Applications (Physics), P. *Introduction to Supersymmetry.* Peter G.O. Freund. Cambridge U Pr, 1986, x + 152 pp, $34.50. [ISBN: 0-521-26880-X] In theoretical physics, the concept of supersymmetry relates (Fermi) matter to (Bose) force. This book presents an introductory description of the mathematical and physical ideas underlying supersymmetry, specifically, Lie superalgebras, supergroups, superspace, supersymmetric field theories, and supergravity. AM

Applications (Physics), T(18: 1, 2), P*. *Superstring Theory.* Michael B. Green, John H. Schwarz, Edward Witten. Mono. on Math. Physics. Cambridge U Pr, 1986. *Volume 1: Introduction,* x + 469 pp, $39.50 [ISBN: 0-521-32384-3]; *Volume 2: Loop Amplitudes, Anomalies and Phenomenology,* xii + 596 pp, $49.50. [ISBN: 0-521-32999-X] A systematic, pedagogical exposition of the present state of knowledge about string theory. String theory involves a mathematical structure incorporating Riemann surfaces, modular forms, infinite dimensional Lie algebras, and a generalization of Riemannian geometry, and has recently been proposed (notably by Witten) in physicists' attempts to reconcile quantum mechanics with gravity. *Volume 1* requires particle physics and quantum field theory, and is a self-contained, detailed introduction to basic ideas. *Volume 2* studies advanced topics, and includes substantial mathematical background on differential and algebraic geometry. Suitable as texts for graduate-level course(s). Likely to become a classic reference. RB

Applications (Physics), P. *Renormalized Supersymmetry.* Olivier Piguet, Klaus Sibold. Prog. in Physics, V. 12. Birkhauser Boston, 1986, xv + 346 pp, $41. [ISBN: 0-8176-3346-4] A systematic summary of supersymmetry (renormalized perturbation theory only), aimed at a search for hard and soft anomalies. Primarily for the specialist who is already initiated into the arcana of quantum field theories. BC

Applications (Physics), P. *Ellipsoidal Figures of*

Equilibrium. S. Chandrasekhar. Dover, 1987, xi + 255 pp, $7.95 (P). [ISBN: 0-486-65258-0] Republication with partial revision of a Nobel laureate's 1969 book expanding upon his 1963 Sillman Lectures: a survey of classical work in the rotation of astronomical bodies. Historical introduction; the virial equations; potentials of homogeneous, heterogeneous ellipsoids; Dirichlet's problem, Dedekind's theorem; Maclaurin spheroids; ellipsoids of Jacobi, Dedekind, Riemann, Roche. RB

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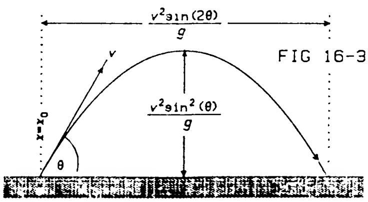

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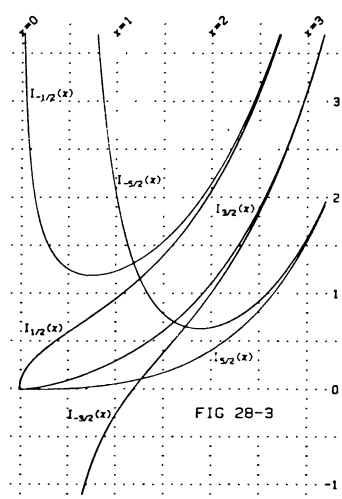

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Advertising Correspondence: Ms. ELAINE PEDREIRA, Advertising Manager, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Subscription correspondence, changes of address, and inquiries about nondelivered or defective issues: Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, DC 20036.

Back Issues: P. and H. BLISS Co., Middletown, CT 06457.

Microfilm Editions: University Microfilms International, Serial Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

The AMERICAN MATHEMATICAL MONTHLY (ISSN 0002-9890) is published monthly except bimonthly June-July and August-September by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, DC 20036 and Montpelier, VT.

The annual subscription price of $24 for the AMERICAN MATHEMATICAL MONTHLY to an individual member of the Association is included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $24. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $70 per year.

Second class postage paid at Washington, DC, and additional mailing offices.

Postmaster: Send address changes to the AMERICAN MATHEMATICAL MONTHLY, Membership/Subscriptions Department, Mathematical Assocation of America, 1529 Eighteenth Street, N.W., Washington, DC 20077-9564.

PRINTED IN THE UNITED STATES OF AMERICA

An Introduction to the Ising Model

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Introduction

This article is an invitation, or advertisement, for readers to work on a problem which is apparently very difficult, yet certainly extremely important. The problem is known generically as the *Ising model*, named after Ernst Ising, who did the first work on it in the early 1920s. Although unpromising in its initial results, the Ising model has turned out to be an exceptionally rich idea. The number of papers written on the subject is staggering; the number which remain to be written is conceivably even more staggering.

The Ising model is concerned with the physics of phase transitions, which occur when a small change in a parameter such as temperature or pressure causes a large-scale, *qualitative* change in the state of a system. Phase transitions are common in physics and familiar in everyday life: we see one, for instance, whenever the temperature drops below 32°F, and another whenever we put a kettle of water on the stove. Other examples include the formation of binary alloys and the phenomenon of ferromagnetism. The latter is also of interest historically: an understanding of ferromagnetism—and especially "spontaneous magnetization"—was the original purpose of the Ising model and the subject of Ising's doctoral dissertation. Partly for this historical significance, we shall use ferromagnetism as a reference point later on for interpreting various features of the model.

In spite of their familiarity, phase transitions are not well understood. One purpose of the Ising model is to explain how short-range interactions between, say, molecules in a crystal give rise to long-range, correlative behavior, and to predict in some sense the potential for a phase transition. The Ising model has also been applied to problems in chemistry, molecular biology, and other areas where "cooperative" behavior of large systems is studied. These applications are possible because the Ising model can be formulated as a *mathematical* problem. Although we shall refer frequently to the physics of ferromagnetism and use language from statistical mechanics, it is the mathematical aspects of the model which will concern us in this article. In particular we shall see that the Ising model has a combinatorial interpretation which is powerful enough in itself to establish some of the basic results concerning phase transitions. There are many other approaches and aspects to the

Ising model, but the combinatorial one makes an especially suitable introduction to the subject.

1. Lattices and the Partition Function

Our starting point for the Ising model is a *lattice*, which for us will be a finite set of regularly spaced points in a space of dimension $d = 1, 2$, or 3. In dimension 1 we simply have a string of points on a line, which we can enumerate from 1 to N ("N" will always denote the number of lattice sites, regardless of dimension):

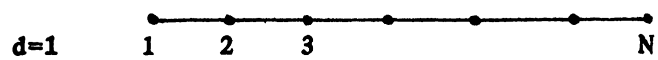

FIG. 1.

In dimension 2 we shall consider the lattice of squares as below:

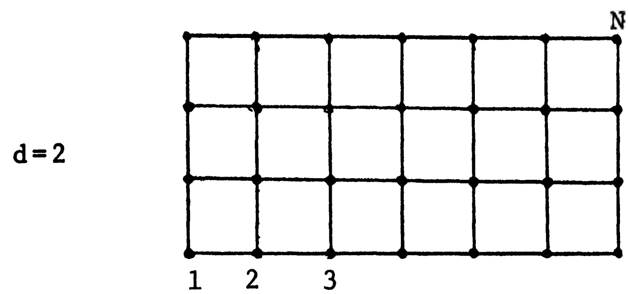

FIG. 2.

In dimension 3 we shall consider the lattice whose repeating units are cubes:

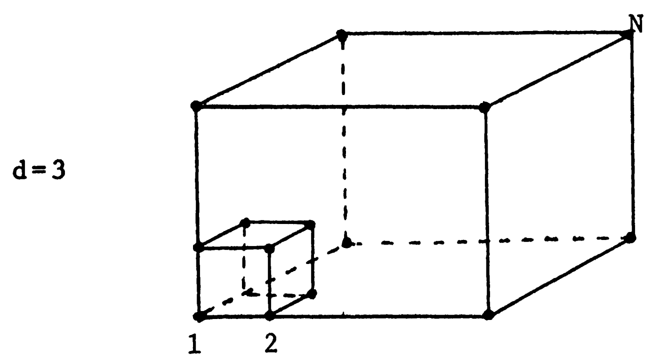

FIG. 3.

In our pictures, each line segment between lattice sites is called a *bond*, and lattice sites are called *nearest neighbors* if there is a bond connecting them. In general, except for lattice sites on the "boundary" of the lattice, each lattice site in a d-dimensional lattice has $2d$ nearest neighbors:

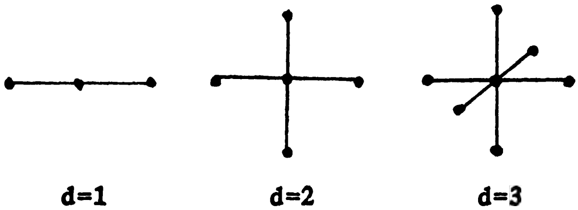

FIG. 4.

The difference between lattice sites on the "boundary" and those in the "interior" of the lattice is mildly annoying. One way to deal with this annoyance is to get rid of the boundary by adopting what might be called a "wrap-around" model: we simply introduce extra bonds connecting lattice sites on opposite sides of the boundary. This amounts to wrapping the one-dimensional lattice into a necklace, the two-dimensional lattice into a doughnut, and the three-dimensional lattice into who knows what.

Although this introduces physically unrealistic long-range interactions (or else requires us to bend a three-dimensional crystal in an impossible manner), physicists will be the first to go along with the idea: intuitively, an extra condition imposed on a "negligibly small" percentage of lattice sites should not affect the overall behavior of the system. (There is an important alternative which we shall mention again later: one can impose "boundary conditions" which *do* influence the behavior of the system, by establishing a preferred direction for the spontaneous magnetization.) Eliminating the boundary also introduces an appealing symmetry into the problem: in the wrap-around Ising model, there are dN bonds connecting the N lattice sites, and every lattice site "looks like" every other lattice site. Therefore we shall henceforth assume that the lattice has been wrapped around.

Our first step is to assign an independent variable σ_i to each lattice site $i = 1, \ldots, N$. The variables σ_i take on only two values, $\sigma_i = \pm 1$, which we shall call the two possible *states* of the lattice site. This reflects the physical assumption that only two possibilities exist at each lattice site, such as up/down or occupied/vacant, as we shall explain below. An assignment of values $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ to each lattice site is called a *configuration* of the system. An essential ingredient in the Ising model will be a *sum* over all possible configurations. Since there are 2^N configurations, this sum clearly has an enormous number of terms if N is at all large. For a macroscopic crystal, with $N \sim 10^{23}$, one should not even contemplate carrying out such a calculation numerically!

In the model of ferromagnetism—Ising's original study—one thinks of the lattice sites as being occupied by atoms of a magnetic material. Each atom has a magnetic moment which is allowed to point either "up" or "down." In a model for binary alloys, the lattice sites are again occupied by atoms, which may be one or the other of the two constituents of the alloy. A third interpretation has the paradoxical name "lattice gas": the lattice sites are points in space which are either occupied or vacant. (The sought-for phase transition here is between a "solid," which has segregated regions of occupied and vacant space, and a "gas" for which the lattice is a homogeneous mixture of the two.) In all cases, the variable σ_i is used to designate which state the ith lattice site is in. Of course one of the many generalizations of the model is to increase the number of states, say to 1,0, and -1, or to a continuum of states.

We next form what is called the *Hamiltonian* of the system. In mathematical physics, the Hamiltonian is the total energy of a system, and it governs the dynamics. For the Ising model, the Hamiltonian is defined after an ideal and apparently very severe assumption is made: we assume that only short-range, "nearest-neighbor" interactions and interactions of the lattice sites with an "external field" contribute to the energy level of the system. For each configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ we have

$$H = H(\sigma) = -\sum_{\langle i,j\rangle} E\sigma_i\sigma_j - \sum_i J\sigma_i, \tag{1.1}$$

where E and J are parameters, the second sum is over all lattice sites, and the first sum is over all pairs of nearest neighbors in the lattice. The parameters E and J correspond to the "energies" associated with nearest-neighbor interactions and interactions with the external field, respectively. For a ferromagnet, E is positive, so that a "magnetized" configuration (with most nearest-neighbor pairs having parallel moments, $\sigma_i = \sigma_j$) has a lower energy level than a non-magnetized configuration. The parameter J corresponds to the presence of an "external magnetic field", which will tend to line up the magnetic moments in the direction of the field, again "favoring" configurations with lower energy levels. Fighting against this, as we shall see below, is thermal agitation. At sufficiently low temperatures, there is not much random motion, and configurations lined up with an external field are highly favored, while at sufficiently high temperatures, the random thermal motion destroys much of the effect of the field.

Partly for its historic interest, let us now explain the nature of the ferromagnetic phase transition which Ising originally sought in his dissertation. The phase transition occurs with the appearance of what is called *spontaneous magnetization*.

Suppose a lattice of magnetic material is placed in a magnetic field and held at a constant temperature. The field will induce a certain amount of magnetization into the lattice—i.e., it will create a tendency for the magnetic moments to point in, say, the "up" direction. The amount of magnetization depends on the strength of the external field and on the (constant) temperature.

Now suppose the external field is slowly turned off. What happens to the lattice? Not surprisingly, for high temperatures, the lattice returns to an unmagnetized condition. But for low temperatures, the lattice retains a degree of magnetism; there is a non-negligible residual tendency for the moments to stay in the "up" position. This is called *spontaneous magnetization*. (Note: it seems that "residual magnetization" would have been a better term, but so be it.) There is a *critical temperature* at which spontaneous magnetization begins to appear, and this is where the phase transition occurs. The figure below shows an (idealized) graph of induced magnetization versus external field strength for three temperatures, including the critical temperature. The curve for the critical temperature is characterized by its having a vertical tangent line at the origin.

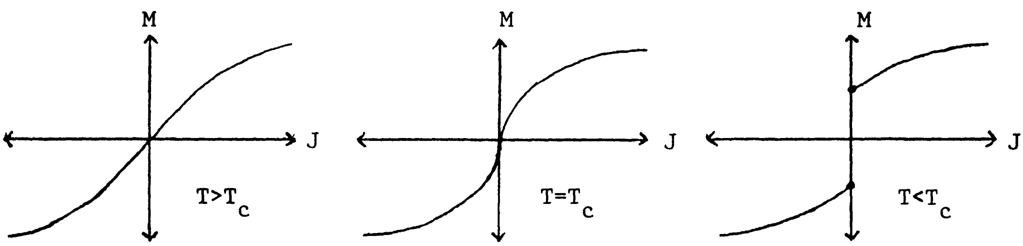

FIG. 5.

As we shall show later, the one-dimensional Ising model does not exhibit a phase transition at any temperature. This negative result, plus some arguments that the same thing would happen in three dimensions, discouraged Ising from pursuing the subject. The Ising model lay dormant for about a decade, until Rudolf Peierls [**37**] in 1936 showed by a very simple argument that, in two dimensions, a phase transition was *guaranteed for some temperature*. In 1941, Hendrick Kramers and Gregory Wannier [**26**] located the phase transition precisely for the two-dimensional model, under the assumption that there is a unique such value. In 1944, Lars Onsager [**36**] gave a complete solution to the two-dimensional Ising model in the "zero-field" ($J = 0$) case. To date, no one has solved any three-dimensional model.

Returning to the Ising model, our third step brings us to the central object in statistical mechanics: the partition function. This is formed by exponentiating the Hamiltonian and then summing over all configurations, which here involves 2^N possible assignments of ± 1 to the N variables $\sigma_1, \dots, \sigma_N$:

$$Z = Z(\beta, E, J, N) = \sum_{\pm 1} e^{-\beta H(\sigma)}. \tag{1.2}$$

The parameter β cancels whatever dimensions the Hamiltonian may have. In statistical mechanics, we typically have $\beta = 1/\mathrm{kT}$, where k is Boltzmann's constant and T is temperature (in absolute degrees).

A simple example may clarify some of the notation. Let's take a very small one-dimensional lattice, consisting of $N = 3$ lattice sites with no wrap-around:

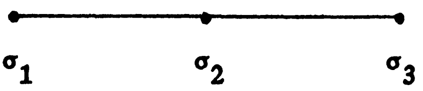

FIG. 6.

The Hamiltonian is

$$H = -E(\sigma_1\sigma_2 + \sigma_2\sigma_3) - J(\sigma_1 + \sigma_2 + \sigma_3).$$

To simplify matters further, we shall set $J = 0$ (the "zero-field" case). The partition function is now

$$\begin{aligned} Z &= e^{-\beta H(1,1,1)} + e^{-\beta H(1,1,-1)} + e^{-\beta H(1,-1,1)} + e^{-\beta H(1,-1,-1)} \\ &\quad + e^{-\beta H(-1,1,1)} + e^{-\beta H(-1,1,-1)} + e^{-\beta H(-1,-1,1)} + e^{-\beta H(-1,-1,-1)} \\ &= e^{\beta E(1+1)} + e^{\beta E(1-1)} + e^{\beta E(-1-1)} + e^{\beta E(-1+1)} \\ &\quad + e^{\beta E(-1+1)} + e^{\beta E(-1-1)} + e^{\beta E(1-1)} + e^{\beta E(1+1)} \\ &= 2e^{2\beta E} + 4 + 2e^{-2\beta E} \\ &= 2^3 \cosh^2 \beta E. \end{aligned}$$

(The final formula in the example is suggestive of what is to come. The reader may want to pause at this point and work out the partition function for the one-dimensional, zero-field model with N lattice sites.)

The partition function plays a fundamental role in statistical mechanics. Essentially, it is the "denominator" in the calculation of probabilities. More precisely, the probability of being in a particular configuration $\sigma = (\sigma_1, \ldots, \sigma_N)$ is given by the formula

$$\operatorname{Prob}(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}. \tag{1.3}$$

The negative sign confers a higher probability on states with lower energy. A small value of β (corresponding to a high temperature, since $\beta = 1/kT$) tends to "flatten out" the distribution, making all configurations more or less equally likely, while a large value of β (corresponding to a low temperature) tends to accentuate the probabilities of the lowest energy states.

From the partition function, one may in principle derive all of the important thermodynamical features of the physical system being modeled: internal energy, specific heat, magnetization and magnetic susceptibility, and so forth. For example, the internal energy is defined as

$$U = \frac{1}{Z} \sum_{\pm 1} H(\sigma) e^{-\beta H(\sigma)}$$

and we easily see that this can be re-expressed as

$$U = -\frac{\partial}{\partial \beta} \log Z.$$

(For more on applications of the partition function, see **[42]** or **[44]**.)

Many of the quantities one computes from the partition function turn out to depend on the logarithm of Z. This is natural, since Z, being a sum over 2^N configurations, tends to grow exponentially with the size of the lattice. This brings us to our last step in setting up the Ising model; we define the "free energy per lattice site" to be

$$F = F(\beta, E, J) = \lim_{N \to \infty} \frac{1}{N} \log Z(\beta, E, J, N). \tag{1.4}$$

The limit as $N \to \infty$ is called the "thermodynamic limit." The main problem of the Ising model is this: *Find a closed-form, analytic expression for the function F.* The idea is that phase transitions will show up as discontinuities in F or in one of its derivatives: a phase transition occurs when some aspect of the system changes radically at certain values of the parameters.

There is, of course, no *a priori* guarantee that the thermodynamic limit F exists. There is also some question as to how the limit is meant to be taken in two or three dimensions, since the lattice can grow at different rates in different directions. We shall not consider these questions any further, but merely assume that the appropriate limits do exist.

For the rest of this article, we shall introduce some elementary steps for analyzing the Ising model and describe what is known about the exact solutions. We shall also present the arguments due to Peierls and to Kramers and Wannier for the existence of phase transitions in two dimensions. The results of these arguments were superseded by Onsager's complete solution (which we do not exposit here), but the techniques and ideas continue to be important. Peierls' argument, in particular, generalizes fairly easily to higher dimensions, where very little else is rigorously known.

2. Elementary Analysis—Some Combinatorics

We shall begin by converting the partition function from transcendental exponentials into a polynomial in two variables with integer coefficients. This is based on the simple observation

$$e^{\pm x} = \cosh x \pm \sinh x = \cosh x(1 \pm \tanh x). \tag{2.1}$$

Since the variables σ_i take on the values ± 1, we have

$$\begin{aligned} Z &= \sum_{\pm 1} e^{\sum_{\langle i,j\rangle} \beta E \sigma_i \sigma_j + \sum_i \beta J \sigma_i} = \sum_{\pm 1} \Big(\prod_{\langle i,j\rangle} e^{\beta E \sigma_i \sigma_j} \Big) \Big(\prod_i e^{\beta J \sigma_i} \Big) \\ &= \sum_{\pm 1} \Big(\prod_{\langle i,j\rangle} \cosh(\beta E)(1 + \sigma_i \sigma_j T) \Big) \Big(\prod_i \cosh(\beta J)(1 + \sigma_i U) \Big) \qquad (2.2) \\ &= (\cosh(\beta E))^B (\cosh(\beta J))^N \sum_{\pm 1} \Big(\prod_{\langle i,j\rangle} (1 + \sigma_i \sigma_j T) \Big) \Big(\prod_i (1 + \sigma_i U) \Big), \end{aligned}$$

where B is the number of bonds, $T = \tanh(\beta E)$ and $U = \tanh(\beta J)$. If we use the "wrap-around" lattice, then $B = dN$ where $d = 1, 2$, or 3 is the dimension of the

model. It is also convenient to make the sum into an *average* over all configurations by introducing a factor 2^N:

$$Z = \left(2\cosh^d(\beta E)\cosh(\beta J)\right)^N \frac{1}{2^N} \sum_{\pm 1} \left(\prod_{\langle i,j \rangle} (1 + \sigma_i \sigma_j T)\right)\left(\prod_i (1 + \sigma_i U)\right). \quad (2.3)$$

The thermodynamic limit is now viewed as consisting of two pieces:

$$F = \lim_{N\to\infty} \frac{1}{N} \log Z = \log\left(2\cosh^d(\beta E)\cosh(\beta J)\right) + \lim_{N\to\infty} \frac{1}{N} \log Z', \quad (2.4)$$

where

$$Z' = \frac{1}{2^N} \sum_{\pm 1} \left(\prod_{\langle i,j \rangle} (1 + \sigma_i \sigma_j T)\right)\left(\prod_i (1 + \sigma_i U)\right). \quad (2.5)$$

The first piece, $\log[2\cosh^d(\beta E)\cosh(\beta J)]$, is always analytic for real (i.e., physical) values of β, E, and J; hence it is a "trivial" contribution exhibiting no discontinuities. We are left with the task of analyzing the "modified" partition function Z'.

Because $\sigma_i^2 = 1$ for all i, we may write

$$\begin{aligned} &\prod_{\langle i,j \rangle} (1 + \sigma_i \sigma_j T) \prod_i (1 + \sigma_i U) \\ &\quad = P(T,U) + \sigma_1 P_1(T, U, \sigma_2, \dots, \sigma_N) \qquad (2.6) \\ &\quad + \sigma_2 P_2(T, U, \sigma_3, \dots, \sigma_N) + \cdots + \sigma_N P_N(T,U) \end{aligned}$$

for polynomials P and $P_1, \dots, P_N$. Note that P is of degree dN in T and N in U, assuming again the wrap-around lattice. When we sum over all configurations, however, each $\sigma_k P_k$ term vanishes by trivial cancellation:

$$\begin{aligned} \sum_{\pm 1} \sigma_k P_k(T, U, \sigma_{k+1}, \dots, \sigma_N) &= \left(\sum_{\sigma_k = \pm 1} \sigma_k \right)\left(\sum_{\pm 1} P_k(T, U, \sigma_{k+1}, \dots, \sigma_N)\right) \\ &= (0)\,(\text{whatever}) = 0. \end{aligned}$$

This leaves the modified partition function

$$Z' = \frac{1}{2^N} \sum_{\pm 1} P(T,U) = P(T,U), \quad (2.7)$$

which is a polynomial in two variables with integer coefficients.

At this point, we shall simplify our discussion by setting $U = 0$. This is called the "zero magnetic field case." In this case the coefficients of the polynomial

$$P(T,0) = 1 + c(1)T + c(2)T^2 + \cdots + c(dN)T^{dN}$$

have a simple combinatorial interpretation. If the lattice is thought of as a graph with lattice sites as the vertices and bonds between nearest neighbors as the edges, then $c(n)$ counts the number of "even" subgraphs with n edges, where "even" means that each vertex has positive, *even* degree. This can be seen by letting the

presence or absence of the bond $\langle i, j\rangle$ in a subgraph correspond to the choice of $\sigma_i\sigma_j T$ or 1 in the expansion of $\prod_{\langle i, j\rangle}(1 + \sigma_i\sigma_j T)$. Each subgraph corresponds to a term in the expansion: $(\prod\sigma_i^{\delta_i})T^n$, where δ_i = degree of vertex i and $n = \frac{1}{2}\Sigma\delta_i$ = number of edges. Only even subgraphs raise each σ_i to an even power, hence only even subgraphs contribute to the modified partition function $Z' = P(T)$ in the zero magnetic field case.

Each connected component of an even subgraph is a closed path in the original lattice. This enables us to solve completely the one-dimensional, zero-field Ising model: in the wrap-around model, there is only one closed path, namely, the complete circuit of length N. Thus $Z' = 1 + T^N$, so that

$$\begin{aligned} F &= \log\left(2\cosh^d(\beta E)\right) + \lim_{N\to\infty}\frac{1}{N}\log(1 + T^N) \\ &= \log\left(2\cosh^d(\beta E)\right), \end{aligned}$$

since $|T| = |\tanh\beta E| < 1$ implies $\lim_{N\to\infty}(1/N)\log(1 + T^N) = 0$. (Note that in the "non-wrap-around" case, there are *no* closed paths, so that $\log(Z') = 0$ directly.)

In dimensions 2 and 3, closed paths obviously do exist, but they must be of even length, unless they are long enough to make use of wrap-around. Also, the shortest paths are of length 4, so we have

$$Z' = 1 + c(4)T^4 + c(6)T^6 + c(8)T^8 + \cdots$$

if the lattice is sufficiently large. For any given n, we can also work out explicitly the coefficient $c(n)$; however this is practical only for small values of n. We shall show this computation (really a counting and bookkeeping argument) for $n = 4$, 6, and 8, and leave $n = 10$, 12, and any higher degrees for the interested (and industrious) reader.

To distinguish between dimensions, let us write

$$Z'_d = 1 + c_d(4)T^4 + c_d(6)T^6 + c_d(8)T^8 + \cdots \tag{2.8}$$

for the modified partition function for the d-dimensional Ising model ($d = 1, 2, 3$). As we pointed out before, $c_1(n) = 0$ for all $n \ll N$. We shall henceforth consider only dimensions $d = 2$ and 3.

An even subgraph with $n = 4$ edges is simply a square. For $d = 2$, the square may be located with a specified (say, lower-left-hand) corner at any lattice site (using again the wrap-around model), so that $c_2(4) = N$. For $d = 3$, we have in addition a choice of orientation, so that $c_3(4) = 3N$.

For $d = 2$, an even subgraph with $n = 6$ edges is a 2×1 rectangle, which can be located at any of the N lattice sites and oriented in two possible ways. Hence $c_2(6) = 2N$. For $d = 3$, in addition to $6N$ "flat" rectangles, there are another $12N$ "bent" rectangles and $4N$ more "twisted" rectangles, for a total $c_3(6) = 22N$.

For $n = 8$, the situation becomes more complicated. For one thing, the subgraphs no longer need to be connected. A disconnected subgraph, however, must consist of two disjoint squares. For $d = 2$, the "first" square may be placed with its lower-

left-hand corner at any of the N lattice sites, while the same corner of the "second" square need only avoid nine lattice sites (see Figure 7).

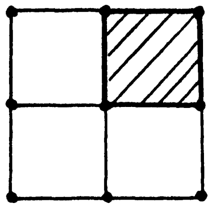

FIG. 7.

Thus for $d = 2$, there are $N(N - 9)/2$ disconnected even subgraphs with 8 edges. (We divide by 2 to eliminate the distinction between the "first" and "second" square.) For $d = 3$, a similar argument shows that there are $3N(3N - 33)/2$ disconnected subgraphs with 8 edges.

The connected paths of length 8 in dimension 2 are easy to count. There are four different types, with a total of 9 orientations, giving, in all, $9N$ connected subgraphs with 8 edges. Thus,

$$c_2(8) = N(N - 9)/2 + 9N = N(N + 9)/2.$$

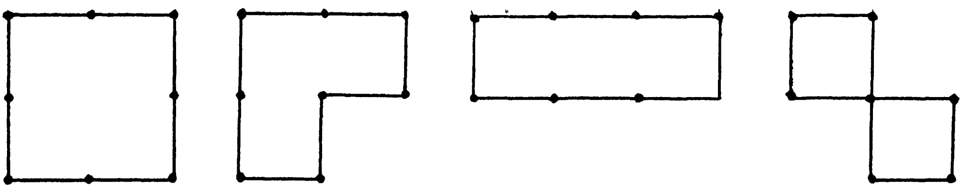

FIG. 8.

The real complication appears for $d = 3$: there are suddenly a lot of different paths of length 8. Classifying them in a manner analogous to the paths of length 6, we have $27N$ $(= 3 \times 9N)$ "flat" graphs, $108N$ graphs with one "bend", $48N$ with two bends, and $48N$ "twisted" graphs, for a total of $231N$ possibilities. Adding in the disconnected subgraphs, we find

$$c_3(8) = 3N(3N - 33)/2 + 231N = (9N^2 + 363N)/2.$$

The reader is invited to look for simpler means of computing these coefficients. This is as far as we shall pursue the matter.

Knowing the first few terms of the partition function allows us to compute corresponding terms in the power series for the thermodynamic limit. We proceed as follows. Since

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \qquad (|x| < 1),$$

we have

$$\begin{aligned}\log(Z') &= \log\left[1 + c(4)T^4 + c(6)T^6 + c(8)T^8 + c(10)T^{10} + \cdots\right]\\ &= \left[c(4)T^4 + c(6)T^6 + c(8)T^8 + c(10)T^{10}\right]\\ &\quad - \frac{1}{2}\left[c(4)T^4 + c(6)T^6\right]^2 + O(T^{12})\\ &= c(4)T^4 + c(6)T^6 + \left[c(8) - \frac{1}{2}c(4)^2\right]T^8\\ &\quad + \left[c(10) - c(4)c(6)\right]T^{10} + O(T^{12}).\end{aligned}$$

From our computations above, we find

$$\frac{1}{N}\log(Z_2') = T^4 + 2T^6 + \frac{9}{2}T^8 + \cdots \tag{2.9}$$

and

$$\frac{1}{N}\log(Z_3') = 3T^4 + 22T^6 + \frac{363}{2}T^8 + \cdots \tag{2.10}$$

Note how the lattice size N has vanished on the right-hand side (at least for the terms we have shown—we expect it to happen for all terms). Taking the limit as $N \to \infty$ gives us a power series expansion for the (modified) free energy F'.

The reader who has had an introductory course in hard analysis should be appalled at what we've just done. In particular, we have not proved the validity of truncating the power series expansion for $\log(1 + x)$ and then letting N tend to infinity. We've also not proved that the N has vanished from all terms on the right-hand side. These objections can be dealt with, however, by taking a formal power series point of view.

That leaves the question of the radius of convergence of the power series as an analytic function around $T = 0$. This is an important question, because it is non-analytic behavior that we look for as the defining characteristic of a phase transition. What we hope will happen is that there will be a phase transition corresponding to some "physical" value of T in the interval $(0, 1)$, and that this will be the closest singularity to the origin. Of course we have no right to think this is what's going to happen. But for $d = 2$ it does.

3. Exact Solutions

To review, we have set $T = \tanh(\beta E)$ and $U = \tanh(\beta J)$, and defined

$$Z_d' = Z_d'(T, U, N) = \frac{1}{2^N}\sum_{\pm 1}\left[\prod_{\langle i, j\rangle}(1 + \sigma_i\sigma_j T)\prod_i(1 + \sigma_i U)\right]$$

for the modified partition function of the d-dimensional Ising model. Let us also

introduce the "modified free energy" function

$$F_d'(T,U) = \lim_{N\to\infty} \frac{1}{N}\log Z_d'(T,U,N)$$

(Recall that the original free energy is

$$F_d = \log\left(2\cosh^d(\beta E)\cosh(\beta J)\right) + F_d'(T,U).)$$

We have seen that Z_d' is a polynomial in T and U with integer coefficients (of degree dN in T and N in U, for the wrap-around model). Assuming that the limit exists, F_d' is, therefore, a power series (at least formally) in T and U, with rational coefficients. If we fix E and J, we may consider T and U as functions of the parameter β. Our objective here is to realize F_d' as an analytic function of β for small β. (Note that T and U are small if β is small.) Remembering that β is inversely proportional to temperature in the physical model, we call the power series in T and U a "high-temperature expansion" for F_d'. Phase transitions occur at the positive real values of β at which F_d' is nonanalytic.

This objective has been met only "halfway." The results are given below, organized according to the dimension of the model and the absence or presence of an "external magnetic field" U.

$$F_1'(T,0) = 0.$$

$$F_1'(T,U) = \log\left[\frac{1+T+\left[(1+T)^2 - 4T(1-U^2)\right]^{1/2}}{2}\right]. \qquad \text{(Ising, 1925)}$$

$$F_2'(T,0) = \frac{1}{2}\int_0^1\int_0^1 \log\left[(T^2+1)^2 - 2T(1-T^2)[\cos(2\pi x)+\cos(2\pi y)]\right]dx\,dy.$$

(Onsager, 1944)

$$F_2'(T,U) = ?$$

$$F_3'(T,0) = ??$$

$$F_3'(T,U) = ???$$

In the next section we shall derive Ising's result for $F_1'(T,U)$. A derivation of Onsager's famous result for $F_2'(T,0)$ is beyond the scope of this article. It has been written up in many forms, and we refer the reader to any or all of **[12]**, **[33]**, and **[42]**. We shall, however, present the beautiful arguments of Peierls **[37]** and Kramers and Wannier **[26]**, which establish the existence of spontaneous magnetization in two dimensions and the precise location of the phase transition for $F_2'(T,0)$ under a mild (and physically reasonable) assumption.

4. Ising's Result—The Transfer Matrix Method

In this section we shall obtain the complete solution to the one-dimensional Ising model. We begin by looking at the "linear" rather than the "wrap-around" model. (As remarked earlier, it should make no difference in the thermodynamic limit

anyway.) Then

$$Z'(N) = \frac{1}{2^N} \sum_{\pm 1} (1 + \sigma_N U) \prod_{i=1}^{N-1} [(1 + \sigma_i \sigma_{i+1} T)(1 + \sigma_i U)]. \tag{4.1}$$

Suppose we write $Z'_+(N)$ for that portion of the partition function summation for which $\sigma_N = +1$, and Z'_- for that portion for which $\sigma_N = -1$. Clearly $Z'(N) = Z'_+(N) + Z'_-(N)$. But also,

$$\begin{aligned} Z'_+(N) &= \frac{1}{2^N} \sum_{\pm 1} (1 + U)(1 + \sigma_{N-1} T)(1 + \sigma_{N-1} U) \\ &\quad \times \prod_{i=1}^{N-2} [(1 + \sigma_i \sigma_{i+1} T)(1 + \sigma_i U)] \qquad (4.2) \\ &= \frac{1}{2}[(1 + U)(1 + T) Z'_+(N-1) + (1 + U)(1 - T) Z'_-(N-1)], \end{aligned}$$

and, likewise,

$$Z'_-(N) = \frac{1}{2}[(1 - U)(1 - T) Z'_+(N-1) + (1 - U)(1 + T) Z'_-(N-1)]. \tag{4.3}$$

We can put this in matrix form:

$$\begin{bmatrix} Z'_+(N) \\ Z'_-(N) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (1 + U)(1 + T) & (1 + U)(1 - T) \\ (1 - U)(1 - T) & (1 - U)(1 + T) \end{bmatrix} \begin{bmatrix} Z'_+(N-1) \\ Z'_-(N-1) \end{bmatrix}. \tag{4.4}$$

Iterating this, and paying careful attention to the initial case $N = 2$, we obtain the formula

$$\begin{bmatrix} Z'_+(N) \\ Z'_-(N) \end{bmatrix} = \frac{1}{2^{N-1}} \begin{bmatrix} (1 + U)(1 + T) & (1 + U)(1 - T) \\ (1 - U)(1 - T) & (1 - U)(1 + T) \end{bmatrix}^{N-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.5}$$

The matrix in these expressions is called the *transfer matrix*. If we denote it by M,

$$M = M(U, T) = \frac{1}{2} \begin{bmatrix} (1 + U)(1 + T) & (1 + U)(1 - T) \\ (1 - U)(1 - T) & (1 - U)(1 + T) \end{bmatrix}, \tag{4.6}$$

then we have

$$Z'(N) = [1 \quad 1] M^{N-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.7}$$

We have derived this expression for $Z'(N)$ using the "linear" model because the derivation is especially simple to explain. However, we now prefer to replace it with the corresponding formula for the "wrap-around" model, but shall leave the derivation of that formula as an exercise for the reader. To distinguish the two

models, we shall write $Z''(N)$ for the wrap-around model in this section:

$$Z''(N) = \frac{1}{2^N} \sum_{\pm 1} \prod_{i=1}^{N} \left[(1 + \sigma_i \sigma_{i+1} T)(1 + \sigma_i U)\right], \tag{4.8}$$

where it is understood that $\sigma_{N+1} = \sigma_1$. The analogue to equation (4.7) is much nicer:

$$Z''(N) = \mathrm{Tr}(M^N), \tag{4.9}$$

where "Tr" denotes the trace and M is still the transfer matrix.

It is now clear from elementary linear algebra what to do: we *diagonalize* M, by finding its eigenvalues, λ_1 and λ_2, and conclude that

$$Z''(N) = \lambda_1^N + \lambda_2^N. \tag{4.10}$$

Furthermore, assuming that the eigenvalues are positive real numbers with $\lambda_1 > \lambda_2$, then

$$\begin{aligned} F_1' &= \lim_{N \to \infty} \frac{1}{N} \log Z''(N) \\ &= \lim_{N \to \infty} \frac{1}{N} \log\left(\lambda_1^N \left(1 + (\lambda_2/\lambda_1)^N\right)\right) \\ &= \log(\lambda_1). \end{aligned} \tag{4.11}$$

(Note: the main idea here is to diagonalize M; this leads to the result $F_1' = \log(\lambda_1)$ even if one sticks with the "linear" model. Our reason for preferring the wrap-around model is purely aesthetic.)

The rest of the solution is routine: One easily sees that $\mathrm{Tr}(M) = 1 + T$ and $\det(M) = T(1 - U^2)$, and, therefore, M has the characteristic equation

$$\lambda^2 - (1 + T)\lambda + T(1 - U^2) = 0,$$

so that the eigenvalues are

$$\lambda = \frac{1 + T \pm \left[(1 + T)^2 - 4T(1 - U^2)\right]^{1/2}}{2}.$$

Note that $|U| = |\tanh(\beta J)| < 1$ for real values of the parameters, and therefore,

$$(1 - T)^2 < \left[(1 + T)^2 - 4T(1 - U^2)\right] < (1 + T)^2.$$

In any case, the eigenvalues are positive real numbers when $0 < T < 1$.

The partition function is of less interest at this point than the free energy. We find

$$F_1' = \log\left[\frac{1 + T + \left[(1 + T)^2 - 4T(1 - U^2)\right]^{1/2}}{2}\right]. \tag{4.12}$$

As a function of β, F_1' is analytic on the positive real axis. We interpret this as

meaning that the one-dimensional Ising model does not exhibit a phase transition at any temperature: a string of iron atoms will not spontaneously magnetize (according to this model, anyway).

That was the discouraging result of Ising's doctoral dissertation. The lack of a phase transition can be understood by thinking of spontaneous magnetization as a *cooperative* phenomenon of the lattice, which requires "communication" between lattice sites. But in the one-dimensional lattice, a single defect destroys the only line of communication. For example, a configuration $\cdots + + + + - - - - \cdots$ is only negligibly more energetic (i.e., less "favorable") than $\cdots + + + + + + + + \cdots$: only one term in the Hamiltonian changes.

According to Brush [**4**], Ising "gave some approximate calculations purporting to show that his model could not exhibit a phase transition in three dimensions either." However, the higher-dimensional models do have phase transitions. The "single-defect" argument does not apply: there are many "lines of communication" connecting each pair of lattice sites.

5. Spontaneous Magnetization in Two Dimensions

In this section we shall present a proof originally due to Peierls [**37**], which shows that the two-dimensional Ising model does have a phase transition—i.e., it exhibits spontaneous magnetization at sufficiently low temperatures. For this purpose we shall forsake some of our previous notation and also return to the "flat" model which has a boundary. We shall exploit the boundary to create a preference for the magnetic moments throughout the lattice.

Recall that sponetaneous magnetization is the tendency for the magnetic moments to remain in, say, the "up" position after an external magnetic field has been turned off. One way to imagine turning off the field is to "impose" a magnet on the boundary of the lattice by *setting* all $\sigma_i = +1$ on the boundary—then letting the boundary "move off to infinity," which is what happens anyway when $N \to \infty$. We can then ask the following question: For a lattice site "O" "deep in the interior", what is the probability that $\sigma_0 = -1$?

If there were no magnetic field, this probability would simply be $1/2$. But the fixed "+" signs on the boundary tend to make the lattice sites near them be positive also, and this creates a "ripple effect" that goes some distance into the lattice. When the temperature is high, this effect is quickly dissipated, but for low temperatures it is possible that the "ripple" will travel a considerable distance inward. What we shall show is that the effect can in fact travel all the way through the lattice; i.e., for sufficiently low temperatures, the probability that $\sigma_0 = -1$ is less than $1/2$ *by an amount which is independent of the lattice size*. The proof is quite beautiful in its elegant use of crude estimates to bound the probability.

Recall that the probability for a given configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ is given by the formula

$$\mathrm{Prob}(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}, \tag{5.1}$$

where H is the Hamiltonian and

$$Z = \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}, \tag{5.2}$$

where Ω is the set of all configurations which are positive on the boundary. (The "± 1" notation is not sufficient here; the heart of Peierls' proof is to consider the sum over various subsets of configurations.) Suppose we label the lattice so that σ_0 corresponds to a lattice site somewhere in the "middle" of the lattice. Then

$$\operatorname{Prob}(\sigma_0 = -1) = \frac{1}{Z} \sum_{\sigma \in \Omega_0} e^{-\beta H(\sigma)}, \tag{5.3}$$

where $\Omega_0 \subseteq \Omega$ is the set of configurations σ for which $\sigma_0 = -1$.

Consider a typical configuration in the set Ω_0, such as the one shown in Figure 9:

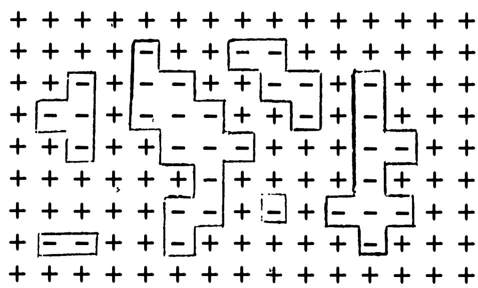

FIG. 9.

Because of the boundary condition, we can think of any configuration as consisting of "islands" of negative signs in a positive "ocean." Some of the islands may have interior "lakes", but they all have "shorelines." Finally, one of the islands contains the site "0."

Now a "shoreline" is a closed path consisting of line segments connecting the midpoints of adjacent squares in the lattice. The main characteristic is that each segment of shoreline separates a positive sign (ocean) from a negative sign (land). Thus a given shoreline corresponds to a set of bonds $\langle i, j\rangle$ for which $\sigma_i\sigma_j = -1$.

Suppose now we draw a shoreline S, creating an island around "0," and say that its length is $n(S)$. Let's consider the set Ω_S of configurations in Ω_0 having S as a shoreline. Then

$$\begin{aligned}
\operatorname{Prob}(\Omega_S) &= \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta H(\sigma)} \\
&= \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta E n(S)} e^{\beta E \sum_{\langle i,j\rangle \notin S} \sigma_i \sigma_j} \\
&= e^{-\beta E n(S)} \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{\beta E \sum_{\langle i,j\rangle \notin S} \sigma_i \sigma_j}.
\end{aligned} \tag{5.4}$$

Given $\sigma \in \Omega_S$, we can form another configuration, σ', by changing all the signs inside the shoreline S. For a fixed shoreline S, the map $\sigma \to \sigma'$ is one-to-one. We shall let Ω'_S denote the image of Ω_S under this mapping. One easily sees that

$$\sum_{\langle i,j\rangle \notin S} \sigma_i\sigma_j = \sum_{\langle i,j\rangle} \sigma'_i\sigma'_j - n(S), \tag{5.5}$$

and, therefore,

$$\sum_{\langle i,j\rangle \notin S} \sigma_i\sigma_j < \sum_{\langle i,j\rangle} \sigma'_i\sigma'_j. \tag{5.6}$$

Plugging this inequality into the previous computation (noting that $\beta E > 0$), we have

$$\operatorname{Prob}(\Omega_S) < e^{-\beta E n(S)}\frac{1}{Z}\sum_{\sigma' \in \Omega'_S} e^{\beta E \Sigma_{\langle i,j\rangle}\sigma'_i\sigma'_j} = e^{-\beta E n(S)}\frac{1}{Z}\sum_{\sigma' \in \Omega'_S} e^{-\beta H(\sigma')}. \tag{5.7}$$

(We have also used the fact that $\sigma \to \sigma'$ is one-to-one, so that the sum over Ω_S can be replaced by the sum over Ω'_S.) But now, since $e^{-\beta H(\sigma)} > 0$ for *all* configurations σ, we can replace the sum over Ω'_S by a sum over *all* configurations! Thus,

$$\operatorname{Prob}(\Omega_S) < e^{-\beta E n(S)}\frac{1}{Z}\sum_{\sigma \in \Omega} e^{-\beta H(\sigma)} = e^{-\beta E n(S)}, \tag{5.8}$$

since $Z = \Sigma_{\sigma \in \Omega} e^{-\beta H(\sigma)}$!

Now consider the set $\mathcal{S}$ of all shorelines which surround the lattice site "0." Then

$$\begin{aligned}\operatorname{Prob}(\sigma_0 = -1) &= \sum_{S \in \mathcal{S}} \operatorname{Prob}(\Omega_S)\\ &< \sum_{S \in \mathcal{S}} e^{-\beta E n(S)}\\ &= \sum_{n=4}^{\infty} s(n)e^{-\beta E n},\end{aligned} \tag{5.9}$$

where $s(n)$ denotes the number of shorelines of length n which surround the lattice site "0." Thus we have one last chore before the denouement: we have to bound $s(n)$. We shall do this in a wonderfully crude manner.

A shoreline, remember, is simply a path in the lattice connecting the midpoints of adjacent squares. Since we required our shorelines to surround the lattice site "0," the path cannot wander too far away from "0": if the shoreline is of length n, it must be contained in a square with sides of length $n/\sqrt{2}$. (See Figure 10.)

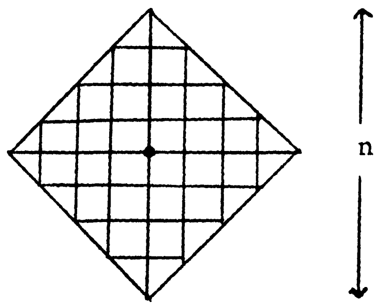

FIG. 10.

Now let $r(n)$ denote the number of "random walks" of length n which originate inside this square. (Minor remark: the random walk, like the shoreline, goes from midpoint to midpoint, not lattice site to lattice site.) It is easy to see that

$$s(n) < \frac{1}{n} r(n). \tag{5.10}$$

(The factor $1/n$ comes from the fact that each shoreline gets counted n times as a random walk, since any point along it can be considered as the origin.) But the random walk has $(n/\sqrt{2})^2 = n^2/2$ possible starting points, and then 4^n possible paths. (This can be reduced to $4 \cdot 3^{n-1}$ if you disallow "backtracking," but there's no real gain in doing so.) Thus

$$s(n) < \frac{1}{2} n 4^n, \tag{5.11}$$

and, therefore,

$$\begin{aligned} \operatorname{Prob}(\sigma_0 = -1) &< \sum_{n=4}^{\infty} \frac{1}{2} n 4^n e^{-\beta E n} \\ &= \frac{1}{2} \sum_{n=4}^{\infty} n (4e^{-\beta E})^n \\ &< \frac{1}{2} \sum_{n=1}^{\infty} n (4e^{-\beta E})^n. \end{aligned} \tag{5.12}$$

The denouement is at hand: recall that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1.$$

Hence, by differentiating and multiplying by x,

$$\frac{x}{(1-x)^2} = x(1 + 2x + 3x^2 + \cdots) = \sum_{n=1}^{\infty} n x^n.$$

Thus, letting $x = 4e^{-\beta E}$, we have

$$\operatorname{Prob}(\sigma_0 = -1) < \frac{1}{2}\left[\frac{4e^{-\beta E}}{(1 - 4e^{-\beta E})^2}\right]. \tag{5.13}$$

The conclusion is clear: by taking β sufficiently large (which corresponds to low temperature), the right-hand side of the inequality can be made arbitrarily small, in a way which is independent of the size of the lattice. Thus spontaneous magnetization is guaranteed at some temperature.

6. The Critical Point in Two Dimensions

Peierls' proof that a phase transition exists for the two-dimensional model can put a lower bound on the critical temperature, but cannot locate it exactly. In this section we shall present a lovely combinatorial argument due to Kramers and Wannier [26] which proves the following result for the two-dimensional Ising model in the zero-field case: if there is a unique phase transition for $F_2'(T,0)$ on the interval $(0,1)$, then it occurs precisely at $T_c = \sqrt{2} - 1$. (The historical progression of results is thus Peierls' 1936 proof that a phase transition exists; Kramers and Wanniers' 1941 proof that it occurs at $T_c = \sqrt{2} - 1$; and Onsager's complete solution in 1944.)

The starting point is the combinatorial interpretation of $c_2(n)$ as the number of "even subgraphs with n edges," whose connected components are closed paths on the lattice. In this section it will be convenient to refer to such subgraphs as "closed paths of length n," even when the "path" has several components.

In general, a closed path of finite length in the plane may be associated with the bounded region which it encloses or, alternatively, with the unbounded region outside of it. If we consider the set of bounded regions and their complements, there is a two-to-one correspondence between such "shaded" regions and closed paths in the plane. (See Figure 11.)

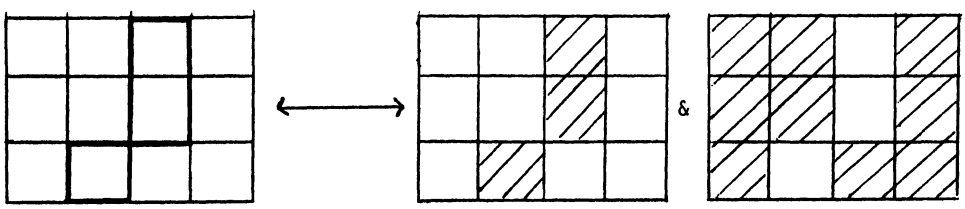

FIG. 11.

On the lattice, a "shaded" region can be designated by enumerating all the *squares* of the lattice, $i = 1, 2, 3, \ldots$, and assigning an independent variable, say, τ_i, to each square: $\tau_i = 1$ if the square i is shaded, and $\tau_i = -1$ if square i is unshaded. Unfortunately, when we restrict regions to a finite lattice, the correspondence is no longer precisely two-to-one: the "zero" path and the path around the boundary both correspond to both the "empty" region and the full lattice region. This can be

fixed by eliminating the boundary with the wrap-around lattice, but then even worse things happen: a simple loop around the lattice does not correspond to *any* region (much less to two of them). One should note, however, that such paths are of very long length— $\sqrt{N}$, if one takes the overall lattice to be a square. The contribution of these paths to the partition function is thus far out in the power series, and hence we expect it to vanish in the thermodynamic limit. In keeping with our disregard for mathematical rigor (but only when its safe to do so!) we shall take it for granted that this is what happens.

In spite of these drawbacks, we shall use the wrap-around model since it simplifies some of the notation. In particular, there are as many squares as there are lattice sites, and we can enumerate them according to, say, the lattice site i in the lower left-hand corner. Each square has four "nearest neighbors" with which it shares an edge: squares i and j are nearest neighbors if and only if lattice sites i and j are nearest neighbors.

Suppose we are given a configuration for the *squares*, $\tau = (\tau_1, \dots, \tau_N)$. How long is the closed-path boundary of the corresponding shaded region on the lattice? We answer this by first noting that the edge joining squares i and j is part of the boundary if and only if $\tau_i\tau_j = -1$—i.e., if and only if one square is shaded and the other is not. Thus the length, n, of the closed path is given by

$$n(\tau) = \sum_{\langle i,j\rangle} \delta(i,j) \quad \text{where} \quad \delta(i,j) = \begin{cases} 1 & \text{if } \tau_i\tau_j = -1 \\ 0 & \text{if } \tau_i\tau_j = 1. \end{cases}$$

Our interest is actually in T^n. We write $T^{n(\tau)} = \prod_{\langle i,j\rangle} T^{\delta(i,j)}$ to begin with, but then notice that we can rewrite

$$\begin{aligned} T^{\delta(i,j)} &= \frac{1}{4}\left[(\tau_i + \tau_j)^2 + (\tau_i - \tau_j)^2 T\right] \\ &= \frac{1}{2}\left[(1+T) + \tau_i\tau_j(1-T)\right] \\ &= \frac{1+T}{2}\left[1 + \tau_i\tau_j\left(\frac{1-T}{1+T}\right)\right]. \end{aligned}$$

Therefore,

$$T^{n(\tau)} = \left(\frac{1+T}{2}\right)^{2N} \prod_{\langle i,j\rangle}\left[1 + \tau_i\tau_j\left(\frac{1-T}{1+T}\right)\right]. \tag{6.1}$$

Now remembering the combinatorial interpretation of $c_2(n)$, remembering that there is a two-to-one correspondence between closed paths and shaded regions, and forgetting that this correspondence breaks down at some point, we have

$$Z'(T) = \sum_{n=0}^{\infty} c_2(n) T^n \cong \frac{1}{2} \sum_{\pm 1} T^{n(\tau)}, \tag{6.2}$$

where the sum is now over all configurations $\tau = (\tau_1, \dots, \tau_N)$. (The approximate

equality ($\cong$) reflects the breakdown of the correspondence between closed paths and shaded regions; more precisely, it means that the formal power series are identical out to a power determined by the size of the lattice.) Plugging in (6.1), we have

$$\begin{aligned} Z'(T) &\cong \frac{1}{2}\left(\frac{1+T}{2}\right)^{2N} \sum_{\pm 1} \prod_{\langle i,j\rangle} \left[1 + \tau_i\tau_j\left(\frac{1-T}{1+T}\right)\right] \\ &\cong \frac{1}{2}\left(\frac{(1+T)^2}{2}\right)^{N} Z'\left(\frac{1-T}{1+T}\right). \end{aligned} \tag{6.3}$$

The partition function reappears on the right-hand side! When we now take the limit $N \to \infty$, the approximate equality becomes exact, and we obtain the result

$$F'(T) = \log\left(\frac{(1+T)^2}{2}\right) + F'\left(\frac{1-T}{1+T}\right). \tag{6.4}$$

Recall that values of T near 0 correspond to high temperatures, while values near 1 correspond to low temperatures. Observe now that when T is near 0, $(1-T)/(1+T)$ is near 1 and vice versa: $T \to (1-T)/(1+T)$ maps the interval onto itself. Thus (6.4) is a formula—or functional equation, if you will—relating high and low temperatures. Viewing F' as an analytic function, (6.4) provides an analytic continuation of F'. In particular, if F' is nonanalytic at T, then it is also nonanalytic at $(1-T)/(1+T)$; i.e., phase transitions will occur in *pairs*. Thus if we assume that there is a unique (physical) phase transition in the interval $(0,1)$, then it can only occur at the solution of the equation

$$T = \frac{1-T}{1+T},$$

which is obviously at $T = T_c = \sqrt{2} - 1$.

We conclude by observing that this result is in agreement with Onsager's solution

$$F'(T,0) = \frac{1}{2}\int_0^1\int_0^1 \log\left[(T^2+1)^2 - 2T(1-T^2)[\cos(2\pi x) + \cos(2\pi y)]\right] dx\, dy.$$

For $0 < T < 1$, we have

$$\begin{aligned} (T^2+1)^2 - 2T(1-T^2)(\cos(2\pi x) + \cos(2\pi y)) &\geqslant (T^2+1)^2 - 4T(1-T^2) \\ &= T^4 + 4T^3 + 2T^2 - 4T + 1 \\ &= (T^2 + 2T - 1)^2 \end{aligned}$$

with equality only when $\cos(2\pi x) = \cos(2\pi y) = 1$. Thus the integrand in Onsager's solution has a singularity if and only if $T^2 + 2T - 1 = 0$—i.e., $T = \sqrt{2} - 1$.

7. Concluding Remarks

The Ising model has become a vast subject. This article has touched only on portions of it, and the simplest ones at that. We have not spoken, for instance, of critical exponents, correlation functions, or renormalization. We have adhered rather strictly to a combinatorial approach, ignoring important algebraic and representation-theoretic techniques. Our purpose here has been to introduce the Ising model to a wider audience, not to expound on what the experts already know; the combinatorial interpretation seems to be the most accessible avenue, and has indeed led to several of the advances in the subject. The author hopes that this article may encourage some of its readers to dig more deeply into the Ising model. There is a lot of gold left in the mine.

The author would like to thank Lynn Steen at St. Olaf and the referee for their helpful comments.

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LETTERS TO THE EDITOR

Editor,

I urge each member of the MAA to read a recent publication, *Toward a Lean and Lively Calculus* (MAA Notes #6), on the future of college calculus, especially in light of the pressures of discrete mathematics and calculators/computers. It contains some very thoughtful position papers and summary reports from a Sloan Foundation sponsored conference (January, 1986).

The participants agreed that: 1) calculus should remain "as the core of the undergraduate mathematics curriculum"; 2) "change is desirable, possible, and even inevitable;" 3) "the syllabus should... contain *fewer topics*, but... have more *conceptual depth*, numerically and geometrically;" and 4) we "should make use of the latest technology but the goals of the calculus must extend far beyond facility with either calculators or computers." They also offer sample syllabi, suggestions on requisite changes in the classroom, and methods of implementation.

We need not agree with the views expressed in the report, but we need extensive discussions for collective decisions about the future of calculus and they have given us a fine basis. *Read the report! And discuss it!*

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Editor:

Underwood Dudley's review of *Why Math* in the May 1987 Monthly showed contempt for his students and for his profession.

He starts off with a little joke about calculus students: "little passion and hardly any imagination." He cannot accept at face value a former student's assertion that he uses mathematics "every day." "I would have been impolite to ask for examples." He reckons the number of students who can see the beauty of mathematics "however dimly?" 5%? 1%? 0.5%? or less?"

I could go on, there are examples in almost every paragraph of this five-page review.

With teachers who think like this, no wonder so many students hate and fear mathematics.

My students have both passion and imagination, but they do not show it to teachers they do not trust. Yes, people in business use mathematics every day, if they have been taught how useful mathematics really is. And 100% of my students can at least dimly appreciate the beauty of mathematics, if they are not treated with a contempt they do not deserve.

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words of length L on a $\mu(U)$-letter alphabet, that avoid U, has a polynomial bound in L. On the other hand, the bound for xx is exponential. McNulty draws our attention to a paper of Zimin (1984).

D. H. Lehmer conjectured that there is no *composite* value of n such that $\varphi(n)$, Euler's totient function, is a divisor of $n-1$, i.e., that for no value of n is $\varphi(n)$ a *proper* divisor of $n-1$ [1973, 192].

Schinzel noted that if $n=p$ or $2p$, where p is prime, then $\varphi(n)+1$ divides n, and asked if the converse is always true. See B37 in Guy (1981).

Sanford Segal observed that Schnizel's question reduces to Lehmer's, that it arises in group theory, and may have been raised by G. Hajós. See Miech (1966), though it is there attributed to Gordon. For the reduction of Schinzel's question to that of Lehmer, see Cohen (tbp).

Bernardo Recamán [1973, 919; and see 1975, 998] asked several questions about Ulam's sequence, $U_1=1$, $U_2=2$, and for $n\geqslant 3$, U_n is the least integer expressible *uniquely* as the sum of two distinct earlier members of the sequence. A remark of Eggleton [1973, 920] shows that $U_{n+1}\leqslant U_n+U_{n-2}$. Hence $U_{n+1}<2U_n$ and it follows [1977, 809] that $\{U_n\}$ is **complete**, i.e. that every positive integer is expressible as the sum of distinct U-numbers. David Zeitlin (see [1977, 815] for reference) conjectured that $\{U_n\}$ is still complete, even after the deletion of one or two members. Robert Stong (wrc) recalls his earlier proofs of this, and of another Zeitlin conjecture, that $\{U_n^*\}$ is complete, where $U_1^*=1$, $U_2^*=2$, and, for $n\geqslant 3$, $U_n^*=U_n+U_{n-2}$ (since $U_{n+1}^*=U_{n+1}+U_{n-1}<2U_n+2U_{n-2}=2U_n^*$).

Molnar [1974, 383] asked for determinants with nonunit integer entries whose value was 1, and remained so when the entries were squared. Several solutions were given [1975, 999–1000; 1977, 809] but the following have not previously appeared. The first two are by Don Coppersmith, the next three by Morris Newman, and the last two by Sadao Saito (wrc).

$$\begin{bmatrix} 27 & 26 & 23 \\ 5 & 5 & 4 \\ -6 & -5 & -6 \end{bmatrix}\begin{bmatrix} 119 & 208 & 277 \\ 9 & 14 & 16 \\ 12 & 21 & 28 \end{bmatrix}\begin{bmatrix} 43257 & 7 & 9 \\ 18544 & 3 & 4 \\ 12376 & 2 & 3 \end{bmatrix}\begin{bmatrix} -386723 & -17 & -23 \\ 68242 & 3 & 4 \\ 45 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 8n^2-8n & 2n+1 & 4n \\ -4n^2-4n & n+1 & 2n+1 \\ -4n^2-4n+1 & n & 2n-1 \end{bmatrix} \qquad \begin{bmatrix} -8n^2-4n+1 & 2n & 2n+2 \\ -8n^2-8n-3 & 2n+1 & 2n+1 \\ -8n^2-4n & 2n & 2n+1 \end{bmatrix}$$

$$\begin{bmatrix} -8n^2-4n+1 & 2n & 4n+4 \\ -8n^2-8n-3 & 2n+1 & 4n+2 \\ -4n^2-2n & n & 2n+1 \end{bmatrix}$$

Richard Macintosh found two examples involving several Fibonacci numbers:

$$\begin{bmatrix} 1167 & 2 & 5 \\ 1698 & 3 & 8 \\ 2866 & 5 & 13 \end{bmatrix} \qquad \begin{bmatrix} 610 & 5 & 13 \\ 1054 & 8 & 21 \\ 1665 & 13 & 34 \end{bmatrix}$$

In Conway's game of Sylver Coinage, players alternately name positive integers, subject to their not being the sum (with repetitions allowed) of previously named

integers: if you name 1, you lose. The last of the twenty questions [1976, 634] was: can the reader complete the table (of good replies to various positions)? The 1983 reprinting of *Winning Ways* showed this (p. 596) largely done by John Francis, who also found the good replies: 28 to $\{16, 24, 5\}$; 6 and 9 to $\{16, 24, 7\}$; and 7 and 11 to $\{16, 24, 9\}$. Francis Voelkle (wrc) has since made considerable advances, finding the good replies: 10 to $\{16, 24\}$; 10, 15, 16, and 21 to $\{12, 18\}$; and 24 to $\{18, 27\}$. However, we still don't know (questions 4 & 5) if $\{16\}$, $\{18\}$ or $\{27\}$ has any good reply, though Voelkle conjectures that 15 may be a good reply to $\{27\}$. We conjectured (questions 6 & 7) that $\{a, b, c, \dots\}$ was a $\mathscr{P}$-position (previous-player-winning) if $\{2a, 2b, 2c, \dots\}$ or $\{3a, 3b, 3c, \dots\}$ was, but Voelkle finds numerous counterexamples: e.g. $\{8, 14\}$ is a $\mathscr{P}$-position, while $\{4, 7\}$ has the good reply 13; $\{12, 15, 18\}$ is a $\mathscr{P}$-position, but $\{4, 5, 6\}$ is not.

A pair (a, b) is called an **α-pair** if $\{5, a, 2a, b, 2b\}$, with one member in each residue class, mod 5, is a $\mathscr{P}$-position: e.g. (2, 3), (7, 8), (17, 18), (22, 23), (39, 41). Question 18 asked about the truth of the *α-hypothesis* ¿$|a - b| = 1$ or 2? Voelkle has now found larger differences.

The following table summarizes the work of Hutchings and Voelkle and shows all known good replies to the first few composite numbers, and (in parentheses) the only other candidates:

$\{4\}$	6	$\{12\}$	$8(4n+2 \geqslant 30, 3n \geqslant 27)$	$\{20\}$	$5(2n \geqslant 16)$
$\{6\}$	4, 9	$\{14\}$	7, 8, 10(26)	$\{21\}$	$7(3n \geqslant 15)$
$\{8\}$	12, 14	$\{15\}$	$5(3n \geqslant 21)$	$\{22\}$	$11(2n \geqslant 18)$
$\{9\}$	6	$\{16\}$	$(20, 2n \geqslant 26)$	$\{24\}$	$(3n \geqslant 15, 2n \geqslant 20)$
$\{10\}$	5, 14(26, 32, 36, 46)	$\{18\}$	$(2n \geqslant 20, 3n \geqslant 30)$	$\{25\}$	5

Voelkle acknowledges extensive use of the VAX 8600 at the École Polytechnique Fédérale de Lausanne.

We quoted [1985, 718] Tunnell's paper in connexion with the congruent number problem [1980, 43] in which he observed that the conjecture of Birch and Swinnerton-Dyer, together with a result of Waldspurger (1981) leads to a conjectured explicit description of congruent numbers. Kramarz (1986) verifies a version of the conjecture and finds all congruent numbers < 2000.

Ernst Selmer (1986) has produced two volumes on the postage stamp problem [1980, 206]. It is convenient to distinguish the (Frobenius, Sylvester) **coin problem**: find the *largest* number of units that cannot be made up from a given set of coin denominations (which *doesn't* include a unit coin), from the **postage stamp problem**: find the *smallest* number of units that cannot be affixed to an envelope with room only for a given number of stamps, chosen from a given set of denominations (which *does* include a unit stamp). Selmer distinguishes between this, the **local** stamp problem, and the **global** one: given the envelope size and the number of different denominations, choose these denominations to maximize the range of consecutive postages that can be stamped. Selmer's encyclopedic work contains 103 references, but there remains a plethora of unsolved problems, requiring interplay of theory and computation.

Many problems remain open concerning "peeling rinds" from a sequence [1982, 113]. We can now complete the bibliographic details for Gibson and Slater (1984) and for Schwenk (1984), who finds the maximum number of rinds that can be peeled from a sequence of n symbols, each occurring twice, for $8 \leqslant n \leqslant 14$, and conjectures that, for all $n \geqslant 8$, the sequence exemplified, for $n = 10$, by

$$4\ 3\ 1\ 2\ 0\ 0\ 1\ 2\ 3\ 4\ 9\ 8\ 7\ 6\ 5\ 5\ 7\ 6\ 8\ 9$$

gives this maximum number.

The $3x + 1$ problem [1983, 35; 1985, 3] remains a hardy annual. Korec and Znám (wrc) write $P \prec Y$ to mean that for every positive integer x, there is some $y \in Y$, and i, j such that $f^i(y) = f^j(x)$ where $f(x) = 3x + 1$ (x odd), $= x/2$ (x even). So all we have to show is $P \prec \{1\}$. If $a(m)$ denotes the set of positive integers $\equiv a (\bmod m)$, they prove that if p is an odd prime, and 2 is a primitive root of p^2, then $P \prec a(p^n)$ for every pair of positive integers n, a with a prime to p.

Yuri Fradkin (wrc) obtains an equivalent problem by defining $g(x)$ for odd x by $(3x + 1)/2$ $(x = 4k - 1)$, $(3x + 1)/4(x = 8k + 1)$, and $(x - 1)/4(x = 8k - 3)$, and defining the set, RO, of **regular odd numbers** by $1 \in \mathrm{RO}$; $4x + 1 \in \mathrm{RO}$ if x does; $(4x - 1)/3 \in \mathrm{RO}$ if $x \equiv 1 \pmod 3$ does; and $(2x - 1)/3 \in \mathrm{RO}$ if $x \equiv 2 \pmod 3$ does. The set RO is similar to those considered by Klarner and Rado (1973, 1974).

Alon and Frankl (1985) prove an old conjecture of Erdős by showing that the number of disjoint pairs in a family of 2^{n+1} subsets of a $2n$-element set is bounded by $(1 + o(1))2^{2n}$. They also verify the conjecture of Daykin and Erdős [1983, 119] and establish the corresponding Erdős-Stone type result by showing that if $\mathscr{F}$ is a family of m distinct subsets of an n-element set, $d(\mathscr{F})$ is the number of disjoint pairs in $\mathscr{F}$, and $d(n, m)$ the maximum of $d(\mathscr{F})$ over all m-element families, then, for $m = 2^{(1/(k+1)+\delta)n}$ and $\delta > 0$, there is a $\beta > 0$ such that

$$d(n, m) < \left(1 - \frac{1}{k}\right)\binom{m}{2} + O(m^{2-\beta\delta^2})$$

We give details of the paper of Giblin and Kingston (1986) which solved Giblin's moving triangle problem [1983, 121].

Yang Yanlin (wrc) of the Beijing Light Industry Institute, proves the conjecture of Borwein and Edelstein [1983, 389] that if A and B are two finite, disjoint sets whose union spans projective $(m + n)$-space, then there is either an A-monochrome m-flat or a B-monochrome n-flat (an affine variety of dimension n spanned by points of B, that contains no points of A).

Rzymowski and Stachura (1986) show that the circle $|z| < rv/(v - 1)$ is the domain of largest area which can be guarded by a defender with destruction radius r and maximum speed 1 against an invader with maximum speed $v > 1$. See Thews [1984, 416].

Roger Nelsen (1987) affirmatively answers Walter Piegorsch's question [1984, 562]: can we generate a bivariate Poisson distribution with a negative correlation? He constructs a bivariate probability function with arbitrary marginal distributions (which need not be members of the same family) and *any* required correlation

between the theoretical minimum and maximum values. His techniques are elementary and can be adapted to simulation studies requiring samples from discrete bivariate distributions.

The note of Broadie and Cottle (1984) on the simplexity of the 5-cube [1984, 628] has appeared.

Forcade and Pollington (wrc), using computer time at Bellcore, NJ, found the counterexample 195 to their conjecture with Lamoreaux [1986, 119], and suspect that 255 may be another. These have three distinct odd prime factors and make a good example of the Strong Law of Small Numbers (1988), parallel to the converse of Fermat's little theorem (Carmichael numbers) and to the size of the coefficients in cyclotomic polynomials.

Bob Guralnick (wrc) answers Feuer's question [1986, 120] negatively with the example, in $G = S_n (n \geqslant 6)$,

$$a = A = (12)(34) \qquad b = (34)(56) \qquad B = (13)(24)$$

so that $\langle A, B\rangle \cong \langle a, b\rangle$ is the Klein 4-group, and either $f(a,b) = f(A,B) = 1$ or $f(a,b)$, $f(A,B)$ are products of two transpositions, and hence are conjugate in G. But $\langle A, B\rangle$, $\langle a, b\rangle$ are not conjugate in G since they do not have the same orbit sizes. He believes that this has been known since the turn of the century, and that variations on the problem have applications to number theory and geometry, as well as group theory.

There have been several small rumblings concerning Hofstadter's sequence [1986, 186], but they are mainly variations on a theme, rather than answers to the original problem. Thanks to John Robertson for noting the misprints in two of the tables: for $k = 7, 8$; $Q(2^k + 1) = 63,\ 143$; $Q(3.2^k + 1) = 135,\ 278$; $f_k = 9, -8$ and $\Delta f_k (= e_k?) = -1, -17$. We look forward to a paper of Golomb (1988).

For the Mahler-Popken problem [1986, 188]: find the least number, $f(n)$, of ones needed to represent n, using only $+$ and $\times$ (and parentheses), Isbell and Myerson suggested that there might be examples of shape $n = (3x+1)^2 + 6$ which required $f(n)$ to be as large as $2(f(x) + 4) + 5 = 2f(x) + 13$. However, John Selfridge (wrc) notes that

$$\begin{aligned} f(n) &= f(3(x(3x+2)+2)+1) \\ &\leqslant 3 + (f(x) + (3 + f(x) + 2 + 2)) + 1 = 2f(x) + 11. \end{aligned}$$

He also gives a negative answer to the second question on p. 189, by observing that $f(2^7) = 14 = f(3^3 5)$, while $3^3 5$ is greater than 2^7, but not of the form $2^{7-3c} 3^{2c}$.

Craig Bailey, Joel Brenner, Kārlis Čerāns (1987), Miklós Laczkovich, François Sigrist, as well as Stanley Rabinowitz and Jack Arrow, all send proofs of Grometstein's inequality [1986, 279].

$$yx^y\{y^x - (y-1)^x\} - xy^x\{x^y - (x-1)^y\} > 0$$

for all real $x > y > 1$ (not $y \geqslant 1$ as originally printed).

Z. Z. Uoiea of Grouse Creek, UT, and a score of others, from as far afield as Novosibirsk, observed that the characteristic function of a single point, say $f(x) =$

$0 (x \neq 0)$, $f(0) = 1$, is a counterexample to Funar's conjecture [1986, 280]. However, many went on to show how nearly it was true. Dan Velleman, of Amherst, proved that if $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of a bijection and an injection, then it can be written as the sum of two bijections, and asked what can be said if we don't assume the axiom of choice. Eric Milner, of Calgary, proved that if S is an infinite abelian group, then any map $f\colon S \to S$ whose range $f[S]$ has the same cardinality as S is expressible as the sum of a bijection and an injection. Arnold Miller, of Wisconsin, showed that the conjecture is true if we ignore at most finitely many values of x.

Carl Ponder [1986, 280] asked for the asymptotic behavior of $\varphi_h(1)$, where $\varphi_h(x)$ is the (polynomial of degree $2^h - 1$) solution of the differential equation

$$\frac{d}{dx}\varphi_h(x) = \{\varphi_{h-1}(x)\}^2$$

with boundary conditions $\varphi_0(x) = \varphi_h(0) = 1$. James B. Shearer (wrc) obtains the quite good bounds

$$c_1\lambda_1^h < \varphi_h(1) < c_2\lambda_2^h$$

with $\lambda_1 = \max_\alpha((\alpha + 1)/2)^{1/\alpha} \approx 1.26107$, $\lambda_2 = 2\ln 2 \approx 1.38629$, $c_1 = 1$ and $c_2 = 2.12$. His methods can be used to improve the upper bound slightly, and to prove the existence of the limit of $\{\varphi_h(1)\}^{1/h}$.

Ih-Ching Hsu [1986, 371] asked for the general solution of the functional equation

$$F(x, y) + F(\varphi(x), \psi(y)) = F(x, \psi(y)) + F(\varphi(x), y) \tag{1}$$

(a) where φ, ψ are given functions, (b) where φ, ψ are given involutions, and (c) where $\varphi(t) = \psi(t) = 1 - t$. This generated correspondence from Roger Howe, Yale; John Snygg, Upsala College; and Dan Velleman, Amherst; and papers from Kouong Law, Longwood College; John S. Lew, IBM, Yorktown Heights; Richard Rice, Seattle; and Mario Taboada, Minnesota.

Taboada answered (c) with $F(x, y) = A(x - 1/2, y - 1/2) + B(x, y - 1/2)$ where A is even in x and odd in y, and B is even in y. Velleman gave a similar solution and added conditions for the solution to be continuous, and to be differentiable, and noted that his method could be used to solve (b). This was done by Snygg. A version of (c) with $\varphi(t) = a - t$, $\psi(t) = b - t$ was solved by Kouong Law with a power series in $x - a/2$, $y - b/2$.

Howe showed that any solution had the form $F_1 + F_2$, where $F_1(x, y) = F_1(\varphi(x), y)$ and $F_2(x, y) = F_2(x, \psi(y))$; that if φ, ψ were involutions, or, more generally, of finite order, then the problem could be analyzed in terms of the representation theory of finite groups; that the solution was purely set-theoretic and didn't concern the topological structure of $\mathbb{R}$, so the general solution might well be discontinuous; that there would be situations in which the general *continuous* solution had the form

$$F(x, y) = f(x) + g(y) \tag{2}$$

noted by Hsu; and that there was a connexion with D'Alembert's solution of the wave equation in one dimension.

Rice used the same equivalence relation as Lew (see next paragraph), calling the classes **φ-orbits**. Then, given φ, K, the equation $F(\varphi(x), y) = F(x, y) + K(x, y)$ has a solution just if K sums to zero on cycles of φ-orbits. He also gave the general solution to (1).

Lew gave the most complete treatment. If $\varphi\colon X \to X$ is an arbitrary mapping on an arbitrary set, define the equivalence relation E, on X, by $x_1 E x_2$ just if $\varphi^m(x_1) = \varphi^n(x_2)$ for some integers $m, n \geqslant 0$. Let C, C' be φ-orbits of X and D, D' be ψ-orbits of Y, then the F-values on $C \times D$ do not affect the F-values on any disjoint $C' \times D'$. If $r(C), s(D)$ are representative points of C, D, then you can assign F-values arbitrarily on every set $[C \times s(D)] \cup [r(C) \times D]$, and extend this, by (1), to a solution. Every solution of (1) has this form. Here, use of r, s assumes the axiom of choice, but in particular cases (e.g., the solutions of Kouong Law, Taboada, and Velleman) the representatives can be chosen constructively. Lew gave conditions under which (2) is the most general *continuous* solution. Examples are (1) $X = Y =$ the complex numbers, with φ, ψ non-linear polynomials, or $az + b$ with $|a| \neq 1$; (2) $X = Y =$ the Riemann sphere, with φ, ψ rational functions whose numerator and denominator are not both linear polynomials; (3) $X = Y =$ the reals, with φ, ψ continuous functions such that the fixed points of $\varphi(\varphi(x))$ and $\psi(\psi(y))$ are nonempty *countable* sets.

Myerson [1986, 457] asked how small a sum of five Nth roots of unity can be. Dean Hickerson (wrc) finds infinitely many N for which $f(5, N) < 8\pi/\sqrt{5}N^2$, and shows that for all sufficiently large N, $f(5, N) \leqslant 8\pi N^{-4/3}$.

Two correspondents suggest that David Dowe's question [1986, 627] "Are Maxwell's equations logically consistent?" makes little sense. They observe that it is equivalent to the question "Is mathematics, say *ZFC*, consistent?" and that there is no "weak" or "strong" notion of consistency in mathematics, nor in applied mathematics. I apologize that, in rewording the question, I may have misinterpreted the referee's remarks. Dowe himself adds the question "When, and in what sense, can a physical theory be said to be logically complete?" Meanwhile, I recall a classical article of Wightman (1976), on Hilbert's sixth problem, which contains 136 references, at least a few of which may be relevant.

Pambuccian [1986, 627] defined $a(n)$ to be the smallest integer a for which there is an integer b, $0 < b < a$, $(a, b) = 1$, with all members of the arithmetic progression $a + b, 2a + b, \dots, na + b$ composite. He conjectured that $a(n)$ was always prime, but Erdős thought not. No surprise that Erdős was right and I was wrong, though I was right to suggest that a computer might settle our differences. Andy Odlyzko (wrc) make some of the earliest and most extensive calculations, among several others, and exhibited $a(135) = 8207 = 29 \times 283$, with $b = 3251$; and $a(150) = 12311 = 13 \times 947$, with $b = 6779$.

Noam Elkies (and independently John Leech and Ian Macdonald) noted that there are generally 2^n spheres touching all $n + 1$ hyperfaces of an n-dimensional simplex, not $n + 2$ as stated in Hatada's problem [1986, 628]. In the regular,

3-dimensional case, three are at infinity. The intended $n+1$ spheres are those which touch n hyperplanes on the same side as the insphere does, and one hyperplane on the opposite side. Hatada let $f(n)$ be the minimum ratio of the content of the simplex formed by these $n+1$ excentres, to the content of the original simplex, and conjectured that $f(n) = 2^n/(n-1)^n$ for $n \geqslant 2$, and that this minimum is attained just when the simplex is regular. Elkies and Macdonald each show that $f(n)$ can be made as small as you like for $n \geqslant 3$, and that Hatada's value is realized for $n = 3$ for just those tetrahedra whose faces split into two pairs with equal area-sum.

Bencsath and Mezei (wrc) relate Corley's problem [1986, 628] to the "hard spheres" problem of statistical mechanics, which leads to work on simulation and approximation. See Barker and Henderson (1976) and Mezei and Beveridge (1986). They later sent an extended bibliography, available from the present writer.

George Andrews [1986, 708], in studying Ramanujan's "lost" notebook, came across the striking q-series

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = \sum_{n=0}^{\infty} S(n)q^n$$
$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots + 8q^{3288} + \cdots$$

about which he made two conjectures:

1. $\limsup |S(n)| = +\infty$?
2. $S(n) = 0$ for infinitely many n?

Dean Hickerson studied a table of values of $S(n)$ and conjectured a "closed form" for $S(n)$ which, if valid, would imply both conjectures. Here are his observations:

Consider the diophantine equation

$$x^2 - 6y^2 = m \tag{3}$$

for positive or negative integers $m \equiv 1 \pmod{24}$ and call two solutions (x, y) and (x', y') **equivalent** if

$$x + y\sqrt{6} = \pm(5 + 2\sqrt{6})^r(x' + y'\sqrt{6})$$

for some integer r. It's easy to show, by induction on $|r|$, that if (x, y) and (x', y') are equivalent, then $x + 3y \equiv \pm(x' + 3y') \pmod{12}$.

Let $T(m)$ be the excess of the number of inequivalent solutions of (3) with $x + 3y \equiv \pm 1 \pmod{12}$ over the number with $x + 3y \equiv \pm 5 \pmod{12}$. Then Hickerson conjectured that

$$¿\, S(n) = T(24n+1)\,?$$

Andrews and Hickerson (tbp) have since proved this conjecture, which raises some further questions:

Is there a partition-theoretic interpretation of $T(m)$ for $m < 0$? Freeman Dyson, at the Ramanujan Centenary Conference, conjectured what it was, and Andrews and Hickerson proved it.

Are there similar expressions in other quadratic fields? Hickerson has found $T(m)$ in terms of $Q(\sqrt{2})$ and $Q(\sqrt{3})$.

Ramanujan would have liked this.

Jim Lawrence (wrc) notes that Shapiro's conjecture [1987, 46] on polyhedral cones is Theorem 8.9(b) of McMullen and Schneider (1983). In collaboration with Jon Spingarn he also gives a quite short direct proof.

My indebtedness to numerous correspondents is obvious. So too, is the usefulness of a clearinghouse for information on problems which are not always quite as unsolved as we thought.

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Edgar Allan Poe on Probability

Nothing, for example, is more difficult than to convince the merely general reader that the fact of sixes having been thrown twice in succession by a player at dice, is sufficient cause for betting the largest odds that sixes will not be thrown in the third attempt. A suggestion to this effect is usually rejected by the intellect at once. It does not appear that the two throws which have been completed, and which lie now absolutely in the Past, can have influence upon the throw which exists only in the Future. The chance for throwing sixes seems to be precisely as it was at any ordinary time—that is to say, subject only to the influence of the various other throws which may be made by the dice. And this is a reflection which appears so exceedingly obvious that attempts to controvert it are received more frequently with a derisive smile than with anything like respectful attention. The error here involved—a gross error redolent of mischief—I cannot pretend to expose within the limits assigned me at present; and with the philosophical it needs no exposure. It may be sufficient here to say that it forms one of an infinite series of mistakes which arise in the path of Reason through her propensity for seeking truth *in detail.*

The Mystery of Marie Roget

[Contributed by Gerald Weinstein of City University of New York]

NOTES

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Selecting Estimators for the Standard Deviation of a Normal Distribution

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In a recent paper by Birch and Robertson **[1]** it was shown that the mean squared error of the sample variance is *less* than the mean squared error of the second sample moment. They also gave strong numerical evidence indicating that the reverse is true for the corresponding estimates of the standard deviation. That is, based on numerical evidence, they conjectured that the mean squared error of the square root of the sample variance is *greater* than the mean squared error of the square root of the second sample moment. In this note we will prove that this numerical conjecture of Birch and Robertson is true in general. In order to give the reader some background we will first briefly state some of the pertinent results of Birch and Robertson.

In their paper, Birch and Robertson state that the normal distribution is a two-parameter distribution and the estimation of the variance, σ^2, depends upon whether or not the mean, μ, is known. Birch and Robertson also pointed out that paradoxes sometimes arise when two different criteria are used for choosing an estimator for the variance of a normal distribution. The two criteria they considered were the principles of maximum likelihood and mean squared error. They explicitly showed that the best estimator for the variance due to the method of maximum likelihood is *not* the best estimator when one uses mean squared error for their criteria.

In particular, they showed that

$$\begin{aligned}\text{MSE(of the sample variance)} &= \left(\frac{2}{n} - \frac{1}{n^2}\right)\sigma^4 < \frac{2\sigma^4}{n}\\ &= \text{MSE(of the second sample moment)}, \qquad (1)\end{aligned}$$

where the second sample moment is defined by

$$S^2(\mu) \equiv \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2, \qquad (2)$$

the sample variance is defined by

$$S^2(\bar{x}) \equiv \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad (3)$$

and MSE is short for mean squared error. The paradox that equation (1) demonstrates can be emphasized by noting that although $S^2(\mu)$ is a minimum variance

unbiased estimator of σ^2, its mean squared error is *greater* than the mean squared error of $S^2(\bar{x})$, which is a biased estimator of σ^2. Birch and Robertson stated that equation (1) goes against one's intuition in that it seems to indicate that when μ is known, one should ignore this fact and use the estimator $\bar{x}$ for μ. The rest of their paper examines this counterintuitive fact in more detail.

Toward the end of their article Birch and Robertson state:

> On the other hand, one might not be so much interested in σ^2 as in σ. Thus we might compare the mean squared errors of $S(\bar{x})$ and $S(\mu)$ as estimators of σ. The distributions of these statistics are not so tractable as those of $S^2(\bar{x})$ and $S^2(\mu)$. However, we calculated their mean squared errors for sample sizes of $1, 2, \ldots, 30, 40, 50, \ldots, 100$, and in each of these instances the mean squared error of $S(\mu)$ was less than that of $S(\bar{x})$. Thus it seems that for the estimators $S(\bar{x})$ and $S(\mu)$ of the standard deviation σ our intuition holds while, for the estimators $S^2(\bar{x})$ and $S^2(\mu)$ of σ^2 it does not hold.

It is the purpose of this note to prove, in general, that our intuition does indeed hold for the above estimators of σ and that one should not ignore the fact that μ is known. That is, it will be shown that the mean squared error of $S(\mu)$ is always less than the mean squared error of $S(\bar{x})$ for arbitrary sample sizes $n \geqslant 2$.

Before launching into our proof, it is worth recalling a few fundamental relationships. We begin by giving the definition of mean squared error and then expressing it as a sum of expected values. Let $\hat{\theta}$ be an estimator for some arbitrary statistical quantity θ. Then the following is easily shown [6].

$$\mathrm{MSE}(\hat{\theta}) \equiv E\big((\hat{\theta} - \theta)^2\big) = E(\hat{\theta}^2) - 2\theta E(\hat{\theta}) + \theta^2, \tag{4}$$

where we have defined the symbol $E(x)$ to mean the expectation of x. Note that an estimator $\hat{\theta}$ is said to be unbiased if $E(\hat{\theta}) = \theta$. Finally, we note that the variance of $\hat{\theta}$ is defined by $\mathrm{Var}(\hat{\theta}) \equiv E[(\hat{\theta} - E(\hat{\theta}))^2]$. Clearly, $\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}(\hat{\theta})$ when $\hat{\theta}$ is an unbiased estimator.

We are now ready to begin our proof. We start by explicitly defining the two estimators for the standard deviation that we will consider.

$$S(\mu) \equiv \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2\right]^{1/2} \tag{5}$$

$$S(\bar{x}) \equiv \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})\right]^{1/2}, \tag{6}$$

where n is an integer (assumed greater than or equal to 2) representing the size of the sample. We note that $S(\mu)$ and $S(\bar{x})$ are the two estimators for σ that Birch and Robertson numerically consider.

The next thing we will do is obtain expressions for the mean squared error of $S(\mu)$ and $S(\bar{x})$. This is most easily done by noting the fact that the distributions of $nS^2(\mu)/\sigma^2$ and $nS^2(\bar{x})/\sigma^2$ are chi-square with n and $n-1$ degrees of freedom,

normal distribution, the variance of any unbiased estimate of σ is always greater than or equal to $\sigma^2/2n$ (Cramér [2] p. 484). Thus, using equation (4) and noting that $\sqrt{n/2}\,S(\mu)/R(n+1)$ is an unbiased estimate for σ, we have

$$\operatorname{MSE}\left(\sqrt{n/2}\,S(\mu)/R(n+1)\right) = \left(\frac{n}{2R^2(n+1)} - 1\right)\sigma^2 \geqslant \frac{\sigma^2}{2n}. \tag{13}$$

Factoring out σ^2, rearranging, and letting n go to $n-1$ yields the desired result:

$$R(n) \leqslant \frac{n-1}{\sqrt{2n-1}}. \tag{14}$$

The final result needed is an upper bound on the ratio of two "adjacent" $R(n)$'s. Specifically, we will show

$$\frac{R(n)}{R(n+1)} \leqslant \frac{2(n-1)}{2n-1}. \tag{15}$$

The proof of (15) is simple. Using (12) we can write immediately

$$\frac{R(n)}{R(n+1)} = \frac{2}{n-1}R^2(n). \tag{16}$$

Next, substitute the bound for $R(n)$ given in (14), and (15) follows.

Now we are ready to continue our proof. In particular we are ready to prove our claim that $T(n)$ is less than one. Using equation (12) we can write

$$T(n) = \frac{R(n)}{\sqrt{2n}\,(n-1)} + \frac{R(n)}{R(n+1)}. \tag{17}$$

Next use the upper bounds given in equations (14) and (15) to obtain

$$T(n) \leqslant \frac{\sqrt{2n-1} + \sqrt{2n}\,(2n-2)}{\sqrt{2n}\,(2n-1)}. \tag{18}$$

Noting that $\sqrt{2n-1}$ is strictly less than $\sqrt{2n}$, we will replace the term $\sqrt{2n-1}$ by $\sqrt{2n}$ and simplify. The result is

$$T(n) < \frac{1 + (2n-2)}{2n-1} = 1. \tag{19}$$

Therefore $T(n)$, as claimed, is less than one and we see from (10) that

$$\operatorname{MSE}(S(\bar{x})) > \operatorname{MSE}(S(\mu)). \tag{20}$$

Equation (20) explicitly demonstrates the numerical conjecture of Birch and Robertson. That is, for arbitrary sample size $n \geqslant 2$, we see that our intuition is indeed upheld and $S(\mu)$ is a better estimator for σ than is $S(\bar{x})$ when one uses mean squared error as their criteria.

Although the results of equation (20) are interesting in their own right, particularly when compared with those of equation (1), it is always nice to be able to

display a real-life example where the questions dealt with here are relevant. The problem we will briefly outline deals with light scattering in normal rabbit corneas.

High magnification electron micrographs reveal that the cornea is composed of uniform diameter collagen fibrils surrounded by an optically homogeneous macromolecular solution called the ground substance. The fibrils lie parallel to each other and extend entirely across the cornea. The electron micrographs show that the fibrils' positions are not random, but are instead correlated with local order extending over distances comparable to the wavelength of light [**4**].

A standard theoretical approach is to treat the fibrils as being infinitely long dielectric cylinders and compute the scattering to be expected from such a model [**4**]. One straightforward procedure is to sum the electric fields scattered by the individual fibrils. Using this approach gives expressions for the ensemble average of the scattered light intensity, $\langle I \rangle$, of the form

$$\langle I \rangle = \langle |\mathbf{E} - \langle \mathbf{E} \rangle|^2 \rangle, \tag{21}$$

where the angled brackets $\langle \quad \rangle$ denote ensemble average (expected value) and $\mathbf{E}$ is the scattered electric field [**3**]. The interesting point is that one knows what the one-particle distribution function is in this case; it is just ρ, the (constant) fibril number density [**5**]. Hence, one can compute $\langle \mathbf{E} \rangle$ exactly. However, $\langle I \rangle$, the quantity of interest, cannot be computed exactly since the two-particle distribution function is not known. Hence, one is faced with the problem of how best to estimate $\langle I \rangle$ given that they know $\langle \mathbf{E} \rangle$.

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Barycentric Representation for the Incenter and Excenters of a Triangle

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A triangle with vertices A, B, C and (opposite) sides of length a, b, c has many special points of interest, among them the incenter I (center of the inscribed circle), and the excenters I_a, I_b, I_c (centers of the three escribed circles, tangent to the sides

display a real-life example where the questions dealt with here are relevant. The problem we will briefly outline deals with light scattering in normal rabbit corneas.

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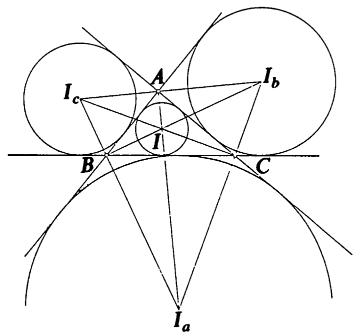
A triangle with vertices A, B, C and (opposite) sides of length a, b, c has many special points of interest, among them the incenter I (center of the inscribed circle), and the excenters I_a, I_b, I_c (centers of the three escribed circles, tangent to the sides

a, b, c, respectively). The purpose of this note is to indicate the simple barycentric representations of these points, which are useful in the classical discussion of the properties of the resulting configuration.

PROPOSITION. *The points I, I_a, I_b, I_c are uniquely given by the formulas*

$$I = \frac{aA + bB + cC}{a + b + c}, \quad I_a = \frac{-aA + bB + cC}{-a + b + c},$$

$$I_b = \frac{aA - bB + cC}{a - b + c}, \quad I_c = \frac{aA + bB - cC}{a + b - c}.$$



Proof. Let r be the radius of the incircle. Then we have for the areas

$$\triangle IBC = \tfrac{1}{2}ar, \qquad \triangle ICA = \tfrac{1}{2}br, \qquad \triangle IAB = \tfrac{1}{2}cr.$$

But it is well known that for any P the areas $\triangle PBC, \triangle PCA, \triangle PAB$ are proportional to the barycentric coordinates of P [**1**]. It follows that for the barycentric coordinates (α, β, γ) of the incenter I we have $\alpha : \beta : \gamma = a : b : c$, which proves the desired formula for I. The formulas for the excenters are established similarly taking care of orientations.

The result and proof extend to the incenter and excenters of a simplex in n-space, with the $(n - 1)$-dimensional content of the simplex opposite a vertex replacing the length.

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The Arithmetic-Geometric Mean Inequality Revisited: Elementary Calculus and Negative Numbers

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Just recently, in teaching theory of interest to actuarial students, I ran into a problem that used the long-known result that the arithmetic mean of n nonnegative numbers exceeds their geometric mean, except when all n numbers are identical in which case the two means coincide. [**1**, p. 39]. So I decided to produce a proof suitable for first year (i.e., one-variable) calculus students which appears to be new or at least little known. The other proofs in the literature require no calculus at all or, on the contrary, some understanding of advanced calculus.

The method produces various corollaries which I did not note in the literature. This in turn inspires a brief discussion of what becomes of the inequality when nonnegativity requirements are relaxed—a point rarely if ever treated heretofore.

Let $x_1, x_2, \ldots, x_n$ be n arbitrary nonnegative numbers. We wish to show

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \cdots x_n} \tag{1}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. All this clearly holds if any $x_i = 0$, so assume each $x_i > 0$. Now (1) is equivalent to

$$f(x_n) \equiv (x_n + k)^n - n^n c x_n \geqslant 0, \tag{2}$$

with $k \equiv x_1 + x_2 + \cdots + x_{n-1} > 0$; $c \equiv x_1 x_2 \cdots x_{n-1} > 0$. We proceed by induction on n; the induction hypothesis is

$$k \geqslant (n-1)c^{1/(n-1)}, \tag{3}$$

with equality if and only if $x_1 = \cdots = x_{n-1}$. This holds trivially for $n = 2, 3$. Now in (2) let x_n be a real variable x regardless of sign. Examining first and second derivatives shows f has its only relative minimum at

$$x_{\min} = nc^{1/(n-1)} - k. \tag{4}$$

Substituting in f and simplifying produces

$$f(x_{\min}) = n^n c\left[k - (n-1)c^{1/(n-1)}\right]. \tag{5}$$

Clearly (3) renders $f(x_{\min}) \geqslant 0$. Now if n is even, $x_{\min}$ is the only critical number for f, and $f(x_{\min})$ is an absolute minimum, whereas if n is odd, f also has a unique relative maximum at some $x_{\max} < 0$ where $x_{\max} < x_{\min}$—in either case (2) surely holds for nonnegative x_n. Also, $f(x_{\min}) = 0$ if and only if (3) is an equality if and only if (induction hypothesis) $x_1 = \cdots = x_{n-1}$; then, by (4), $x_{\min}$ (i.e., x_n) $= x_{n-1}$ also. This ends the proof.

Here are the promised corollaries. We leave others as exercise. What formula (2) and the resulting proof show is that *negative* x (i.e., x_n) can be such that (1) can still be satisfied. We have

COROLLARY 1a. *Let any $n-1$ of the n numbers $\{x_1, x_2, \ldots, x_n\}$ be fixed and exceed zero. (Without loss of generality, let x_n be the remaining number.) Let f be as in* (2).

Then, if n is odd

(a) *if at least two x_i, $i < n$, are unequal, f has a unique zero r, which is negative; also, inequality* (1) *holds if and only if $x \equiv x_n \geqslant r$, and* (1) *is an equality only at r.*

(b) *if all x_i are equal, $i \geqslant 2$, then f has precisely two zeroes $r_1 < 0$ and $r_2 > 0$; further,* (1) *holds if and only if $x \geqslant r_1$, with equality only at r_1 and r_2.*

If n is even, f is always nonnegative.

This yields unnoted extensions of the arithmetic–geometric mean inequality:

COROLLARY 1b. *Let n nonzero (for simplicity) numbers: $x_1, x_2, \ldots, x_n$, not necessarily all positive, have the property that some $n-1$ of them $\{x_1, x_2, \ldots, x_{n-1}\}$ have positive sum and positive product satisfying* (3). *Then the arithmetic-geometric mean inequality holds*

(a) *for odd n, for all $x \equiv x_n \geqslant r$ (or r_1) of Corollary* 1a.

(b) *for even n, for all $x_n \geqslant 0$. Further, if we raise both sides of inequality* (1) *to the nth power, then in that form the arithmetic-geometric mean inequality always holds, regardless of the sign of x_n.*

Perhaps this note will spark further investigation of the arithmetic-geometric inequality where the nonnegativity requirements are relaxed. Now we leave the following as exercises. Again, these seem unnoted in literature. Let

$$g(x_1, x_2, \ldots, x_n) \equiv (x_n + x_2 + \cdots + x_n)^n - n^n x_1 x_2 \cdots x_n \tag{6}$$

be a function of all n variables.

COROLLARY 2. *Let n be odd and let the x_i be any real numbers. In each of the $2^n - 2$ orthants of R^n, excepting the first and all-negative orthants, g is unbounded in both directions. If $c \leqslant 0$ in* (2), *then $g = 0$ renders x_n a function of x_i, $i < n$, if precisely an odd number of these x_i are nonpositive or if any is zero. (In this sense, $g = 0$ is a "unique" $n-1$ dimensional surface.)*

COROLLARY 3. *Let n be even. On each line in Cartesian n-space parallel to a coordinate axis, inequality* (1) *with both sides raised to the nth power holds except possibly for a single finite interval. Also, in each orthant, except the first and all negative orthants, g is unbounded in both directions.*

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Matrices Satisfying $AB - BA = I$

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If A and B are n by n matrices over a commutative ring with unity, under what conditions will $AB - BA = I$? This note gives necessary and sufficient conditions for the existence of such a pair and along the way shows how such pairs can be constructed.

The commutator relation $[A, B] = AB - BA = \lambda I$, λ a scalar, is one which has attracted attention from the standpoint of infinite matrices and then of linear operators [1], [5]. Max Born and Pascual Jordan observed that their "sharpened quantum conditions" $PQ - QP = (h/2\pi i)I$, could not be satisfied by finite matrices over the complex numbers because the trace of $PQ - QP$ is zero and that of $(h/2\pi i)I$ is not. This trace argument can be used for matrices over commutative rings to determine when $[A, B]$ cannot be I and to suggest when $[A, B] = I$ is possible. An argument making use of properties of free modules establishes the suggested existence result.

THEOREM. *Let R be a commutative ring with unity and $M_n(R)$ the full ring of n by n matrices over R. Then there exists $A, B \in M_n(R)$ such that $[A, B] = I$ if and only if the characteristic of R divides n.*

Proof. Since $0 = \text{trace } [A, B]$ and $\text{trace} I = n \cdot 1$, if $\text{char} R$ does not divide n, then $[A, B] \neq I$.

Let F be a free (unitary, left) module over R. Then every free basis for F has the same cardinality, which we call the dimension of F [2]. Take the dimension of F to be km, where $m = \text{char} R$, select a free basis for F, and write the basis as the disjoint union of the sets $B_j = \{b_{ij}: i = 0, \ldots, m-1\}$, $j = 1, \ldots, k$. Define α and β on $\cup B_j$ via their action on a typical B_j:

$$b_{0j}\alpha = 0; \qquad b_{ij}\alpha = ib_{i-1,j}, \qquad i = 1, \ldots, m-1; \qquad b_{ij}\beta = b_{i+1,j},$$
$$i = 0, \ldots, m-2; \qquad b_{m-1,j}\beta = 0.$$

Then,

$$b_{ij}(\beta\alpha - \alpha\beta) = b_{ij} \text{ (in particular, } b_{m-1,j}(\beta\alpha - \alpha\beta) = (1-m)b_{m-1,j},$$

which clearly shows the point where "$\text{char} R$ divides $\dim F$" is used). Next, extend α and β to R-endomorphisms α' and β' respectively and note that $[\beta', \alpha'] = 1_M$. The algebra of R-endomorphisms on M is isomorphic to $M_{km}(R)$, giving the desired matrices [4, p. 448].

Each step in the existence proof can be made explicitly constructive and an algorithm for building such pairs then follows. Clearly, many such pairs can be

*The author wishes to acknowledge the helpfulness of conversation and correspondence with Professor P. R. Halmos.

constructed for a given $M_n(R)$ by rearranging the basis elements. As an example, there are exactly twenty-four such pairs for $M_2(Z_2)$. It would be of interest to count and classify all such pairs where the matrices are over finite fields.

The more general (and more difficult) problem of determining conditions under which some $[A, B] = 1$, where A, B are from $M_n(R)$, R noncommutative, or where A, B are endomorphisms on modules, is addressed in [3].

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Consequences of the Memoryless Property for Random Variables

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Transistors and fledged robins, along with certain other physical objects and biological organisms, share a common property: they seem not to "age." The failure or death of the item appears to be the result of some sudden event rather than of gradual wear or deterioration. In other words, if the item has survived at least a time units, then the (conditional) probability that it survives at least an additional b units is independent of a. The item appears to "forget" how long it has survived or lived, in the sense that the probabilities for remaining life do not depend on the age of the item. Such a property is commonly called the memoryless or lack-of-memory property. We will consider the question of what sorts of distributions a memoryless random variable can have.

Formally, the random variable X is said to have the *memoryless property on* S if for all a, b in S,

$$P(X > a + b | X > a) = P(X > b). \tag{1}$$

Although in many situations, such as in the introductory examples, X represents a "lifetime" of an object or organism and S is thus a subset of $[0, \infty)$, we will consider X to be an arbitrary random variable and S to be any specified set of real numbers. We should also note here that no ordering of a and b is implied by equation (1), as we could just as well have written

$$P(X > a + b | X > b) = P(X > a).$$

It is a standard exercise in a beginning mathematical statistics course to show that if X is exponentially distributed, then X has the memoryless property on $(0, \infty)$, and if X is a geometric random variable, then X has the memoryless property on the set of positive integers (see [**1**, pp. 168–169 and 197] and [**2**, pp. 88 and 145]). The converses of these statements are rarely considered; if they are, it is usually a limited converse, such as: If X is a continuous random variable such that $P(X > 0) = 1$ and $P(X > x) > 0$ for $x > 0$, and if X satisfies the memoryless property on $(0, \infty)$, then X must be exponentially distributed [**3**, pp. 262–263].

In this article, we will investigate more general converses involving the memoryless property. We will show how the single assumption that X has the memoryless property on the set $\mathbf{R}$ of reals forces X to be exponential. We will also discover how this assumption must be weakened in order to obtain a converse in the discrete case.

well-known result concerning the solution of a certain functional equation. We include a proof for completeness:

LEMMA. *Suppose g is a continuous function on $[0, \infty)$ which satisfies the equation $g(x+y) = g(x) + g(y)$ for $x, y > 0$. Then $g(x) = g(1)x$.*

Proof. Suppose there is an $x_0 > 0$ such that $g(x_0) \neq g(1)x_0$. If we set $h(x) = g(x) - g(1)x$, then h satisfies the same functional equation as g, and $h(x_0) \neq 0$. Thus for any positive integer n, $h(nx) = nh(x)$, and so $\lim_{n\to\infty}|h(nx_0)| = \infty$. But we will show that this cannot be, since h is bounded. We note that $h(1) = 0$, so $h(x+1) = h(x)$. Hence h is periodic. Since h is continuous on $[0,1]$, it is bounded there, and by periodicity, bounded on $[0, \infty)$.

Our first converse is now at hand:

THEOREM 2. *Suppose X is a random variable which satisfies the memoryless property on $\mathbf{R}$. Then X is exponential.*

Proof. Let $g(x) = \ln[1 - F(x)] = \ln[P(X > x)]$. Part (c) of Theorem 1 and equation (2) guarantee that g satisfies the hypotheses of the Lemma, so $\ln[1 - F(x)] = (-1/\theta)x$ where $(-1/\theta) = \ln[1 - F(1)]$, with $\theta > 0$ since $F(1) > 0$. Thus $F(x) = 1 - \exp(-x/\theta)$ for $x \geqslant 0$. Hence, X is an exponentially distributed random variable.

In order to obtain a converse in the discrete case, clearly we will have to weaken the hypothesis that equation (1) holds for all reals. Before doing so, we recall that if X is a discrete random variable, then its range $R_X = \{x | P(X = x) > 0\}$ is a (nonempty) discrete set; that is, the intersection of R_X with any bounded interval contains at most a finite number of elements. Suppose we first weaken the hypothesis that X has the memoryless property on $\mathbf{R}$ to the statement that (1) holds whenever a or b is in R_X (the other still may be any real number). This now implies that R_X is a subset of $(0, \infty)$, for if R_X contains a nonpositive element b, then for any real $a, a + b \leqslant a$, so that $P(X > b) = P(X > a + b | X > a) = 1$. Thus $P(X = b) = 0$, a contradiction. But if R_X contains only positive reals, it must have a smallest positive element t. We will assume that $P(X = t) < 1$, since otherwise $R_X = \{t\}$, which is a trivial case.

Since $P(X > 0) = 1$, we again need only consider the memoryless property for positive a and b, and thus we will again use the probability statement given by (2) for the memoryless property. We can now obtain the desired converse for discrete random variables:

THEOREM 3. *Let X be discrete with range R_X. Suppose (2) holds for positive a, b such that a or b (or both) are in R_X. Then*

(a) *there is a $t > 0$ such that $R_X = \{nt | n = 1, 2, 3, \ldots\}$; and*

(b) *if we set $Y = X/t$, then Y is a geometric random variable.*

Proof. (a) Since R_X has a smallest positive element t, clearly $t \in R_X$. We will use mathematical induction to show that $nt \in R_X$ for $n \geqslant 1$. Assume $kt \in R_X$ for

$k = 1, 2, \ldots, n$; $n \geqslant 1$. Then

$$P(X = (n+1)t) = \lim_{\varepsilon \to 0^+}[P(X > (n+1)t - \varepsilon) - P(X > (n+1)t)].$$

But $P(X > (n+1)t - \varepsilon) = P(X > nt)P(X > t - \varepsilon) = P(X > nt)$ for all ε in $(0, t)$, and $P(X > (n+1)t) = P(X > nt)P(X > t)$, so that $P(X = (n+1)t) = P(X > nt)[1 - P(X > t)]$. Using (2) n times, we obtain $P(X = (n+1)t) = [P(X > t)]^n F(t) > 0$, so that $(n+1)t \in R_X$. To show that multiples of t are the only elements of R_X, we need to show that if $nt < x < (n+1)t$, then $F(x) = F(nt)$, or $P(X > x) = P(X > nt)$. But $x = (n+1)t - \varepsilon$ for some ε in $(0, t)$, and it has just been shown that $P(X > (n+1)t - \varepsilon) = P(X > nt)$.

(b) The range of Y is clearly $\{1, 2, 3, \ldots\}$. Let $P(Y > 1) = 1 - p$, $0 < p < 1$, and let G denote the distribution function for Y. Then $P(Y > n) = [P(Y > 1)]^n = (1 - p)^n$, so that $G(n) = 1 - (1 - p)^n$ for $n \geqslant 1$, and thus Y is geometric.

The second part of the conclusion in Theorem 3 may at first glance appear surprising, in that a function of X is geometric rather than X itself. This is in contrast to the result in Theorem 2, where we obtained just the exponential family of random variables. But note that if Y is exponentially distributed with parameter θ, and if $Y = X/t$ for $t > 0$, then X is also exponentially distributed, but with parameter $t\theta$.

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How Electronics Ended the Poisson Approximation to the Binomial Distribution

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1. Background. Sooner or later every serious student of probability is exposed to the Poisson approximation to the binomial distribution [1]. Simply stated, if n is "large" and p is "small," then, for $0 \leqslant k \leqslant n$ (k an integer), the binomial probability,

$$b_k = \binom{n}{k} p^k q^{n-k},$$

$k = 1, 2, \ldots, n$; $n \geqslant 1$. Then

$$P(X = (n+1)t) = \lim_{\varepsilon \to 0^+}[P(X > (n+1)t - \varepsilon) - P(X > (n+1)t)].$$

But $P(X > (n+1)t - \varepsilon) = P(X > nt)P(X > t - \varepsilon) = P(X > nt)$ for all ε in $(0, t)$, and $P(X > (n+1)t) = P(X > nt)P(X > t)$, so that $P(X = (n+1)t) = P(X > nt)[1 - P(X > t)]$. Using (2) n times, we obtain $P(X = (n+1)t) = [P(X > t)]^n F(t) > 0$, so that $(n+1)t \in R_X$. To show that multiples of t are the only elements of R_X, we need to show that if $nt < x < (n+1)t$, then $F(x) = F(nt)$, or $P(X > x) = P(X > nt)$. But $x = (n+1)t - \varepsilon$ for some ε in $(0, t)$, and it has just been shown that $P(X > (n+1)t - \varepsilon) = P(X > nt)$.

(b) The range of Y is clearly $\{1, 2, 3, \ldots\}$. Let $P(Y > 1) = 1 - p$, $0 < p < 1$, and let G denote the distribution function for Y. Then $P(Y > n) = [P(Y > 1)]^n = (1 - p)^n$, so that $G(n) = 1 - (1 - p)^n$ for $n \geqslant 1$, and thus Y is geometric.

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$$b_k = \binom{n}{k} p^k q^{n-k},$$

is given, to a good degree of approximation, by the Poisson probability,

$$p_k = e^{-np}\frac{(np)^k}{k!}.$$

The approximation is generally felt to be good when $p \leqslant 0.10$ and np is of moderate size, say, no more than 20. The same result may be used when p is "large" simply by interchanging the meaning of p and $q = 1 - p$ in the problem. Such an approximation was extremely valuable in the past because of the rather extensive tabulation of Poisson probabilities **[2]** and the lack of availability of binomial tables **[3]** in precisely those parameter circumstances where the approximation applies.

With the advent of high-speed computing devices, however, such approximations become less useful as the capability of replacing tables altogether becomes more common. The purpose of this paper is to show precisely how this happens for the particular case under discussion.

2. Electronic Tables. Both the binomial and Poisson probability functions admit a simple recursion relation wherein the kth term may be expressed in terms of the preceding one. In the binomial case, it is straightforward to discover the relationship

$$b_k = \frac{(n-k+1)p}{kq} * b_{k-1} \qquad k = 1,2,\ldots,n.$$

Using an initial value of

$$b_0 = q^n$$

it is then possible to generate binomial probabilities recursively, summing them if the cumulative probabilities, here denoted $B(k)$, are desired.

Similarly, Poisson probabilities may be generated using the recursion relation

$$p_k = \frac{np}{k} * p_{k-1}, \qquad k = 1,2,\ldots,$$

with initial value

$$p_0 = e^{-np}$$

(np is replaced by a parameter μ for the general case). Cumulative Poisson probabilities will be denoted $P(k)$. In each of these cases it might be noted that the computation of factorials is obviated.

Both of these relationships are easily programmed in a variety of programming languages for various computing devices. These include programmable hand-held calculators (here viewed as miniature computers) such as the HP 41-C series **[4]**. If storage is available and parameter choices are reasonable, a table of probabilities, including cumulative, may be stored and recalled on demand. This is the method often used by some of the current commercial software packages such as Minitab **[5]**. When storage is at a premium, we can think of the "table" as available on demand for any choice of k but not entirely visible. In that case the required

probabilities must be calculated over and over again for each choice of k. That would seem cumbersome and inefficient were it not for the speed with which such calculations can be carried out by modern computers.

The only real restriction on the computing device, then, will be its ability to calculate the initial values in the two cases. In the binomial case this is q^n, which is typically computed internally as $e^{n \ln q}$. The Poisson requirement, similarly, is to be able to calculate e^{-np}. Either way, computation will depend upon the computer's ability to handle e^r. To illustrate some cases, the IBM PC using BASIC requires $r \geqslant -88$, roughly, while the HP 41-C will allow $r \geqslant -228$. With the 8087 Numeric Data Processor chip installed in the IBM PC, software packages such as APL [**6**] and True BASIC [7] will extend the range to about $r \geqslant -700$. For most applications even the minimum configuration provides a generous assortment of parameter choices.

3. Some Examples. To gain an appreciation for the range of parameter choices, and to provide a basis for further discussion, the following examples are provided. Each is a slight modification of one found in a current textbook as an instance where the Poisson approximation is called for because the binomial calculation is impractical if not "impossible."

Suppose you are dealing with 8,000 Bernoulli trials in which there is a constant probability of $p = 0.0003$ for "success." Computing, say, the probability of obtaining as many as 5 successes is a problem that, on face value, would be a natural choice for the Poisson approximation. For p is small and $np = 2.4$ is moderate and within the reach of most textbook tables. (It would only take a slight modification to take it out of that range, however.) Even term-by-term calculation of the Poisson probabilities would not be too difficult (perhaps bothersome) in this case to find the Poisson value of $P(5) = 0.9643$. But the more exact value $B(5) = 0.9644$ is calculated by the HP calculator in about six seconds; the IBM PC produces the answer about as quickly as the parameters can be entered. Equating turning on the computer and entering the parameters to locating a set of tables and finding the right page, row, and column, there is little, if any, gain in time using the Poisson approximation; there is, however, a little loss in accuracy, albeit not serious in this example.

To cite another case, for $n = 200$ trials with $p = 0.05$ the value of $B(9)$ is required. Here, $np = 10$ and the Poisson approximation of $P(9) = 0.4579$ can be easily found in most Poisson tables. For this case the HP calculator slows to 10 seconds to give the more exact binomial answer $B(9) = 0.4547$; again, the IBM PC produces this answer almost instantly.

Still another example involves $n = 1{,}000$, $p = 0.1$ and $B(85)$ is required. The binomial answer using the HP calculator is 0.0607 and this time it takes just over a minute to compute it. Ordinary BASIC with the IBM PC cannot handle the problem because the initial value is out of range. But a compiled version of BASIC using the 8087 chip produces the answer almost instantly. The Poisson approximation in this case will not be easy to find in tables with $np = 100$ so large. Using the

computer routine just mentioned, it was found to be 0.0708, not terribly good in this case since np is not moderate. The normal approximation of 0.0632 is (expectedly) better but still unnecessary since the more accurate value is so easily computed.

4. Conclusions. When will the Poisson approximation to the binomial be useful then? It will be hard to find a case. There is a simple mathematical reason for this. Using L'Hôpital's rule, it is easy to verify that

$$\lim_{p \to 0} \frac{\ln(1 - p)}{p} = -1.$$

Hence, for every n, the quantities $n \ln q$ and $-np$ are approximately equal for very small values of p. For the application at hand this means that, roughly, the initial values b_0 and p_0 are either both computable or else neither can be computed. In short, precisely when the Poisson approximation is needed, due to the fact that the binomial probability cannot be computed, the Poisson probability will usually not be calculable either. Moreover, when it can, the parameter choices will involve a large binomial mean np. In that case, the normal approximation is a better option anyway. Otherwise, as noted in the examples, there is nothing to be gained by resorting to the approximation.

It might be argued that not everyone has an electronic device at hand when a calculation is needed. But that is getting to be less and less true every day. Current technology promises even more powerful and faster machines at less and less cost. As it becomes more commonplace to be working in such an environment, it is surely no more trouble to turn on a computer than to reach for a set of tables. And the computer program will consistently produce more accurate answers with less chance for numerical error.

Does this mean that the Poisson approximation to the binomial distribution should be abandoned? Certainly not. It is one of the few elementary limit theorems we have in probability theory and it has intrinsic value on that account alone. But, as practical computation for otherwise intractable binomial parameters, there is little to commend it. This is but one example of the impact of the electronic age both on what we say in the classroom and what we write in textbooks.

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Some Applications of the Bounded Convergence Theorem for an Introductory Course in Analysis

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The Arzela bounded convergence theorem is the special case of the Lebesgue dominated convergence theorem in which the functions are assumed to be Riemann integrable.

THE BOUNDED CONVERGENCE THEOREM. *Suppose (f_n) is a sequence of functions which are Riemann integrable on an interval $[a, b]$, suppose that the sequence (f_n) converges pointwise to a function f, and suppose that there exists a number K such that $|f_n(x)| \leqslant K$ for all $n \in \mathbb{Z}^+$ and $x \in [a, b]$. Then the sequence of integrals $\int_a^b f_n(x)\,dx$ converges, and in the event that the function f is also Riemann integrable on $[a, b]$, we have*

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Because the proof of this theorem has traditionally been perceived as quite hard, or dependent on concepts which lie beyond a first course in analysis, the theorem is presently omitted in such courses, and its applications at this level have therefore been somewhat neglected. However, a recent paper **[3]** of the author shows that the bounded convergence theorem can be proved quite easily in a first course, and it is therefore worth knowing what its applications might be. In this paper we shall show how the bounded convergence theorem may be used to obtain simple proofs of some quite sharp forms of the theorems which concern differentiation under the integral sign and inversion of repeated integrals. We shall obtain versions of these theorems which are distinctly sharper than the results usually found in an undergraduate text.

Differentiation under the integral sign. In a typical first course in analysis, the theorems on differentiation under the integral sign are given for continuous functions only (see, for example, Buck **[2]** Theorems 10 and 29, or Apostol **[1]**, Theorem 7.40). However, using the bounded convergence theorem, it is easy to drop the requirement of continuity, and obtain sharper theorems of the type one might expect to see at a more advanced level using Lebesgue integrals. A theorem of the sharper type may be found in [**1**, Theorem 10.39], three chapters beyond Theorem 7.40, in the chapter on Lebesgue integration.

THEOREM ON DIFFERENTIATING UNDER THE INTEGRAL SIGN. *Suppose $f\colon [a, b] \times S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, and that for every point $y \in S$, the Riemann integral*

$$\phi(y) = \int_a^b f(x, y)\,dx$$

exists. Suppose y_0 is both a point of S and a limit point of S, and that for every

$x \in [a,b]$, the partial derivative

$$D_2 f(x, y_0) = \lim_{y \to y_0} \left[\frac{f(x, y) - f(x, y_0)}{y - y_0} \right]$$

exists, and suppose that the Riemann integral $\int_a^b D_2 f(x, y_0)\, dx$ exists. Suppose finally that there exists a number K such that for all $x \in [a, b]$ and $y \in S \setminus \{y_0\}$, we have

$$\left\| \left[\frac{f(x, y) - f(x, y_0)}{y - y_0} \right] \right\| \leqslant K.$$

Then

$$\phi'(y_0) = \int_a^b D_2 f(x, y_0)\, dx$$

Proof. We deduce this theorem directly from the bounded convergence theorem. Given any sequence (y_n) in $S \setminus \{y_0\}$, converging to y_0, we have

$$\frac{\phi(y_n) - \phi(y)}{y_n - y} = \int_a^b \left[\frac{f(x, y_n) - f(x, y_0)}{y_n - y_0} \right] dx \to \int_a^b D_2 f(x, y_0)\, dx \quad \text{as } n \to \infty.$$

A somewhat weaker but less clumsy form of this theorem is:

Suppose $f\colon [a, b] \times S \to \mathbb{R}$, where S is an interval, and that for every point $y \in S$, the Riemann integral

$$\phi(y) = \int_a^b f(x, y)\, dx$$

exists. Suppose that for every $x \in [a, b]$ and for every $y \in S$, the partial derivative $D_2 f(x, y)$ exists, and that the Riemann integral $\int_a^b D_2 f(x, y)\, dx$ exists. Suppose finally that there exists a number K, such that for all $x \in [a, b]$, and $y \in S$, we have $|D_2 f(x, y)| \leqslant K$. Then for every $y \in S$, we have

$$\phi'(y) = \int_a^b D_2 f(x, y)\, dx.$$

The useful analogues of this theorem for improper Riemann integrals can be deduced almost as simply, using an obvious "dominated convergence" analogue of the bounded convergence theorem which would apply to improper Riemann integrals. As an example of the sort of result that can be obtained, we cite the following:

THEOREM ON DIFFERENTIATING AN IMPROPER INTEGRAL UNDER THE INTEGRAL SIGN. *Suppose $-\infty < a < b \leqslant \infty$, S is an interval, and that $f\colon [a, b) \times S \to \mathbb{R}$. Suppose that for every point $x \in [a, b)$, the function $f(x, \cdot)$ is differentiable on S, and that for every point $y \in S$, the functions $f(\cdot, y)$ and $D_2 f(\cdot, y)$ are improper Riemann integrable on $[a, b)$, and suppose finally that there exists an improper Riemann integrable function g on $[a, b)$ such that for all $x \in [a, b)$ and $y \in S$, we have $|D_2 f(x, y)| \leqslant g(x)$.*

Then if we define

$$\phi(y) = \int_a^{\to b} f(x, y)\, dx$$

for all $y \in S$, we have

$$\phi'(y) = \int_a^{\to b} D_2 f(x, y)\, dx$$

at every point $y \in S$.

Inversion of repeated integrals. The sharpest known result on inversion of iterated Riemann integrals is the elegant result that was proved in 1913 by G. Fichtenholz. We shall state three versions of Fichtenholz's theorem. The first of these is the easiest to prove, the second is the best possible result for Riemann integrable functions, and the third form is the ultimate theorem on the inversion of iterated integrals for a bounded function defined on a rectangle. In this third form of the theorem, we see that the theorem remains true even if some of the integrals are only assumed to be Lebesgue integrals.

FICHTENHOLZ'S THEOREM ON INVERSION OF ITERATED INTEGRALS FIRST FORM. *Suppose f is a bounded function on the rectangle $[a, b] \times [c, d]$. Then the identity*

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

will hold if both sides exist as repeated Riemann integrals.

SECOND FORM. *Suppose f is a bounded function on the rectangle $[a, b] \times [c, d]$. Suppose that for every point $x \in [a, b]$, the function $f(x, \cdot)$ is Riemann integrable on $[c, d]$, and that for every point $y \in [c, d]$, the function $f(\cdot, y)$ is Riemann integrable on $[a, b]$. Then*

(a) *The function $\phi\colon [a, b] \to \mathbb{R}$ defined by $\phi(x) = \int_c^d f(x, y)\, dy$ for all $x \in [a, b]$, is Riemann integrable on $[a, b]$,*
(b) *The function $\psi\colon [c, d] \to \mathbb{R}$ defined by $\psi(y) = \int_a^b f(x, y)\, dx$ for all $y \in [c, d]$, is Riemann integrable on $[c, d]$,*
(c) *$\int_a^b \phi(x)\, dx = \int_c^d \psi(y)\, dy$, in other words,*

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

THIRD FORM. *Suppose f is a bounded function on the rectangle $[a, b] \times [c, d]$. Suppose that for every point $x \in [a, b]$, the function $f(x, \cdot)$ is Riemann integrable on $[c, d]$, and that for every point $y \in [c, d]$, the function $f(\cdot, y)$ is Lebesgue measurable on $[a, b]$. Then*

(a) *The function $\phi\colon [a, b] \to \mathbb{R}$ defined by $\phi(x) = \int_c^d f(x, y)\, dy$ for all $x \in [a, b]$, is Lebesgue measurable on $[a, b]$,*
(b) *The function $\psi\colon [c, d] \to \mathbb{R}$ defined by $\psi(y) = \int_a^b f(x, y)\, dx$ for all $y \in [c, d]$, is Riemann integrable on $[c, d]$,*

(c) $\int_a^b \phi(x)\,dx = \int_c^d \psi(y)\,dy$, *in other words*,

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

Proof of the first form. For each natural n, denote as $\mathscr{P}_n$ the regular n-partition of $[c, d]$. For $i = 1, \ldots, n$, the ith point of $\mathscr{P}_n$ is, of course, $c + i(d - c)/n$, but for simplicity, we shall denote this as y_{ni}. For each natural n and $x \in [a, b]$, define

$$\phi_n(x) = \sum_{i=1}^{n} f(x, y_{ni})(y_{ni} - y_{ni-1}).$$

Since the function $f(x, \cdot)$ is Riemann integrable for every $x \in [a, b]$ and since $\|\mathscr{P}_n\| \to 0$, it follows from Darboux's theorem that $\phi_n(x) \to \phi(x)$ for each $x \in [a, b]$. Since we have also assumed that ϕ is Riemann integrable on $[a, b]$, it follows from the bounded convergence theorem that

$$\int_a^b \phi_n(x)\,dx \to \int_a^b \phi(x)\,dx = \int_a^b \int_c^d f(x, y)\,dy\,dx \quad \text{as} \quad n \to \infty.$$

But for each n, we have

$$\begin{aligned}\int_a^b \phi_n(x)\,dx &= \int_a^b \sum_{i=1}^{n} f(x, y_{ni})(y_{ni} - y_{ni-1})\,dx \\ &= \sum_{i=1}^{n} \left[\int_a^b f(x, y_{ni})\,dx\right](y_{ni} - y_{ni-1}) \\ &= \sum_{i=1}^{n} \psi(y_{ni})(y_{ni} - y_{ni-1}),\end{aligned}$$

and since ψ is Riemann integrable on $[c, d]$, the latter expression approaches $\int_c^d \psi(y)\,dy$ as $n \to \infty$.

This shows that $\int_a^b \phi(x)\,dx = \int_c^d \psi(y)\,dy$ which is what we had to prove.

Proof of the second form. The difference between this second form of the theorem and the first form, is that the Riemann integrability of the functions ϕ and ψ is now part of the conclusion. What we have to show therefore, is that ϕ and ψ are automatically Riemann integrable on $[a, b]$ and $[c, d]$, respectively. As above, let $\mathscr{P}_n$ be the regular n-partition of $[c, d]$ for each natural n, and denote the ith point of $\mathscr{P}_n$ as y_{ni}. To show that ψ is Riemann integrable on $[c, d]$, we shall show that there is a number L such that for every possible choice of numbers t_{ni} in the intervals $[y_{ni-1}, y_{ni}]$ we have

$$\sum_{i=1}^{n} \psi(t_{ni})(y_{ni} - y_{ni-1}) \to L \quad \text{as} \quad n \to \infty.$$

Let us look for the moment at one possible choice of the nunbers t_{ni}. For each

natural n and $x \in [a, b]$, define

$$\phi_n(x) = \sum_{i=1}^{n} f(x, t_{ni})(y_{ni} - y_{ni-1}).$$

Since the function $f(x, \cdot)$ is Riemann integrable for every $x \in [a, b]$ and since $\|\mathscr{P}_n\| \to 0$, it follows from Darboux's theorem that $\phi_n(x) \to \phi(x)$ for each $x \in [a, b]$. It, therefore, follows from the bounded convergence theorem that the sequence of integrals $\int_a^b \phi_n(x)\,dx$ converges. The limit of this sequence of integrals is obviously independent of the choice of numbers t_{ni} for if t_{ni}^* is another choice, and the functions ϕ_n^* are defined analogously by

$$\phi_n^*(x) = \sum_{i=1}^{n} f(x, t_{ni}^*)(y_{ni} - y_{ni-1}) \quad \text{for} \quad x \in [a, b],$$

then we also have $\phi_n^*(x) \to \phi(x)$ for all $x \in [a, b]$ and the bounded convergence theorem implies that $\int_a^b [\phi_n(x) - \phi_n^*(x)]\,dx \to 0$. Now for each n, we have

$$\int_a^b \phi_n(x)\,dx = \sum_{i=1}^{n} \psi(t_{ni})(y_{ni} - y_{ni-1})$$

and, therefore, the latter expression tends to a limit as required. This shows that ψ is Riemann integrable on $[c, d]$. The proof that ϕ is Riemann integrable on $[a, b]$ is similar.

Proof of the third form. As in the proof of the second form, we need to show that ψ is Riemann integrable on $[c, d]$. The proof we use now is similar to the one used before except that this time, we have to make use of the Lebesgue dominated convergence theorem. As before, denote as $\mathscr{P}_n$ the regular n-partition of $[c, d]$ and the ith point of $\mathscr{P}_n$ as y_{ni}. We shall prove the theorem by showing that ϕ is measurable on $[a, b]$, and that for every possible choice of numbers t_{ni} in the intervals $[y_{ni-1}, y_{ni}]$ for $n = 1, \ldots$ and $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} \psi(t_{ni})(y_{ni} - y_{ni-1}) \to \int_a^b \phi(x)\,dx.$$

Suppose then, that the numbers t_{ni} have been chosen. For each natural n and $x \in [a, b]$, define

$$\phi_n(x) = \sum_{i=1}^{n} f(x, t_{ni})(y_{ni} - y_{ni-1}),$$

and notice that each function ϕ_n, being a linear combination of Lebesgue measurable functions, is Lebesgue measurable on $[a, \text{b}]$. As above, it follows from the Riemann integrability of the functions $f(x, \cdot)$ that $\phi_n(x) \to \phi(x)$ for each $x \in [a, b]$. Therefore ϕ is Lebesgue measurable on $[a, b]$ and it follows from the Lebesgue dominated convergence theorem that

$$\int_a^b \phi_n(x)\,dx \to \int_a^b \phi(x)\,dx$$

and the result follows as before from the identity

$$\int_a^b \phi_n(x)\,dx = \sum_{i=1}^{n} \psi(t_{ni})(y_{ni} - y_{ni-1}).$$

An interesting (and possibly surprising) feature of Fichtenholz's theorem is the fact that it makes no requirement of integrability of f jointly in the two variables x and y. The theorem is, therefore, quite different in character from Fubini's theorem and from the theorems on pages 111–114 of Buck [**2**] and those in Section 7.25 of Apostol [**1**]. As is well known, if the Continuum Hypothesis is assumed, then the analogue of Fichtenholz's theorem for Lebesgue integrals is not even true; see Rudin [**5**, page 152]. This means that the above requirement of Riemann integrability of the function with respect to at least one of its variables is really needed. Some further counterexamples may be found in Luxemburg [**4**], which also contains a significant generalization of Fichtenholz's theorem to some abstract theories of integration. But it should be mentioned that one of the examples cited by Luxemburg is incorrect, possibly a result of a misreading of Proposition C_{49} in Sierpiński [**6**]. Luxemburg cites the incorrect example as a counter example to the above third form of Fichtenholz's theorem.

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of the decimal digits of n, and $\tau(n)$ is the number of positive divisors of n.

E 3239. *Proposed by M. S. Klamkin, University of Alberta.*

Show that if $\mathbf{A}$ is any three-dimensional vector and $\mathbf{B}, \mathbf{C}$ are unit vectors, then

$$[(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} + \mathbf{C})] \times (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{B} + \mathbf{C}) = 0.$$

Interpret the result as a property of spherical triangles.

E 3240. *Proposed by C. L. Mallows, AT & T Bell Laboratories, Murray Hill, NJ.*

The tennis players in a certain club compete in singles matches. After each match is played, a ranking of the players is computed according to the following rule. Starting with the most recent match, and working back through time, use the match results to build up a partial order. Ignore any match that is inconsistent with more recent results. Stop when the partial order is a complete order.

Prove or disprove: a player cannot improve his position by intentionally losing a match.

E 3241. *Proposed by Gregory P. Wene, University of Texas at San Antonio.*

Suppose that a, b, c are given natural numbers with $a < b < c$.

(i) Show that a function $f\colon N \to N$ is uniquely defined by the following pair of formulas:

$$f(n) = n - a \qquad \text{if } n > c,$$

$$f(n) = f(f(n + b)) \qquad \text{if } n \leqslant c.$$

(ii) Determine a necessary and sufficient condition for f to have at least one fixed point.

(iii) Give such a fixed point explicitly in terms of a, b, and c.

E 3242. *Proposed by J. B. Miles and L. A. Rubel, University of Illinois at Urbana-Champaign.*

It can be proved, using Picard's theorem or Nevanlinna theory, that if f is an entire function which is real precisely when z is real, then $f(z) = az + b$, where a and b are real numbers and $a \neq 0$. Find a more elementary proof.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Convergence of Means

E 2937 [1982, 213]. *Proposed by A. N. Philippou and G. D. Stamatelos, University of Patras, Greece.*

Suppose γ_n is a sequence such that $\gamma_{2n} \to \alpha$, $\gamma_{2n+1} \to \beta$. Show that

$$\frac{1}{n}\sum_{j=1}^{n} \frac{j}{j+1}\gamma_j \to \frac{1}{2}(\alpha+\beta).$$

Solution by Artin Boghossian, University of Petroleum and Minerals, Dhahran, Saudi Arabia. Let $x_j = (j/(j+1))\gamma_j$ and $y_j = (x_j + x_{j+1})/2$ and observe the identity

$$\frac{1}{n}\sum_{j=1}^{n} \frac{j}{j+1}\gamma_j = \frac{1}{n}\sum_{j=1}^{n} x_j = (x_1 - x_{n+1})/(2n) + \frac{1}{n}\sum_{j=1}^{n} y_j.$$

Since the x_n are bounded, $(x_1 - x_{n+1})/(2n) \to 0$. It follows from our assumptions that $y_{2j} \to (\alpha+\beta)/2$ and $y_{2j+1} \to (\alpha+\beta)/2$; therefore, $y_j \to (\alpha+\beta)/2$. But a theorem of Cauchy says that if $a_j \to a$, then $(1/n)\sum_{j=1}^{n} a_j \to a$. An application of this result to the sequence y_j completes the proof.

Editorial note. Many of those who submitted solutions remarked that the factor $j/(j+1)$ is superfluous. For any sequence

$$b_j \to b, \qquad \frac{1}{n}\sum_{j=1}^{n} b_j\gamma_j \to \frac{1}{2}(b\alpha + b\beta);$$

just replace the sequence x_j by $\hat{x}_j = b_j\gamma_j$ and replace α and β by $\hat{\alpha} = b\alpha$ and $\hat{\beta} = b\beta$ in the argument above.

Also solved by 45 other readers and the proposers.

Bandwidth of the Integer Simplex

E 3003 [1983, 400]. *Proposed by J. L. Brenner, Palo Alto, California.*

The 20 balls in a 4-layered pyramid (of balls) are labeled with the integers 1–20 (each integer being used). The maximum of the difference between the labels of two kissing balls is the "discrepancy." Find the *minimum possible discrepancy* (= bandwidth) under all 20! labelings. *What is the bandwidth for the pyramid of $k(k+1)(k+2)/6$ balls in k layers?

Solution by Douglas B. West, University of Illinois, Urbana, IL. For $k = 4$ the answer is 8, and in general $\left(\binom{k+2}{3} - 1\right)/(k-1)$ is a lower bound and $\lceil (k+3)/2 \rceil \lfloor (k+1)/2 \rfloor$ is an upper bound.

More generally, consider the problem of labeling the n vertices of a graph G with labels $1, \ldots, n$, attempting to minimize the maximum difference between adjacent labels. This minimum, over all possible labelings, is called the *bandwidth* $b(G)$; the term comes from the bandwidth of the adjacency matrix. The problem posed is to compute the bandwidth of the "adjacency graph" of the k-layered pyramid, which we denote G_k. The graph G_k can be described as follows. The vertices correspond to

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by at most 7. In other words, $b(G_4) = 8$. To prove this, suppose that G_4 has a vertex labeling f with maximum difference 7; we will obtain a contradiction.

We want to show we may assume that f assigns labels in order by Types. Within distance two of labels 1 and 2 we have at most the 14 labels $2, \dots, 15$ and the 15 labels $\{1, \dots, 16\} - \{2\}$, respectively; similarly for labels 20 and 19. Let us compute how many vertices are within distance two for each of the three types of vertices (permutations of 3000, 2100, and 1110). Note that $d(v, w) = 3$ if and only if v, w are non-zero on disjoint sets of coordinates. Since there are $\binom{r+i-1}{i-1}$ compositions of r restricted to i coordinates, these vertices have $10, 4, 1$ vertices at distance three, respectively, or $9, 15, 18$ at distance one or two. Hence $f^{-1}(1)$ and $f^{-1}(20)$ must be extremes; we may assume they are 3000 and 0003, respectively. Similarly, $f^{-1}(2)$ and $f^{-1}(19)$ must be extremes or permutations of 2100. If $f^{-1}(2)$ is not an extreme, then the 15 vertices within distance two of $f^{-1}(2)$ must include $f^{-1}(1)$. By symmetry in the central coordinates, we may thus assume $f^{-1}(2) \in \{0300, 2100, 1200\}$ and $f^{-1}(19) \in \{0030, 0012, 0021\}$.

Now, vertices labeled $17, 18, 19, 20$ must have distance three from both $f^{-1}(1)$ and $f^{-1}(2)$; hence these must be the four Type 0 vertices $\{0012, 0021, 0030, 0003\}$ that are non-zero only where both $f^{-1}(1)$ and $f^{-1}(2)$ are zero. Similarly, $1, 2, 3, 4$ label the Type 3 vertices $\{3000, 0300, 2100, 0012\}$. Furthermore, if v is Type 2, then v has two Type 3 neighbors. Therefore, v has a neighbor w with $f(w) \leqslant 3$, and hence $f(v) \leqslant 10$. Similarly $f(v) \geqslant 11$ if v has Type 1, so Type 2 vertices have the labels $5, \dots, 10$ and Type 1 vertices have the labels $11, \dots, 16$.

Next we restrict the values of $f^{-1}(16)$ and $f^{-1}(5)$. A Type 1 vertex v of the form $xy11$ has four neighbors of Type 2; hence it has a neighbor w with $f(w) \leqslant 7$. This implies $f(v) \leqslant 14$, so $f^{-1}(15)$ and $f^{-1}(16)$ must have the form $xy02$ or $xy20$. Since $d(f^{-1}(1), f^{-1}(16)) = 3$, we have $f^{-1}(16) \in \{0102, 0120\}$. In either case, $f^{-1}(16)$ has two neighbors of Type 2 (these are $\{1101, 0201\}$ or $\{1110, 0210\}$), which must have labels $9, 10$. Either of these adjacent to $f^{-1}(2)$ must have label 9. Thus the choices for locating $2, 16$ determine the locations for $9, 10$; the four possible locations for $2, 9, 10, 16$ are (0300, 0201, 1101, 0102), (0300, 0210, 1110, 0120), (2100, 1101, 0201, 0102), and (2100, 1110, 0210, 0120). Note that $f^{-1}(2) = 1200$ is no longer possible, since 1200 is adjacent to all potential Type 2 neighbors of $f^{-1}(16)$. By the complementary argument, the possible locations for $19, 12, 11, 5$ are (0030, 1020, 1011, 2010), (0030, 0120, 0111, 0210), (0012, 1011, 1020, 2010), and (0012, 0111, 0120, 0210).

We have shown $f^{-1}(16) \in \{0102, 0120\}$ and $f^{-1}(5) \in \{2010, 0210\}$. By symmetry, we have three cases for $(f^{-1}(16), f^{-1}(5))$: (0120, 0210), (0102, 2010), and (0102, 0210). The first yields adjacent labels with difference 11. The second also falls quickly. Recalling the analysis above, $f(0102) = 16$ forces $\{f^{-1}(9), f^{-1}(10)\} = \{0201, 1101\}$ and $f(2010) = 5$ forces $\{f^{-1}(12), f^{-1}(11)\} = \{1020, 1011\}$. Because 15 and 6 must label a Type 2 and Type 1 vertex with one coordinate two, the remaining options are $f^{-1}(15) \in \{1002, 0120\}$ and $f^{-1}(6) \in \{2001, 0210\}$. Since these labels cannot be adjacent, we may assume by symmetry that $f(1002) = 15$ and $f(0210) = 6$. Unfortunately, the labels $13, 14$ must go on the remaining Type 1 vertices 0120 and

0111, and 0210 is adjacent to both of them. (Putting 7, 8, 13, 14 on 1110, 2001, 0120, 0111 leads to a labeling with only one adjacent pair differing by 8.)

We are left with the possibility $f(0102) = 16$, $f(0210) = 5$. Consider 1011; its Type 2 neighbors are $\{1110, 1101, 2001, 2010\}$, which have labels in $\{6, \ldots, 10\}$. Because $f(0102) = 16$ forces $\{f^{-1}(9), f^{-1}(10)\} = \{0201, 1101\}$, one of $\{9, 10\}$ does not appear on a neighbor of 1011, and 6 does appear. Hence $f(1011) \leqslant 13$, and equality must hold because $f^{-1}(5) = 0210$ implies $\{f^{-1}(12), f^{-1}(11)\} = \{0120, 0111\}$.

The fact that $f^{-1}(13)$ is not a neighbor of $f^{-1}(20)$ forces $d(f^{-1}(6), f^{-1}(20)) = 3$; i.e., $f^{-1}(6)$ has fourth coordinate 0. Also, $f(1110) > 6$, because 1110 has four Type 1 neighbors. This leaves $f^{-1}(6) = 2010$. The only remaining choices for the label on vertex 1020 are 14 and 15. Unfortunately, 1020 is adjacent to $2010 = f^{-1}(6)$, which completes the contradiction. (The labeling can be completed in eight ways with (2010, 1020) being the only adjacent pair differing by 8.)

Editorial comment. The problem of computing the bandwidth of an arbitrary input graph is "hard", in the sense that it is NP-complete (even if the graphs are restricted to be trees!). Nevertheless, this does not eliminate the possibility that there is a formula for the bandwidths of the graphs G_k. Indeed, it is possible that the lexicographic labeling is optimal.

A labeling of G_4 with maximum difference 8 was given jointly by D. P. Mehendale and M. R. Modak, and also by the proposer.

Regressing in L_1

E 3079 [1985, 215]. *Proposed by James Chew, North Carolina Agricultural and Technical State University.*

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in R^2, no two of which have the same x-coordinate. Say that $y = Ax + B$ is a *least absolute-value line* if the function

$$g(a, b) = \sum_{i=1}^{3} |ax_i + b - y_i|$$

attains a minimum at $(a, b) = (A, B)$. Must a least absolute-value line pass through two of the three points?

Solution I by O. P. Lossers, Eindhoven University of Technology, The Netherlands. The answer is yes.

Denote the point (x_i, y_i) by P_i and assume $x_1 < x_2 < x_3$. Each of the lines $y = ax + b$ divides the plane into an upper half plane $U_{a,b}$ and a lower half plane $L_{a,b}$. If a is fixed, $g(a, b)$ is a decreasing function of b when $U_{a,b}$ contains two or more of the points P_i in its interior, but $g(a, b)$ is an increasing function of b when $L_{a,b}$ contains two or more of the points P_i in its interior. Therefore, for fixed a, the

minimum of $g(a, b)$ as a function of b occurs when the line $y = ax + b$ passes through a vertex of the triangle $P_1P_2P_3$ and a point on the opposite side.

For the line $y = a(x - x_r) + y_r$ passing through a fixed vertex (x_r, y_r) the function

$$g(a, y_r - ax_r) = \sum_{i=1}^{3} |a(x_i - x_r) - (y_i - y_r)|$$

is a linear function of a as long as the line does not pass through a second vertex. Also $\lim_{a \to \pm\infty} g(a, y_r - ax_r) = +\infty$. Thus, under the restriction that $y = ax + b$ passes through the fixed vertex (x_r, y_r), the minimum of $g(a, b)$ occurs when the line contains a second vertex. A simple geometric argument then shows that the vertical distance from P_i to the line through the other two points is minimized for $i = 2$. Thus the least absolute-value line is the line through P_1 and P_3.

Solution II (and generalization) by O. P. Lossers, Eindhoven University of Technology, The Netherlands. Let $(x_1, y_1), (x_2, y_2), \dots, (x_{n+2}, y_{n+2})$ be $n + 2$ points in $\mathbb{R}^2$ such that $x_1 < x_2 < \cdots < x_{n+2}$. Our problem is to minimize over π_n, i.e., the set of polynomials of degree at most n, the discrete L_1-norm

$$E(p) := \sum_{i=1}^{n+2} |p(x_i) - y_i| \quad \text{for } p \in \pi_n.$$

It is a known result in approximation theory (cf. *), that the set $B \subset \pi_n$ of polynomials that minimizes $E(p)$ is a convex set having a finite number of extreme elements. Moreover, every extreme element of B is a polynomial p such that $p(x_i) = y_i$ in at least $n + 1$ points of $\{x_1, \dots, x_{n+2}\}$. Now, let $q \in \pi_{n+1}$ be the polynomial of degree at most $n + 1$ such that $q(x_i) = y_i$ $(i = 1, \dots, n + 2)$. Then the polynomial $p_j \in \pi_n$ for which $p_j(x_i) = y_i$ $(i \neq j)$ satisfies the relation

$$q(x) = p_j(x) + C\prod_{i \neq j}(x - x_i),$$

where C is a constant not depending on j. Hence,

$$\begin{aligned} E(p_j) &= |p_j(x_j) - y_j| = |q(x_j) - p_j(x_j)| \\ &= |C|\prod_{i \neq j}|(x_j - x_i)|. \end{aligned}$$

If $C = 0$, then $q \in \pi_n$. Therefore, $E(q) = 0$ and q is the only polynomial that minimizes $E(p)$ over π_n.

If $C \neq 0$, then one easily verifies that $E(p_1) > E(p_2)$ and $E(p_{n+2}) > E(p_{n+1})$. So we may conclude that the extreme elements of B belong to the set $\{p_2, \dots, p_{n+1}\}$. Since $p_j(x_1) - y_1 = p_j(x_{n+2}) - y_{n+2} = 0$ for $j = 2, \dots, n + 1$, every extreme element of B, and thus every element of B, has the property $p(x_1) - y_1 = p(x_{n+2}) - y_{n+2} = 0$.

Our final result is that a polynomial $p \in \pi_n$ that minimizes $E(p)$ must satisfy

$$p(x_1) - y_1 = p(x_{n+2}) - y_{n+2} = 0.$$

If $n = 1$, the case which corresponds to the stated problem, then the mentioned condition implies that the straight line is unique and passes through two of the three points.

For $n > 1$, uniqueness fails to hold.

REFERENCE

(*) J. Rivlin, An introduction to the approximation of functions, Dover Publications, Inc., New York, 1969, p. 77.

Editorial comment. D. S. Rubin (The University of North Carolina at Chapel Hill) and I. Perez (Université de Dijon, France) independently offered a generalization of the problem that is different from the generalization above. For $k \geqslant 1$ and i such that $1 \leqslant i \leqslant n + k$, if the points $\bar{x}_i$ are in $\mathbb{R}^n$ for $n > 2$, then standard linear programming results allow the conclusion that there is at least one least absolute-value hyperplane that passes through n of the points $(\bar{x}_i, y_i)$. Perez gives the reference, "On L_1 and Chebyshev estimation," by G. Appa and C. Smith, *Mathematical Programming* 5 (1973), pp. 73–87. Rubin gives the reference, *Linear Programming*, by Vašek Chvátal, W. H. Freeman & Co., 1983.

Also solved by I. Bivens, L. King, and B. Klein, M. Bowron, W. Cheung, J. Fukuta (Japan), K. Gough, L. Kuipers (Switzerland), P. McCray, M. Meyerson, D. Neuenschwander (Switzerland), W. A. Newcomb, R. Rosentrater, D. Singmaster (England), J. Ward, S. Wentzig (West Germany), and P. Y. Wu (Republic of China).

An Equality with 1, 2, 3

E 3097 [1985, 428]. *Proposed by Florentin Smarandache, Craiova, Romania.*

Find all real numbers x for which

$$(x+1)^x + (x+2)^x = (x+3)^x.$$

Solution by J. B. M. Melissen, Philips CAD-Centre, Eindhoven, The Netherlands. I will show that $x = 2$ is the only real solution. (We assume that only values of x greater than -1 are to be considered.) Let

$$f(x) = \left(1 - \frac{2}{x+3}\right)^x + \left(1 - \frac{1}{x+3}\right)^x - 1.$$

Then f is strictly decreasing, because

$$\begin{aligned} f'(x) = {} & \left(1 - \frac{2}{x+3}\right)^x \left[\log\left(1 - \frac{2}{x+3}\right) + \frac{2x}{(x+1)(x+3)}\right] \\ & + \left(1 - \frac{1}{x+3}\right)^x \left[\log\left(1 - \frac{1}{x+3}\right) + \frac{x}{(x+2)(x+3)}\right] \end{aligned}$$

$$< \left(1 - \frac{2}{x+3}\right)^x \left[-\frac{2}{x+3} + \frac{2x}{(x+1)(x+3)} \right]$$
$$+ \left(1 - \frac{1}{x+3}\right)^x \left[-\frac{1}{x+3} + \frac{x}{(x+2)(x+3)} \right]$$
$$= -\frac{2}{(x+1)(x+3)}\left(1 - \frac{2}{x+3}\right)^x - \frac{2}{(x+2)(x+3)}\left(1 - \frac{1}{x+3}\right)^x$$
$$< 0$$

for $x > -1$. Since f has a zero for $x = 2$,

$$f(x) > 0 \quad \text{if} \quad x \in (-1, 2)$$
$$f(x) < 0 \quad \text{if} \quad x \in (2, +\infty).$$

Thus $(x+1)^x + (x+2)^x = (x+3)^x$ has exactly one root: $x = 2$.

Also solved by R. Cuculière (France), B. M. M. de Weger (The Netherlands), W. Janous (Austria), L. Kuipers (Switzerland), R. E. Shafer, and S. Wentzig (Federal Republic of Germany). Partially solved by N. Gauthier (Canada), M. Kantrowitz (student), G. N. Lewis, D. S. Rubin, and P. Tracy.

Janous and Rubin mentioned generalizations: if $0 < a < b < c$ satisfy $e^a + e^b < e^c$ then the equation $(x+a)^x + (x+b)^x = (x+c)^x$ has a unique solution. (The argument above can easily be adapted to give this generalization.)

Shafer's solution was based on the midpoint quadrature theorem, which allows $(x+3)^x - (x+1)^x = x\int_{-1}^{+1}(x+2+t)^{x-1}\,dt$ to be approximated by the quantity $2x(x+2)^{x-1}$.

A Minimum Under a Constraint

E 3099 [1985, 507]. *Proposed by Weixuan Li and Edward T. H. Wang, Wilfrid Laurier University, Canada.*

Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ be n nonnegative reals ($n \geqslant 2$) such that

$$\sum_{i=1}^{n} a_i a_{i+1} = 1 (a_{n+1} = a_1).$$

Determine the minimum value of $\sum_{i=1}^{n} a_i$.

Solution by Charles H. Jepsen, Grinnell College. We use Lagrange multipliers to minimize $\sum_{i=1}^{n} a_i$ subject to the constraint $\sum_{i=1}^{n} a_i a_{i+1} = 1$. Let

$$G = \sum_{i=1}^{n} a_i - \lambda\left(\sum_{i=1}^{n} a_i a_{i+1} - 1\right)$$

and consider two cases, the case of an extremum on the boundary and the case of an interior extremum, respectively.

Case 1. $a_1 = 0$, $n \geqslant 3$. Clearly a_{n-1} and a_n must be positive and so we may minimize with respect to these two variables. Since the minimum occurs when

$$\frac{\partial G}{\partial a_{n-1}} = 1 - \lambda(a_{n-2} + a_n) = 0$$

and

$$\frac{\partial G}{\partial a_n} = 1 - \lambda a_{n-1} = 0,$$

we see that $a_{n-2} + a_n = a_{n-1}$. Then $0 \leqslant a_{n-2} \leqslant a_{n-1} \leqslant a_n$ yields $a_n \leqslant a_{n-2} + a_n = a_{n-1} \leqslant a_n$, which forces $a_{n-2} = 0$ and $a_{n-1} = a_n$. From $1 = \sum_{i=1}^n a_i a_{i+1} = a_{n-1}a_n$, we have $a_{n-1} = a_n = 1$ and, hence, $\sum_{i=1}^n a_i = a_{n-1} + a_n = 2$.

Case 2. $a_1 \neq 0$. If we let $a_0 = a_n$, then setting

$$\frac{\partial G}{\partial a_i} = 1 - \lambda(a_{i-1} + a_{i+1}) = 0 \quad \text{for} \quad i = 1, \dots, n,$$

we see that $a_{i-1} + a_{i+1}$ is the same for each i. Except for the case $n = 4$, this forces all a_i to be equal. A direct argument works for $n = 2$ or 3. For $n \geqslant 5$, we have $a_1 = a_5$ and $a_4 = a_n$. Since $a_i \leqslant a_{i+1}$ for all i, it follows that $a_1 = \cdots = a_n$. From

$$1 = \sum_{i=1}^{n} a_i a_{i+1} = na_i^2,$$

we get each $a_i = 1/\sqrt{n}$ and $\sum_{i=1}^n a_i = \sqrt{n}$.

In the exceptional case $n = 4$, the fact that $a_{i-1} + a_{i+1}$ is the same for each i yields only $a_1 + a_3 = a_2 + a_4$. It follows that $a_1 = a_2$ and $a_3 = a_4$. From $1 = \sum_{i=1}^4 a_i a_{i+1}$ we see that $a_1 + a_3 = 1$ and thus $\sum_{i=1}^4 a_i = 2$.

Our conclusion is summarized as follows: For $n = 2$ or 3, the minimum value of $\sum_{i=1}^n a_i$ is $\sqrt{n}$ and occurs when each $a_i = 1/\sqrt{n}$. For $n = 4$, the minimum value of $\sum_{i=1}^n a_i$ is 2 and occurs whenever $a_1 = a_2$, $a_3 = a_4$, $a_1 + a_3 = 1$. For $n \geqslant 5$, the minimum value of $\sum_{i=1}^n a_i$ is 2 and occurs when $a_1 = \cdots = a_{n-2} = 0$, $a_{n-1} = a_n = 1$.

Also solved by R. C. Bernstein, R. Breusch, W. Cross (student), J. S. de Cani, K. Ekblaw, R.-Fr. Gloden (Italy), R. Heller, E. Hertz, G. A. Heuer, C. Hurd, M.-Y. Lee (student, Korea), O. P. Lossers (The Netherlands), E. Morgantini (Italy), T. S. Norfolk, V. Pambuccian (Romania), T. M. Trung (Norway), and the proposers.

A Quadratic Factorial Quotient

E 3123 [1986; 59]. *Proposed by R. M. Grassl and T. Porter, University of New Mexico.*

Prove that

$$\frac{(n^2)!}{\binom{n}{n}\binom{n+1}{n}\binom{n+2}{n}\cdots\binom{2n-1}{n}(n!)^n}$$

is an integer for $n \in \{1, 2, 3, \dots\}$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. By a well-known theorem of Frame, Robinson, and Thrall (cf. D. Knuth, *The Art of Computer Programming*, Vol. 3, Addison-Wesley, 1973, page 63), the number A_n of n by n arrays filled with the integers 1 to n^2 in such a way that all rows and columns increase is $(n^2)!$ divided by the product of all 'hook-lengths'. Here the length of the hook on position (i, j) is the number of positions (i', j') with $i' = i$, $j' \geqslant j$ or $i' \geqslant i$, $j' = j$, which for the square array equals $2n + 1 - (i + j)$. The product of the hooks in column i is $(2n - i)!/(n - i)!$. So we find that A_n equals the expression of the problem. Since A_n is an integer, the result follows.

Solution by the editors. We prove more generally that

$$\frac{(mn)!0!1!\cdots(n-1)!}{m!(m+1)!\cdots(m+n-1)!}$$

is an integer for m, $n \geqslant 1$. (This more general result also follows from the Frame-Robinson-Thrall Theorem mentioned above.) For any prime p, the power of p dividing $q!$ is $\sum_{k\geqslant 1}\lfloor q/p^k\rfloor$. Letting $e(m, n; r) = \lfloor mn/r\rfloor + \sum_{i=0}^{n-1}\lfloor i/r\rfloor - \sum_{i=m}^{m+n-1}\lfloor i/r\rfloor$, our task is to show that $\sum_{k\geqslant 1}e(m, n; p^k) \geqslant 0$ for every p.

We prove more generally that $e(m, n; r) \geqslant 0$ for every m, n, r. Since replacing m by $m + r$ increases both the positive and negative contributions by n, we have $e(m + r, n; r) = e(m, n; r)$. Furthermore, we can write $e(m, n; r)$ symmetrically as

$$e(m, n; r) = \lfloor mn/r\rfloor + \sum_{i=0}^{n-1}\lfloor i/r\rfloor + \sum_{i=0}^{m-1}\lfloor i/r\rfloor - \sum_{i=0}^{m+n-1}\lfloor i/r\rfloor,$$

so $e(m, n; r) = e(n, m; r)$. Hence it suffices to consider the case $0 \leqslant m$, $n < r$. In this case, if $m + n \leqslant r$, then $e(m, n; r) = \lfloor mn/r\rfloor \geqslant 0$. On the other hand, if $m + n > r$, then $e(m, n; r) = \lfloor mn/r\rfloor - (m + n - r) = \lfloor(r - m)(r - n)/r\rfloor \geqslant 0$.

Also solved by Walther Janous, R. E. Shafer, Michael Vowe (Switzerland), Benne de Weger (The Netherlands), C. Wildhagen (The Netherlands), and the proposer. Partially solved by I. Paasche (Germany). The graphical method of solution used for 6514 [1987, p. 1012] is not applicable here.

Coupled Differential Equations

E 3124 [1986, 59]. *Proposed by N. Gauthier, P. Rochon, and J. R. Gosselin, Royal Military College of Canada, Kingston, Ontario.*

Solve the following set of two coupled non-linear differential equations:

$$\frac{d}{dt}\frac{x}{(1-r^2)^{1/2}} = yf,$$

$$\frac{d}{dt}\frac{y}{(1-r^2)^{1/2}} = f - xf.$$

By definition, $r^2 = x^2 + y^2$; by assumption, $x = y = 0$ at $t = 0$ and $f = f(t)$ is t-integrable but otherwise arbitrary.

The above set of equations arose in an effort to explain the physical phenomenon of radiation pressure in elementary terms. See the original statement of the problem for more details.

Solution by C. Georghiou, University of Patras, Greece. We introduce new variables u, v

$$u = x(1-r^2)^{-1/2},$$

$$v = y(1-r^2)^{-1/2}.$$

Then

$$x = u(u^2+v^2+1)^{-1/2}$$

$$y = v(u^2+v^2+1)^{-1/2}$$

and the differential system becomes

$$\frac{du}{dt} = fv(u^2+v^2+1)^{-1/2}$$

$$\frac{dv}{dt} = f\left(1 - u(u^2+v^2+1)^{-1/2}\right)$$

with $u = v = 0$ at $t = 0$, which gives

$$\frac{du}{dv} = v/\left((u^2+v^2+1)^{1/2} - u\right).$$

The solution of the above equation, subject to the initial condition $u = 0$ when $v = 0$, is

$$u = v^2/2.$$

Substitution into the second equation of the differential system gives

$$\frac{dv}{dt} = \frac{2f}{v^2+2}$$

and its solution is

$$\frac{1}{3}v^3 + 2v = 2\int_0^t f(\tau)\,d\tau.$$

By solving the above cubic equation in v we obtain

$$v = (\Phi + 3F)^{1/3} - (\Phi - 3F)^{1/3},$$

where

$$F = \int_0^t f(\tau)\,d\tau \quad \text{and} \quad \Phi = (9F^2 + 8)^{1/2}.$$

The solution of the given system can be found from

$$x = \frac{v^2}{v^2 + 2}, \quad y = \frac{2v}{v^2 + 2}.$$

Also solved by K. F. Andersen (Canada), Robert Baran and James Coughlin, John Dalbec, Lawrence Dickson, Raoul-Fr. Gloden (Italy), Cliff Holroyd, P. L. Hon (Hong Kong), Hans Kappus, Kee-Wai Lau (Hong Kong), Gilbert Lewis, Deborah Lockhart, O. P. Lossers (The Netherlands), William Newcomb, Garrett Sylvester, A. Tissier (France), and the proposers.

A Putnam Extremal Problem

E 3136 [1986, 215]. *Proposed by D. M. Bloom, Brooklyn College of CUNY.*

The 1982 Putnam Exam proposed the following problem: If a, b, c, d are positive integers with $a + c = 1982$, and if $r = 1 - a/b - c/d > 0$, show that $r > (1983)^{-3}$. The stated lower bound for r is not best possible.

(i) Find the actual minimum value of r.

(ii) More generally, if $N > 1$ is a positive integer, find, as a function of N, the minimum positive value of $r = 1 - a/b - c/d$ over all positive integers a, b, c, d such that $a + c = N$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We first consider (ii). Without loss of generality we may assume that $b \leqslant d$. Then there are the following four cases.

I. $\underline{b = d}$. This implies $r = 1 - N/b$, an expression that attains the minimal positive value $(N + 1)^{-1}$ for $b = N + 1$.

II. $\underline{b < d \leqslant N}$. Now $r < 1 - (a + c)/N = 0$, so we do not get any positive value at all for r.

III. $\underline{N \leqslant b < d}$. In this situation $r = 1 - a/b - (N - a)/d \geqslant 1 - a/N - (N - a)/(N + 1) = 1/(N + 1) - a/\{(N + 1)N\}$, and r attains its minimal positive value $(N^2 + N)^{-1}$ for $(a, b, c, d) = (N - 1, N, 1, N + 1)$.

IV. $\underline{b < N < d}$. In this case b can have the values $a + 1, a + 2, \ldots, N - 1$. We distinguish two subcases.

IVa. If $\underline{b = a + 1}$ then we obtain

$$r = 1 - \frac{a}{a + 1} - \frac{N - a}{d} = \frac{d - (a + 1)(N - a)}{(a + 1)d}.$$

For fixed a this expression is an increasing function of d, so its minimal posi-

Relatively Prime Variables

E 2953 [1982, 424; 1986, 299]. *Proposed by John J. Wahl, Mt. Pocono, PA.*

Let A, B, X, Y be variables subject to the condition $AX - BY = 1$.

(a) Find explicit polynomials u and v in A, B, X, Y with integer coefficients such that $A^4u - B^4v = 1$.

(b) Prove in fact that for any positive m and n there exist u and v such that $A^m u - B^n v = 1$.

Editorial comment. The proposer's name should be added to the list of solvers.

Approximating a Sine Function on a Square

E 3006 [1983, 401; 1986, 484]. *Proposed by J. R. Kuttler, Johns Hopkins University.*

Find a function of the form $\alpha(x) + \beta(y)$ which best approximates $\sin x \sin y$ on the square $S = \{(x, y): 0 \leqslant x \leqslant 2\pi, 0 \leqslant y \leqslant 2\pi\}$ in the sup norm, i.e., so that

$$\sup_{(x,y)\in S} |\sin x \sin y - \alpha(x) - \beta(y)|$$

is as small as possible.

Editorial comment. Victor Pambuccian (Romania) has pointed out that the solution to this problem is contained in a theorem given by Leopold Flatto, The approximation of certain functions of several variables by sums of functions of fewer variables, this MONTHLY 73 (1966), no. 4, part II, 131–132.

A Colossal Blunder

E 3085 [1985, 287; 1987, 383]. *Proposed by T. C. Lim, George Mason University, Fairfax, VA.*

Let $g(\mu)$ be the unique nonnegative solution of

$$\{\mu + g(\mu)\}^p + |\mu - g(\mu)|^p = 2\mu,$$

where $1 < p < 2$ and $0 \leqslant \mu \leqslant (1/2)$. Prove that

$$\{1 - \mu + g(\mu)\}^p + |1 - \mu - g(\mu)|^p \leqslant 2(1 - \mu).$$

Editorial Comment. Unfortunately the solution published in the April, 1987 issue is incorrect. The factor $1 - y$ in the seventh line of that solution should actually be $1 + y$; with that change the subsequent argument becomes inapplicable.

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Solution by the Proposer. Put $\lambda = 1 - \mu$. Note that $\mu \leqslant g(\mu) \leqslant \lambda$ (because $(2\mu)^p \leqslant 2\mu$ and $\mu + \lambda = 1 \geqslant 2\mu$). Therefore,

$$\begin{aligned}
&(\lambda + g(\mu))^p + |\lambda - g(\mu)|^p \\
&\quad = (1 + g(\mu) - \mu)^p + (1 - (\mu + g(\mu)))^p \\
&\quad = 1 + p(g(\mu) - \mu) + \binom{p}{2}(g(\mu) - \mu)^2 + \binom{p}{3}(g(\mu) - \mu)^3 + \cdots \\
&\qquad +1 - p(\mu + g(\mu)) + \binom{p}{2}(\mu + g(\mu))^2 - \binom{p}{3}(\mu + g(\mu))^3 + \cdots .
\end{aligned}$$

For n even, $\binom{p}{n}$ is positive and

$$(g(\mu) - \mu)^n + (\mu + g(\mu))^n \leqslant (g(\mu) - \mu)^p + (\mu + g(\mu))^p = 2\mu.$$

For n odd and $\geqslant 3$, $\binom{p}{n}$ is negative and

$$0 \leqslant (\mu + g(\mu))^n - (g(\mu) - \mu)^n \leqslant (\mu + g(\mu)^p + (g(\mu) - \mu)^p = 2\mu.$$

Therefore,

$$\begin{aligned}
&(\lambda + g(\mu))^p + |\lambda - g(\mu)|^p \\
&\quad \leqslant 2 - 2p\mu + 2\mu\left(\binom{p}{2} - \binom{p}{3} + \binom{p}{4} - \cdots\right) \\
&\quad = 2 - 2p\mu + 2\mu(p - 1) \\
&\quad = 2 - 2\mu = 2\lambda.
\end{aligned}$$

ADVANCED PROBLEMS

6560. *Proposed by Ambati Jaya Krishna (student), Johns Hopkins University, Baltimore, MD, A. Murali Mohan Rao, Iona College, New Rochelle, NY, and Gomathi S. Rao, Baltimore, MD.*

If x and y are odd positive integers, evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tan\left(\frac{n\pi}{x}\right) \tan\left(\frac{n\pi}{y}\right).$$

Solution by the Proposer. Put $\lambda = 1 - \mu$. Note that $\mu \leqslant g(\mu) \leqslant \lambda$ (because $(2\mu)^p \leqslant 2\mu$ and $\mu + \lambda = 1 \geqslant 2\mu$). Therefore,

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For n even, $\binom{p}{n}$ is positive and

$$(g(\mu) - \mu)^n + (\mu + g(\mu))^n \leqslant (g(\mu) - \mu)^p + (\mu + g(\mu))^p = 2\mu.$$

For n odd and $\geqslant 3$, $\binom{p}{n}$ is negative and

$$0 \leqslant (\mu + g(\mu))^n - (g(\mu) - \mu)^n \leqslant (\mu + g(\mu)^p + (g(\mu) - \mu)^p = 2\mu.$$

Therefore,

$$\begin{aligned}
&(\lambda + g(\mu))^p + |\lambda - g(\mu)|^p \\
&\quad \leqslant 2 - 2p\mu + 2\mu\left(\binom{p}{2} - \binom{p}{3} + \binom{p}{4} - \cdots\right) \\
&\quad = 2 - 2p\mu + 2\mu(p - 1) \\
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6561. *Proposed by Heinz-Jürgen Seiffert, Berlin.*

For complex numbers z and w with $Re(z) > 0$ and $Re(w) > 0$ consider the matrices

$$c_n(z, w) := (B(z + j + k, w))_{j,k=0,\dots,n},$$

$$D_n(z, w) := \left(\frac{1}{B(z + j + k, w)}\right)_{j,k=0,\dots,n},$$

where B denotes the beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1}(1 - t)^{y-1}\,dt.$$

Show that

(a) $\det C_n(z, w) = nB(z + w + n - 1, n)B(z + n, w + n)\det C_{n-1}(z, w),$

(b) $\det D_n(z, w) = q_n(w)\dfrac{B(z + n - 1, n)}{B(z + n, w)}\,\dfrac{\Gamma(z + n)}{\Gamma(z + 2n)}\det D_{n-1}(z, w),$

where

$$q_n(w) = n\prod_{k=0}^{n-1}(k - w).$$

6562. *Proposed by George E. Andrews, Pennsylvania State University.*

Let $Q(q) = \prod_{n=1}^{\infty}(1 - q^n)$ for $|q| < 1$. Euler's Pentagonal Number Theorem asserts that

$$Q(q) = \sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}(1 - q^{2n+1}).$$

Jacobi showed that

$$Q(q)^3 = \sum_{n=0}^{\infty}(-1)^n(2n + 1)q^{n(n+1)/2}.$$

Prove that

$$Q(q)^2 = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(1 - q^{2n+2})p_n(q),$$

where $p_n(q) = \sum_{r=0}^{n} q^{r(n-r)}$.

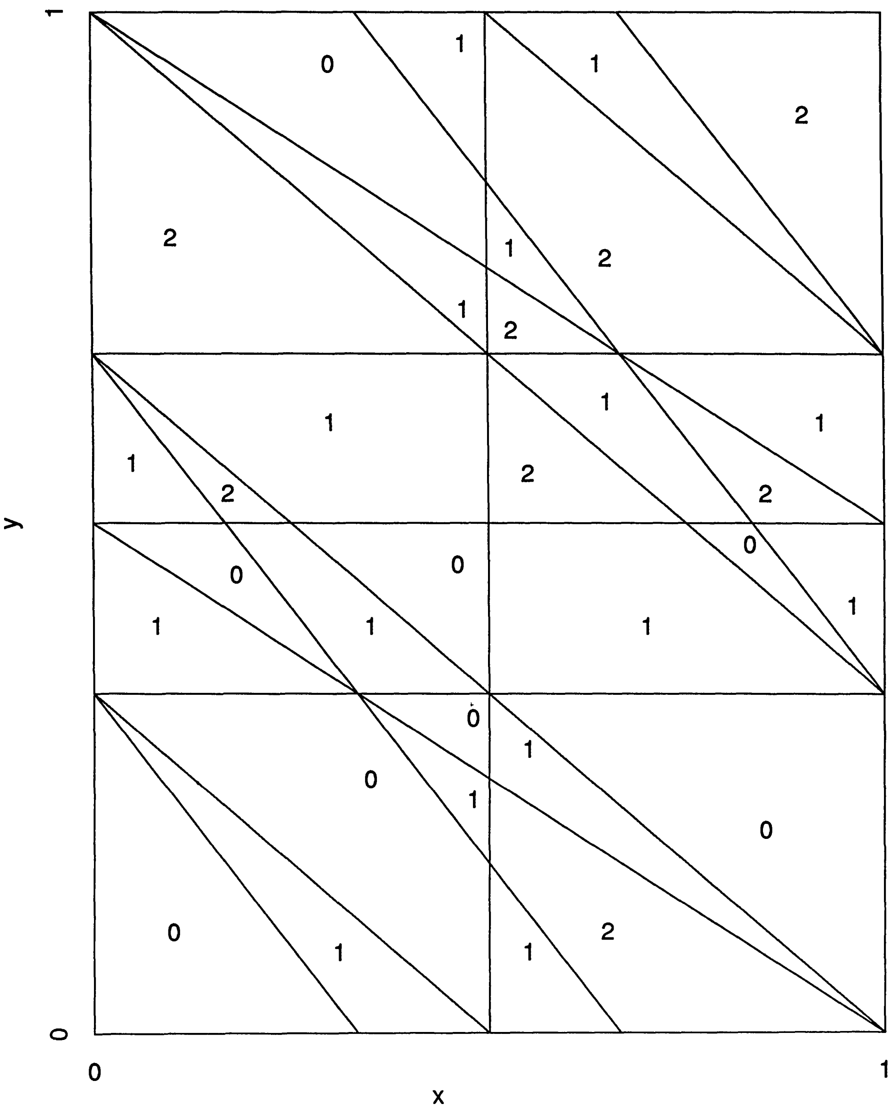

Figure. Value of $f(x, y)$

Conversely, if (*) fails in a neighborhood of (x, y) in $[0,1) \times [0,1)$, then $P(m, n)$ is not an integer for

$$m = [xp], \quad n = [yp],$$

where p is a sufficiently large prime.

Editorial Comments. The theorem stated and proved above goes back at least as far as Landau, *Nouvelles Annales de Math.*, (3) 19 (1900) 344–362, 576 and *Nouvelles Annales de Math.*, (4) 13 (1913) 353–356. See also Landau's *Collected Works*, Thales Verlag, Vol. 1, 116–135 and Vol. 6, 17–19. The result is quoted incorrectly in Dickson's *History of the Theory of Numbers*, 1918, Vol. 1, p. 268, but is given correctly in Bachmann's *Niedere Zahlentheorie I*, (1902) p. 64 (Chelsea, 1968).

The diagram printed (due to C. L. Mallows of AT & T Laboratories, Murray Hill, New Jersey) is essentially the same as Patruno's, except that Patruno includes the two coincident lines $x + y = 1$ and $3x + 3y = 3$ *All* solutions received used the method described by Patruno. However, the editor feels there is still room for other methods, involving perhaps combinatorial interpretations or manipulation of generating functions. In this particular case, the proposer remarks that $C(m, n)$ should be the constant term of the Laurent polynomial

$$\begin{aligned}&(1-x)^m(1-x^{-1})^m(1-y)^m(1-y^{-1})^m(1-xy)^m(1-x^{-1}y^{-1})^m\\&\quad\times(1-xy^{-1})^n(1-x^{-1}y)^n(1-x^2y)^n(1-x^{-2}y^{-1})^n(1-xy^2)^n\\&\quad\times(1-x^{-1}y^{-2})^n,\end{aligned}$$

an expectation related to the Macdonald-Morris constant term identity conjectures.

While Patruno's method solves any integrality problem of this sort affirmatively or negatively, how can the set of such problems with an affirmative solution (and perhaps some degree of elegance) be efficiently enumerated or characterized? What is the fastest algorithm for determining whether or not $f(x, y) \geqslant 0$ on the unit square? At any rate, we hope that Patruno's sharp remarks will raise the level of discussion of this sort of problem.

Also solved by Charles Vanden Eynden, Ira Gessel, Kee-wai Lau (Hong Kong), C. L. Mallows, G. Turnwald (West Germany), Daniel Ullman, and B. M. M. de Weger (The Netherlands).

Infinitely Summable Sequences

6516 [1986, 305]. *Proposed by Erwin Kronheimer, Birkbeck College, University of London, England.*

Do there exist real numbers $s_0, s_1, s_2, \ldots$, not all zero, such that each of the series

$$\begin{gathered}s_0+s_1+s_2+\cdots,\\ s_0+(s_0+s_1)+(s_0+s_1+s_2)+\cdots,\\ s_0+(s_0+(s_0+s_1))+(s_0+(s_0+s_1)+(s_0+s_1+s_2))+\cdots,\\ \text{etc.}\end{gathered}$$

converges?

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$$s_0 + (s_0 + (s_0 + s_1)) + (s_0 + (s_0 + s_1) + (s_0 + s_1 + s_2)) + \cdots,$$
$$\text{etc.}$$

converges?

Solution I by R. A. Vitale, Claremont Graduate School, Claremont, California. Yes. We begin by observing that each continuous function f on $[0,1]$ is determined by its moments

$$s_k = \int_0^1 t^k f(t)\,dt, \qquad k = 0,1,2,\ldots, \tag{1}$$

but a continuous function g on $[0,\infty)$ is not determined by its moments

$$\sigma_n = \int_0^\infty x^n g(x)\,dx, \qquad n = 0,1,2,\ldots,. \tag{2}$$

The first observation can be proved with the help of the Weierstrass polynomial approximation theorem, while the second follows from the fact that both the identically zero function and

$$g(x) = \sin(x^{1/4})\exp(-x^{1/4}), \qquad x \geqslant 0, \tag{3}$$

have $\sigma_n = 0$ for all n. To see this, make the change of variable $u = t^{1/4}$, express the sine function in terms of complex exponentials, and integrate by parts. (This example is due to Stieltjes; see D. V. Widder, *The Laplace Transform*, Princeton, 1946, pp. 125–126. Widder's book contains an excellent introduction to moment problems.) We shall show below that the numbers s_k defined by (1) have the desired property when g is the function defined by (3) (or indeed any similar function for which $\sigma_n = 0$ for all n) and

$$f(t) = \begin{cases} \dfrac{1}{1-t}\,g\left(\dfrac{t}{1-t}\right) & 0 \leqslant t < 1, \\ 0 & t = 1. \end{cases}$$

Clearly $f(0) = 0$, and for any $c > 0$ the function $f(t)\,(1-t)^{-c}$ is continuous on $[0,1]$ and converges to zero as $t \to 1-$. Also by our very first observation, the s_k cannot all be zero.

In terms of f the equation $\sigma_n = 0$ is

$$\int_0^1 \frac{t^n}{(1-t)^{n+1}} f(t)\,dt = 0.$$

Thus

$$\begin{aligned} s_0 + s_1 + \cdots + s_N &= \sum_{k=0}^{N} \int_0^1 t^k f(t)\,dt \\ &= \int_0^1 \frac{1-t^{N+1}}{1-t} f(t)\,dt = -\int_0^1 \frac{t^{N+1}}{1-t} f(t)\,dt. \end{aligned}$$

by virtue of $\sigma_0 = 0$. Since $f(t)/(1-t)$ is bounded, the series $s_0 + s_1 + s_2 + \cdots$ converges to

$$0 = \lim_{N\to\infty} (-1)\int_0^1 \frac{t^{N+1}}{1-t} f(t)\,dt.$$

The terms of the next series are the partial sums of the first, so its $N+1$st partial sum is

$$\sum_{k=0}^{N}(-1)\int_0^1\frac{t^{k+1}}{1-t}f(t)\,dt = -\int_0^1\frac{1-t^{N+1}}{(1-t)^2}tf(t)\,dt$$
$$= \int_0^1\frac{t^{N+1}}{(1-t)^2}tf(t)\,dt \to 0$$

as $N\to\infty$. Here we used $\sigma_1=0$. By proceeding inductively in this fashion, we find that the $N+1$st partial sum I_N of the jth series is

$$(-1)^j\int_0^1\frac{t^{N+1}}{(1-t)^j}t^{j-1}f(t)\,dt.$$

Since this tends to zero as $N\to\infty$, the result follows. In fact, since

$$|I_N - I_{N+1}| \leqslant M_j\int_0^1|t^{N+1}-t^N|\,dt = O(N^{-2}),$$

we even have an affirmative solution for absolute convergence.

Solution II (sketch only) by William S. Zwicker, Union College, Schenectady, New York. Given finite sequences $A=(a_1,\ldots,a_n)$ and $B=(b_1,\ldots,b_m)$, define, for r real and $h\geqslant 1$ integral,

$$rA = (ra_1,\ldots,ra_n),$$
$$A|B = (a_1,\ldots,a_n,b_1,\ldots,b_m)$$

and

$$[A]^h = A|A\ \cdots\ |A,$$

where there are h copies of A on the right. Set

$$P(A) = (a_1, a_1+a_2, a_1+a_2+a_3,\ldots,a_1+\cdots+a_n)$$

and

$$\sum(A) = a_1+\cdots+a_n.$$

Observe that $P(rA)=rP(A)$ and that

$$P(A|B) = [P(A)]|[P(B)]$$

Solution III by M. J. Pelling, Forest Gate, London, England, and Eric Willekens, Katholieke Universiteit Leuven, Leuven, Belgium (independently). If

$$G(z) = \sum_{j=0}^{\infty} s_j z^j$$

is the generating function of the s_k, then

$$\frac{G(z)}{(1-z)^k} = \sum_{j=0}^{\infty} s_j^{(k)} z^j, \qquad k = 0, 1, 2, \ldots$$

is the generating function of the $(k+1)$st series of the problem. Suppose that there exists a function G that is continuous on the closed unit disc and tends to zero very rapidly as $z \to 1$ along any path within the disc. Then one shows that, for each fixed k, $s_j^{(k)} \to 0$ as $j \to \infty$, i.e., $s_0^{(k-1)} + s_1^{(k-1)} + \cdots$ converges to zero.

Pelling chose

$$G(z) = \exp\left\{ -\left(\frac{1+z}{1-z} \right)^{1/2} \right\},$$

with the branch of the square root selected to be positive at the origin. The function

$$z \mapsto \left(\frac{1+z}{1-z} \right)^{1/2}$$

sends the unit disc into the sector $\{w\colon |\arg w| \leqslant \pi/4\}$. Thus, for $|z| \leqslant 1$, $z \neq 1$, we have

$$\operatorname{Re}\left(\frac{1+z}{1-z} \right)^{1/2} \geqslant \frac{1}{\sqrt{2}} \left| \frac{1+z}{1-z} \right|^{1/2} \geqslant \frac{1}{\sqrt{2}} |1-z|^{-1/2}$$

if we further require that $\operatorname{Re} z \geqslant 0$.

It follows that, for each positive integer k, $G(z)/(1-z)^k$ is continuous on $\{z\colon |z| \leqslant 1\}$ if we define its value to be zero at $z = 1$. In particular, we have $|G(z)(1-z)^{-k}| \leqslant M_k$, $k = 0, 1, 2, \ldots$, $|z| \leqslant 1$.

For $j = 0, 1, 2, \ldots$ and $r \in (0, 1)$, Cauchy's formula gives

$$s_j^{(k)} = \frac{1}{2\pi i} \int_{|z|=r} \frac{G(z)}{(1-z)^k} z^{-j-1}\, dz.$$

Let $r \to 1-$. By uniform continuity,

$$s_j^{(k)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{G(e^{i\theta})}{(1-e^{i\theta})^k} e^{-ij\theta}\, d\theta.$$

The last expression tends to zero as $j \to \infty$ by the Riemann-Lebesgue lemma, which establishes our claim that $s_0^{(k-1)} + s_1^{(k-1)} + \cdots$ converges.

Editorial comment. An argument similar to that of Solution III was proposed with the function $G_1(z) = \exp(-1/(1-z))$. The editors note that the associated

series did not converge in this case. The problem is that $G_1(z)$ fails to approach zero as $z \to 1$ along certain paths in the unit disc. Indeed as $z \to 1$ along the path

$$P: \{z = re^{i\theta}: \theta^2 = 1 - r \to 0+\},$$

then $|G_1(z)| \to \exp(-3/2)$. If we set

$$G_1(z) = \sum_{j=0}^{\infty} g_j z^j, \qquad \frac{G_1(z)}{(1-z)^k} = \sum_{j=0}^{\infty} g_j^{(k)} z^j,$$

and if $\sum_{j=0}^{\infty} g_j^{(2)}$ converges, then $g_j^{(2)} \to 0$ as $j \to \infty$. We would then have

$$\sum_{j=0}^{\infty} g_j^{(2)} z^j = \sum_{j=0}^{\infty} o(1) z^j = o\left(\frac{1}{1-|z|}\right) = o\left(\frac{1}{1-r}\right).$$

However, as $z \to 1$ along P,

$$\frac{G_1(z)}{(1-z)^2} \sim \frac{e^{-3/2}}{1-r} \neq o\left(\frac{1}{1-r}\right).$$

It follows that $g_0^{(2)} + g_1^{(2)} + \cdots$ does not converge.

The editors feel that it is likely that there are other interesting things to be said about these "infinitely summable" sequences.

Also solved by Daniel Ullman by combinatorial methods. His method can be used to extend an arbitrary finite sequence to an infinitely summable infinite sequence.

POSTMORTEM COMMENTS ON ADVANCED PROBLEMS

This section contains further remarks on advanced problems for which solutions have already been published. It will appear in the December issue each year if needed.

Circles with Collinear Centers

3887 [1938, 432; 1983, 486]. *Proposed by V. Thebault, La Mans, France.*

(Paraphrased) Let P be a quadrilateral inscribed in a circle O and let Q be the quadrilateral formed by the centers of the four circles internally touching O and each of the two diagonals of P. Then the incenters of the four triangles having for sides the sides and diagonals of P form a rectangle R inscribed in Q.

Editorial comment. A more concise solution than that briefly sketched in the 1983 MONTHLY was published by Gerhard Turnwald, *Elemente der Mathematik*, 41 (1986) 11–13.

Singular Matrices

6057 [1975, 942; 1977, 495]. *Proposed by Anon, Erewhon-upon-Yarkon.*

Let A, B, C, D be $n \times n$ matrices such that $CD' = DC'$, where the prime denotes transpose. Prove that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD' - BC'|.$$

Editorial comment. Both of the solutions published in 1977 obtained the result of the problem for nonsingular D and then appealed to "continuity" to cover the case of singular D.

M. J. Pelling has pointed out to us that such an appeal to continuity is risky and is actually invalid in the case of the following closely related problem:

$$\text{"If } CD' = -DC', \text{ is it true that } \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD' + BC'|?\text{"} \tag{*}$$

The assertion of (*) is correct if D is nonsingular by an easy manipulation of matrices as in the published solution of 6057. But it is not always correct if D is singular; for example, if $n = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then the left-hand determinant of (*) is $+1$ but the right-hand determinant is -1.

Accordingly the published solutions of 6057 are incomplete without a detailed justification of the appeal to continuity. This is provided by the following lemma.

LEMMA. *If C and D are n by n matrices such that*

$$CD' = DC', \tag{1}$$

then $D = \lim_{\varepsilon \to 0} D_\varepsilon$, where D_ε is an n by n nonsingular matrix such that

$$CD_\varepsilon' = D_\varepsilon C'. \tag{2}$$

Proof. It suffices to prove the assertion of the lemma when C, D are replaced by the pair PCQ, PDQ'^{-1} for suitable nonsingular matrices P, Q, since

$$(PCQ)(PDQ'^{-1})' = P(CD')P' \text{ and } (PDQ'^{-1})(PCQ)' = P(DC')P'.$$

Thus we may suppose that

$$C = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

for some r $(0 \leqslant r \leqslant n)$, where I_r is the r by r identity matrix, D_{11} is r by r, D_{21} is

$n-r$ by r, and so on. The condition (1) is equivalent to

$$D'_{11} = D_{11}, \quad D_{21} = 0. \tag{3}$$

We now take $D_\varepsilon = D - \varepsilon I_n = \begin{pmatrix} D_{11} - \varepsilon I_r & D_{12} \\ D_{21} & D_{22} - \varepsilon I_{n-r} \end{pmatrix}$. Note that (3) still holds if we replace D by D_ε, i.e., if we replace D_{11} by $D_{11} - \varepsilon I_r$ and D_{22} by $D_{22} - \varepsilon I_{n-r}$. Thus C, D_ε satisfy (2). Furthermore D_ε is nonsingular if ε avoids the finite number of eigenvalues of D. Since $D = \lim_{\varepsilon \to 0} D_\varepsilon$, the lemma is proved.

Note that the proof of the lemma would break down if we were to replace (1) by $CD' = -DC'$.

We are grateful to Everett C. Dade for providing us with the lemma and its proof.

Nonextreme Unique Critical Points

6483 [1984, 652; 1986, 307]. *Proposed by J. Arias de Reyna, University of Seville.*

Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a differentiable function with a unique critical point at which it has a local minimum. For which values of n is the minimum necessarily an absolute minimum?

In our discussion of the solutions of the above problem we neglected to mention that it was also solved by the proposer.

$n - r$ by r, and so on. The condition (1) is equivalent to

$$D_{11}' = D_{11}, \quad D_{21} = 0. \tag{3}$$

We now take $D_\varepsilon = D - \varepsilon I_n = \begin{pmatrix} D_{11} - \varepsilon I_r & D_{12} \\ D_{21} & D_{22} - \varepsilon I_{n-r} \end{pmatrix}$. Note that (3) still holds if we replace D by D_ε, i.e., if we replace D_{11} by $D_{11} - \varepsilon I_r$ and D_{22} by $D_{22} - \varepsilon I_{n-r}$. Thus C, D_ε satisfy (2). Furthermore D_ε is nonsingular if ε avoids the finite number of eigenvalues of D. Since $D = \lim_{\varepsilon \to 0} D_\varepsilon$, the lemma is proved.

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REVIEWS

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Differential Equations with Linear Algebra. By Zbigniew H. Nitecki and Martin M. Guterman. Saunders, Philadelphia, 1986. xii + 596 pp.
Introduction to Differential Equations. By Richard K. Miller. Prentice-Hall, Englewood Cliffs, 1987. x + 628 pp.
Fundamentals of Differential Equations. By R. Kent Nagle and Edward B. Saff. Benjamin/Cummings, Menlo Park, 1986. xiv + 624 pp.
Introduction to Differential Equations with Applications. By Fred Brauer and John A. Nohel. Harper and Row, New York, 1986. xii + 572 pp.
Ordinary Differential Equations. By Morris Tenenbaum and Harry Pollard. Dover, New York, 1985. xvii + 808 pp.

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In looking over the five books under review one is struck immediately by the similarities in style, content, and perspective. Individual differences occur mainly in the addition or omission of topics that are tangential, in general, to the stated purpose of each text, which is to present basic techniques for solving various classes of ordinary differential equations. Such similarity is to be expected, since over the last thirty years or so the content of the usual freshman-sophomore, postcalculus course on differential equations for mathematics, physics, and engineering students has assumed a more or less fixed form. It is not surprising, then, that writers of textbooks for such classes respect this form in their expositions of the subject matter and embellish them with topics of interest to themselves or their students. Thus each book under review serves up a nourishing, if somewhat mundane, menu of techniques for solving linear and nonlinear differential equations that arise in many physical and biological problem areas, providing (when appropriate) motivation for the introduction of a particular technique and a particular class of "real-world" applications.

Without going into too much detail let me now outline the major topics each book considers, usually within the confines of a single chapter. There is, of course, the obligatory introductory chapter on the history of differential equations and on why student and instructor alike will devote a quarter or semester of their time to the task of "integrating" differential equations. This is followed by a chapter on first-order scalar differential equations in which one learns the usual methods for finding exact solutions and, perhaps, becomes a little cocky because the calculus works so well on a seeming multitude of problems. Some of the books show, however, that all is not as it appears in this paradise of exact solutions by including a discussion of the need for a basic existence and uniqueness theorem that tells us when the initial-value problem, $dy/dx = f(x, y)$, $y(x_0) = y_0$, has a unique solution. The difficulty of finding exact solutions becomes even more apparent in succeeding

chapters on second- and higher-order linear equations, where it is shown that there is, indeed, a limit to the number of problems (even linear ones!) that are solvable in closed form. At this point the authors call upon the basic theory of linear algebra to aid them in solving second- and higher-order, scalar linear differential equations with constant coefficients and possibly with forcing terms. For example, one can write the solution of $d^2y/dx^2 + a_1\, dy/dx + a_2 y = f(x)$, where a_1 and a_2 are constants and f is a smooth function, in a convenient form that involves evaluating a certain integral containing f. Thus if f is not too complicated, then exact solutions can be found, as in the case of certain first-order equations. It is, of course, a different matter to solve an equation of this type if a_1 or a_2 is a general function of x. This is the motivation for seeking solutions of second-order linear differential equations with variable coefficients in the form of power series in x. Each book has a chapter on the use of power series for finding representations of solutions near special points in the domain of interest that are either regular or singular points. The analysis in a neighborhood of a singular point is illustrated best by the Bessel equation $x^2 d^2y/dx^2 + x\, dy/dx + (x^2 - c^2)y = 0$ in the interval $x > 0$, say, where c is a constant. The point $x = 0$ is a singular point and, using standard techniques, one finds formulas for the coefficients a_n in the power series representation of the solution $y(x) = \Sigma_{n \geq 0} a_n x^n$, which is nothing more than a Bessel function of order c. Another chapter in each of the books introduces the Laplace transform technique for solving linear differential equations, especially ones with nonsmooth forcing functions f. Again, this is standard fare that centers around converting the differential equation into an algebraic equation in the transformed variables, solving this algebraic equation, and then recovering the solution of the differential equation by inverting the solution of the algebraic problem. Finally, each book offers a chapter on solving systems of first-order linear equations, either by using Gaussian elimination to reduce the system to a single equation in one unknown function or by applying matrix theory to construct fundamental solutions expressed in terms of matrix exponential functions. The introduction of matrix theory into elementary courses on differential equations has taken place over the last twenty years or so, and it has allowed one to bring the immense power of linear algebra to bear on solving quite general linear systems. The book by Tenenbaum and Pollard, which is a Dover reprint of a text originally published in 1963, contains virtually no matrix theory, and so one can see how linear algebra has changed dramatically (and permanently!) the way we solve such linear systems by looking at the "modern" treatments in the other four books.

Despite the large degree of overlap among these five books, alluded to in the previous paragraph, there are differences that may recommend individual books to certain types of instructors and classes. On the one hand, Tenenbaum and Pollard and Miller present some basic techniques for solving differential equations numerically. There is nothing flashy here, just a sample of the time-honored methods for converting a continuous problem into a discrete one for an associated difference equation. On the other hand, Brauer and Nohel and also Miller have chapters on solving basic *partial* differential equations such as Laplace's equation and the wave equation by means of Fourier series. The idea here is to notice that a solution of,

say, Laplace's equation $u_{xx} + u_{yy} = 0$ can be expressed in the form $u(x, y) = \sum_{k \geq 0} X_k(x) Y_k(y)$, where X_k and Y_k satisfy second-order *ordinary* differential equations. Again this is the rudimentary theory, but it does reveal to the student the power he or she has gained to attack important physical problems through the study of ordinary differential equations. Finally, Brauer and Nohel, Miller, and Nagle and Saff (and to a lesser extent Tenenbaum and Pollard) have chapters on the modern qualitative theory of ordinary differential equations developed by Poincaré and Lyapunov at the turn of this century. There are readable, informative discussions of elementary stability theory, phase-plane analysis of autonomous systems and periodic solutions. This will be without doubt the most interesting part of the course for the instructor since this material is the most analytical (or topological) part of these books. The discussions are elementary enough, though, that students should find them comprehensible and useful as well.

In conclusion, then, I should like to say that judging by the appearance of these five books (and others of similar nature), the subject of elementary differential equations is alive and well, and any instructor can find with little difficulty a text that suits precisely his or her needs. The nine authors reviewed here seem to agree fairly closely on what constitutes the core material for this type of course and on what additional topics will serve the needs of particular groups of students best. Therefore, I do not hesitate in recommending all—and none of them!

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

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C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatibilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

General, L*, P*.** *Encyclopedic Dictionary of Mathematics, Second Edition, Volumes I-IV.* Ed: Kiyosi Itô. MIT Pr, 1986, $350 set. [ISBN: 0-262-09026-0] *Volume I: A-E,* xix + 561 pp; *Volume II: F-N,* 583 pp; *Volume III: O-Z,* 570 pp; *Volume IV: Appendices and Indexes,* 426 pp. Translation of the expanded and revised *Third Edition* of Iwanami Sūgaku Ziten (ISZ3), a thorough, modern, yet amazingly accessible encyclopaedia of mathematics prepared by the Mathematical Society of Japan. This edition, based on articles written between 1980 and 1982, updates the earlier edition (TR, February 1978; ER, September 1978) with new articles (especially in applied mathematics), expansion of explanations in the interest of clarity, coverage of recent work, and reorganization of older topics. Published in four slim volumes (instead of two fat ones), the *Dictionary* concludes with an entire volume of tables, indices and references, including a 230-page subject index to the *Dictionary's* 450 articles. A superb reference to modern mathematics; there is nothing else like it, and no substitute for it. LAS

Precalculus, T(13: 1). *Precalculus: A Problems-Oriented Approach, Second Edition.* David Cohen. West, 1987, xiv + 785 pp, $26.56 [ISBN: 0-314-26209-1]; *Instructor's Manual,* vi + 1419 pp, (P). [ISBN: 0-314-34724-0] Emphasizes word problems, applications, graphing, analytic geometry and trigonometry; extensive use of examples. Changes in this edition include expanded treatment of inverse functions, inequalities and trigonometry; earlier treatment of polynomial and rational functions; introduction of calculator exercises. Huge *Instructor's Manual* contains solutions for every exercise and test question in the text. JNC

Precalculus, T(13: 1), S. *College Algebra.* Steven Roman. Harcourt Brace Jovanovich, 1987, xii + 525 pp, $30.95 [ISBN: 0-15-507890-9]; *Precalculus,* xiii + 679 pp, $32.95. [ISBN: 0-15-571052-4] A well-written series of texts with good exercises and an abundance of figures in two colors. *College Algebra* contains a longer introduction to elementary algebra and a section on counting, whereas *Precalculus* contains threee chapters on trigonometry and a chapter on conic sections. The books have identical chapters on topics such as polynomial functions, exponential and logarithmic functions, systems of linear equations, complex numbers, sequences and series. CEC

Finite Mathematics, T*(13: 1). *Mathematics and Its Applications to Management, Life, and Social Sciences With Finite and Discrete Mathematics.* Margaret B. Cozzens, Richard D. Porter. DC Heath, 1987, xii + 703 pp, $25. [ISBN: 0-669-09368-8] An attractive and lucid presentation of traditional finite mathematics with added discrete mathematics topics (e.g., graph theory and difference equations); numerous applications including RNA and DNA chains, Markov chains, scheduling problems, etc. JNC

Education, S(15), P. *Geometry for Grades K-6: Readings from the Arithmetic Teacher.* Ed: Jane M. Hill. NCTM, 1987, v + 173 pp, $9.50 (P). [ISBN: 0-87353-237-6] Articles published since 1974 that feature a hands-on approach with emphasis on development of concepts like area, perimeter and volume, and on classifying shapes. JNC

Education, P. *Studies in Mathematics Education: The Mathematical Education of Primary-School Teachers, V. 3.* Ed: Robert Morris. UNESCO, 1984, 258 pp, (P). [ISBN: 92-3-102141-9] Seventeen chapters by authors from sixteen countries ex-

amine the changing nature of primary school mathematics, the responsibilities of teachers in teaching it, and the implications for teacher education. Chapters on computers/calculators, concept learning and teaching, teacher education and assessment, and descriptions of primary teacher support programs in several countries. MW

Education, S(15-18), P, L*. *Estimation and Mental Computation, 1986 Yearbook.* Harold L. Schoen, Marilyn J. Zweng. NCTM, 1986, viii + 248 pp, $16. [ISBN: 0-87353-226-0] Many ideas here for why estimation is indeed a basic skill, when it is appropriate or even necessary, how to do it, and how to teach it. Emphasizes interplay between estimation and conceptual understanding. Articles set theoretical foundation, describe instructional activities, and review research on teaching, learning, and evaluating estimation. MW

Education, S(15-18), P. *Alternative Courses for Secondary School Mathematics.* Marilyn N. Suydam, *et al.* NCTM, 1985, v + 57 pp, $4.50 (P). [ISBN: 0-87353-222-8] Profiles 74 courses identified in an NCTM survey as different from traditional high school mathematics courses. Categorized by mathematical focus, descriptions include content, teaching modes, target audience and contact person. Authors found less "flexibility, diversity, and uniqueness" than anticipated, but there are ideas here for innovators. MW

History, P, L. *Beno Eckmann Selecta.* Ed: Max-Albert Knus, Guido Mislin, Urs Stammbach. Springer-Verlag, 1987, xii + 835 pp, $107.50. [ISBN: 0-387-17518-0] Reprints of Eckmann's major papers, on the occasion of his 70th birthday, together with a complete bibliography and a brief autobiographical note. LAS

Logic, T(17-18: 1, 2), S, P. *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets.* Robert I. Soare. Perspectives in Math. Logic. Springer-Verlag, 1987, xviii + 437 pp, $35. [ISBN: 0-387-15299-7] From the Preface: "This book represents a kind of progress report over the last forty years on the programs, ideas, and hopes expressed by E.L. Post in 1944." (A recursively enumerable (r.e.) degree is an equivalence class of sets, essentially all equally difficult to compute, one of which is an r.e. set—i.e., listable.) BC

Discrete Mathematics, T*(13: 1), S, L. *An Introduction to Discrete Mathematics and Its Applications.* Kenneth Kalmanson. Addison-Wesley, 1986, x + 486 pp, $34.95. [ISBN: 0-201-14947-8] A solid introduction which emphasizes algorithmic reasoning. Topics include sets, number systems, logic, counting, graph theory, trees, digraphs, networks, relations, Boolean algebra, and recursion. Well-written with lots of good examples and exercises. Also includes many optional programming exercises. CEC

Discrete Mathematics, T(13-14: 1), L. *Discrete Mathematics for Computer Science.* Angela B. Shiflet. West, 1987, xxii + 450 pp, $28.16 [ISBN: 0-314-28513-X]; *Instructor's Manual,* v + 130 pp, (P). [ISBN: 0-314-35404-2] Much more directed toward computer science students than most of the recent books in discrete mathematics. No computer language is required, but Pascal would be helpful. In addition to standard topics, there are chapters on subscripts, matrices, Boolean algebra, binary and hexadecimal arithmetic, and analysis of algorithms. Appendices, glossary, index. JS

Discrete Mathematics, T(1), S. *Discrete Mathematical Structures.* Mario Benedicty, Frank R. Sledge. Harcourt Brace Jovanovich, 1987, xvi + 529 pp, $29. [ISBN: 0-15-517683-8] An introductory text for mathematics and computer science majors. A pragmatic approach stressing syntax and the need for abstraction. Topics include decimal, binary and octal arithmetic, arithmetics and algebras, elementary combinatorics, set theory, structures and logic, functions, algebraic structures, recursion and induction, graph theory, coding theory, Boolean functions and finite state acceptors. JNC

Number Theory, S(18), P. *Exponential Diophantine Equations.* T.N. Shorey, R. Tijdeman. Tracts in Math., V. 87. Cambridge U Pr, 1986, x + 240 pp, $44.50. [ISBN: 0-521-26826-5] Exponential diophantine equations are of the form $Ax^m + By^m = C$ where x, y are rational integers and $m \geq 2$ is a variable rational integer. This book provides a unified approach to the problem of determining the number of solutions of these equations. Results include 1) there are only a finite number of perfect squares in the Fibonacci sequence, and 2) Fermat's equation $x^n + y^n = z^n$ has only finitely many solutions when $x - y$ is composed of fixed primes. MR

Number Theory, T*(14), S, L. *Elementary Introduction to Number Theory, Third Edition.* Calvin T. Long. Prentice-Hall, 1987, xii + 292 pp. [ISBN: 0-13-257502-7-01] This new edition includes computer-oriented problems, a stress on computational aspects when appropriate, a brief discussion on computational complexity, some comments on primality testing and factoring, and a short chapter on public key encryption. To keep the text from growing too large, some problem sets have been shortened and a section on decimal expansions has been dropped. (*Second Edition,* TR, January 1973.) CEC

Number Theory, P. *Lecture Notes in Mathematics-1245: L-Functions and the Oscillator Representation.* Stephen Rallis. Springer-Verlag, 1987, xv + 239 pp, $23.60 (P). [ISBN: 0-387-17694-2] Relates work of Waldspurger on the Shimura correspondence to the general theory of the oscillator representation

on dual reductive pairs. Aims at developing exact formulas for special values of L-functions of automorphic representations. BC

Linear Algebra, T(14). *Elementary Linear Algebra with Applications.* Howard Anton, Chris Rorres. Wiley, 1987, xvi + 788 pp, $42.50. [ISBN: 0-471-85104-3] The first nine chapters constitute a slightly rewritten version of the *Fifth Edition* of Anton's *Elementary Linear Algebra.* Chapter 10 incorporates much of the material from what has previously appeared as a separate paperback, *Applications of Linear Algebra,* by Rorres and Anton. The ninth chapter on numeric methods is supported by Anton's *Linear Kit,* a software package that will run on most popular microcomputers. AWR

Linear Algebra, T*(14: 1). *Elementary Linear Algebra.* Leslie Hogben. West, 1987, xi + 570 pp [ISBN: 0-314-28497-4]; *Instructor's Solutions Manual,* v + 107 pp, (P). [ISBN: 0-314-34795-X] A careful and very readable presentation of the standard topics: discusses R^n prior to abstract vector spaces and uses algorithms for computational methods; optional sections cover applications from a variety of disciplines. JNC

Calculus, T(13: 2). *Brief Calculus With Applications, Second Edition.* Roland E. Larson, Robert P. Hostetler. DC Heath, 1987, xii + 903 pp, $29 [ISBN: 0-669-12060-X]; *Brief Calculus With Applications, Alternate Second Edition.* 1987, xi + 728 pp, $27. [ISBN: 0-669-12186-X] This edition includes revision of many existing sections, new chapter summaries and review exercises plus new sections on related rates, differentials, definition of the integral, elementary topics in probability and statistics, first order linear differential equations, p-series and ratio test. Supplementary computer software available. (*First Edition,* TR, March 1984.) The *Alternate Edition* omits the chapters on differential equations, series and trigonometric functions. (*First Edition,* TR, March 1984.) JNC

Differential Equations, P. *Patterns and Waves: Qualitative Analysis of Nonlinear Differential Equations.* Ed: Takaaki Nishida, Masayasu Mimura, Hiroshi Fujii. Stud. in Math. & Its Applic., V. 18. Elsevier Science, 1986, xii + 692 pp, $150. [ISBN: 0-444-70144-3] This is a collection of papers of former students and other mathematicians influenced by Masaya Yamaguti. Watch out for the price! AWR

Numerical Analysis, P. *Lecture Notes in Mathematics-1237: Rational Approximation and its Applications in Mathematics and Physics.* Ed: J. Gilewicz, M. Pindor, W. Siemaszko. Springer-Verlag, 1987, xii + 350 pp, $31.90 (P). [ISBN: 0-387-17212-2] Four surveys and 22 research papers on continued fractions and rational approximation. Replacing Taylor (polynomial) with Padé (rational) approximation makes theoretical life more difficult (or interesting), but the practical gain in accuracy can be spectacular. BC

Numerical Analysis, T(15-16), S, L. *Numerical Methods for Scientists and Engineers, Second Edition.* R.W. Hamming. Dover, 1986, ix + 721 pp, $14.95 (P). [ISBN: 0-486-65241-6] Unabridged republication of the 1973 *Second Edition* (TR, February 1974) of a 1962 classic text that moved numerical analysis into the computer age with its focus on sampling frequency as the key to understanding. Uncommonly well written; over-flowing with insight and good sense that is still valuable 25 years after its original publication. LAS

Numerical Analysis, P. *Methods of Functional Analysis in Approximation Theory.* Ed: C.A. Micchelli, D.V. Pai, B.V. Limaye. ISNM 76. Birkhauser Boston, 1986, 410 pp, $51. [ISBN: 0-8176-1761-2] The proceedings of a conference held December 16-20, 1985 in Bombay, India. Contains 27 papers surveying recent research trends. AO

Functional Analysis, S. *Stable Solution of Inverse Problems.* Johann Baumeister. Adv. Lect. in Math. Friedr. Vieweg & Sohn, 1987, viii + 253 pp, (P). [ISBN: 3-528-08961-X] "Designed to provide the main ideas and methods [on inverse problems]," the author's intention to provide a clear explanation is marred throughout by an annoying awkwardness in the use of the English language; it needs the help of an English-speaking edition. AWR

Functional Analysis, T(17). *Nonlinear Operators and Differential Equations in Banach Spaces.* Robert H. Martin, Jr. Robert E Krieger, 1987, xi + 440 pp, $49.50. [ISBN: 0-89874-803-8] A nice development of basic ideas (normed linear spaces, functions defined on such spaces, fixed point theorems, examples) followed by material on solving differential equations. Presupposes an undergraduate background in real and complex analysis. (1976 Wiley Edition, TR, January 1977.) AWR

Algebraic Geometry, P. *Regular Solids and Isolated Singularities.* Klaus Lamotke. Adv. Lect. in Math. Friedr. Wieweg & Sohn, 1986, ix + 224 pp, (P). [ISBN: 3-528-08958-X] Polynomial equations associated with the groups of the regular solids describe isolated singularities of complex surfaces. The structure of the singularities is studied via commutative algebra, algebraic geometry, differential and algebraic topology, and several complex variables. SS

Algebraic Geometry, S(17-18), P. *Zariski Surfaces and Differential Equations in Characteristic* $p > 0$. Piotr Blass, Jeffrey Lang. Pure & Appl. Math., V. 106. Dekker, 1987, viii + 441 pp, $79.75. [ISBN: 0-8247-7637-2] A set of articles on Zariski surfaces, which are closely related to equations of the form $z^p = f(x, y)$, but in fields of characteristic p. Computational emphasis, with many explicit examples. BC

Operations Research, T(16-17: 2). *Introduction to Operations Research, Fourth Edition.* Frederick S. Hillier, Gerald J. Lieberman. Holden-Day, 1986, xviii + 888 pp, $48.95. [ISBN: 0-8162-3871-5] A new edition of a standard work, reorganized to place more emphasis on mathematical modelling, to group mathematical programming together, and to rewrite the introduction to the simplex method. Likely to remain a standard. (*First Edition,* TR, February 1968; *Second Edition,* TR, June-July 1976; *Third Edition,* TR, May 1981.) AWR

Dynamical Systems, S(16-18), P, L. *Order Within Chaos: Towards a Deterministic Approach to Turbulence.* Pierre Bergé, Yves Pomeau, Christian Vidal. Transl: Laurette Tuckerman. Wiley, 1986, xv + 329 pp, $49.95. [ISBN: 0-471-84967-7] An introduction to dissipative dynamical systems, centered on concrete problems in physics and chemistry. Dissipation (a.k.a. friction) necessitates the existence of attractors, some of which may be strange. BC

Control Theory, T(17-18: 1). *Theory and Practice of Recursive Identification.* Lennart Ljung, Torsten Söderström. MIT Pr, 1987, xix + 529 pp, $19.95 (P). [ISBN: 0-262-62058-8; 0-262-12095-X] Paperback edition of a work formerly available only in hardcover. A systematic and coherent treatment of algorithms for on-line adaptive systems. AO

Stochastic Processes. *Lecture Notes in Statistics-41: Palm Probabilities and Stationary Queues.* François Baccelli, Pierre Brémaud. Springer-Verlag, 1987, vii + 106 pp, $15.80 (P). [ISBN: 0-387-96514-9]

Statistics, S*(13-18), L*. *Elementary Decision Theory.* Herman Chernoff, Lincoln E. Moses. Dover, 1986, xv + 364 pp, $8.95 (P). [ISBN: 0-486-65218-1] Reprint of a 1959 elementary introduction to a decision theoretic approach to statistics. FLW

Statistics, T(13-14: 1). *Understandable Statistics: Concepts and Methods, Third Edition.* Charles Henry Brase, Corrinne Pellillo Brase. DC Heath, 1987, xiv + 591 pp, $23. [ISBN: 0-669-12181-9] This new edition has many new problems, new material on stem-and-leaf displays, box-plots, *p*-values, and using computers. A computer software package for PC's is available for use with the book. (*First Edition,* TR, March 1979.) FLW

Statistics, T(13-14: 1). *Basic Business Statistics.* Bernard J. Landwehr. Robert E Krieger, 1985, xi + 234 pp, $20.50. [ISBN: 0-89874-823-2] Brief treatment of most of the usual topics plus chapters on index numbers and time series. "Only an understanding of basic algebra is necessary." FLW

Statistics, S(13-18), P, L. *Statistical Method from the Viewpoint of Quality Control.* Walter A. Shewhart, W. Edwards Deming. Dover, 1986, ix + 155 pp, $6 (P). [ISBN: 0-486-65232-7] Reprint of Shewhart's 1938 lectures. Foreword by W. Edwards Deming. FLW

Statistics, T(16-18: 1), S, P, L. *Statistical Design For Research.* Leslie Kish. Wiley, 1987, xxii + 267 pp, $34.95. [ISBN: 0-471-08359-3] Aims to bring statistics into the early phases of data collection. Chapter topics include sample surveys, design for comparisons, controls for disturbing variables, samples and censuses, and sample designs over time. FLW

Statistics, T(14-16), S, P, L.** *Some Theory of Sampling.* William Edwards Deming. Dover, 1966, xvii + 602 pp, $14.95 (P). [ISBN: 0-486-64684-X] Corrected, unabridged republication of the original 1950 classic text that introduced students, managers, and theoreticians to the practical nature of statistical sampling. Requires little or no calculus. Includes hundreds of illustrations based on actual large-scale samples undertaken in industry and government. LAS

Computer Literacy, T(13-16). *Computer Education: Literacy and Beyond.* William E. Schall, Lowell Leake, Jr., Donald R. Whitaker. Ser. in Comp. Educ. Brooks/Cole, 1986, xvii + 317 pp, $23.75 (P). [ISBN: 0-534-06102-8] Based on 1980 Association for Computing Machinery (ACM) computer competencies for all teachers. Four chapters on BASIC programming, one each on LOGO and Pascal. List of software types in schools omits spreadsheets, statistical analysis, symbolic manipulation, and graphing software. Perhaps useful as generic introduction; not enough for mathematics teachers. MW

Computer Literacy, T*(14-15: 1), S*, L. *Algorithmics: The Spirit of Computing.* David Harel. Addison-Wesley, 1987, xiv + 425 pp, $22.95 (P). [ISBN: 0-201-19240-3] A superb exposition of the fundamental principles of computer science. Stresses concepts without syntax, understanding without hardware. Algorithms, correctness, efficiency, intractability, undecidability, parallelism, probabilistic algorithms, intelligence. Extensive bibliography with helpful notes; no exercises. Excellent source for a serious course in computer literacy, and for background reading by curious laymen. LAS

Programming, T*(13). *Condensed Pascal.* Doug Cooper. WW Norton, 1987, xxviii + 611 pp, $24.95 (P). [ISBN: 0-393-95540-0] A condensed and reorganized version of *Oh! Pascal!* emphasizing applications in mathematics and the physical sciences. Includes the complete *Standard Pascal User Reference Manual* as an appendix. AO

Computer Graphics, P. *Advances in Computer Graphics I.* Ed: G. Enderle, M. Grave, F. Lillehagen. EurographicSeminars. Springer-Verlag, 1986, xii + 512 pp, $72. [ISBN: 0-387-13804-8] Twenty-two papers drawn primarily from recent Eurographics Conferences (Copenhagen, 1984 and Nice, 1985) consisting of research articles and tutorial surveys

of contemporary developments in computer graphics. Introduction to the field, in French. Mathematical fundamentals of computer graphics, specialized hardware and basic algorithms, standards for computer graphics, applications, all in English. RB

Computer Graphics, P. *Advances in Computer Graphics II.* Ed: F.R.A. Hopgood, R.J. Hubbold, D.A. Duce. EurographicSeminars. Springer-Verlag, 1986, x + 186 pp, $45. [ISBN: 0-387-16910-5] Material presented in the tutorial program of the Eurographics '86 Conference (Lisbon, Portugal, August 1986). Human factors of color displays, automated cartography and geographical information systems, techniques for modelling and displaying 3-D scenes, interfacing standards for storage and communication of computer graphics information, VLSI-oriented graphics system design. RB

Computer Science. *Lecture Notes in Computer Science-252: VDM '87: VDM—A Formal Method at Work.* Ed: D. Bjørner, *et al.* Springer-Verlag, 1987, ix + 422 pp, $30.60 (P). [ISBN: 0-387-17654-3] Proceedings of a conference held in March 1987 in Brussels, Belgium. VDM (Vienna Development Method) is a formal software development methodology. AO

Applications (Biological Science), T(16-17), P, L. *Stochastic Processes in Demography and Their Computer Implementation.* C.J. Mode. Biomath., V. 14. Springer-Verlag, 1985, xvii + 389 pp, $70. [ISBN: 0-387-13622-3] Intended for the lone reader in research or as a text at the undergraduate or graduate level. Treats stochastic modeling in demography with attention paid to computer implementation of the models. Topics include fecundity, survivorship, competing risks, and population projection methodology based on stochastic processes. KK

Applications (Economics), P. *Lecture Notes in Economics and Mathematical Systems-274: Duality in Stochastic Linear and Dynamic Programming.* Willem K. Klein Haneveld. Springer-Verlag, 1986, vii + 295 pp, $30.30 (P). [ISBN: 0-387-16793-5] A linear programming problem in which some of the parameters are random variables is called a stochastic linear programming problem (SLP); if the parameters are well defined as a function of time, then the problem is a stochastic dynamic programming problem (SDP). The relationship of these two types of problems is emphasized in this book in which duality theory plays a key role and is developed in a full chapter. AWR

Applications (Engineering), P, L. *Finite Element Handbook.* Ed: H. Kardestuncer, D.H. Norrie. McGraw-Hill, 1987, xxiv + 1380 pp, $96.50. [ISBN: 0-07-033305-X] A comprehensive survey of the theoretical foundations of the finite element method and its use in engineering. The contributions of 96 authors are collected in sections devoted to mathematical background, fundamentals, applications, and computational aspects. AO

Applications (Engineering), P. *Computer Aided Optimal Design: Structural and Mechanical Systems.* Ed: Carlos A. Mota Soares. NATO ASI Ser. F, V. 27. Springer-Verlag, 1987, xiii + 1029 pp, $139. [ISBN: 0-387-17598-9] Edited version of the lectures and some of the papers presented at the NATO Advanced Study Institute held June 29-July 11, 1986 in Tróia, Portugal. AO

Applications (Information Theory), P, L*. *Fuzzy Sets and Applications: Selected Papers by L.A. Zadeh.* Ed: R.R. Yager, *et al.* Wiley, 1987, 684 pp, $49.95. [ISBN: 0-471-85710-6] A fuzzy set is a class of objects with a continuum of grades of membership intended to model important aspects of human thought. This volume contains seventeen papers by Lofti Zadeh on the subject he created, beginning with his seminal 1965 paper. The volume opens with a 1984 interview with Zadeh from *Comm. ACM* on the relation between fuzzy sets and artificial intelligence. LAS

Applications (Physics), T(13: 2). *Mathematics for Electricity and Electronics.* Gene Waring. Robert E Krieger, 1987, xiii + 672 pp, $41.50. [ISBN: 0-89874-006-4] Chapters covering use of calculators, fractions, decimals, scientific notation, algebra, trigonometry and logarithms are interwoven with chapters on elementary electrical theory; reprint of the 1984 edition. JNC

Applications (Simulation), T(16: 1), S, P, L. *Stochastic Simulation.* Brian D. Ripley. Appl. Prob. & Stat. Wiley, 1987, xi + 237 pp, $32.95. [ISBN: 0-471-81884-4] A comprehensive guide to various simulation methods in statistics and operations research. Covers generation of random numbers, non-uniform random variates and stochastic processes, variance reduction, output analysis, analysis of shift register generators, sensitivity analysis of normal variate generators, analysis of simulation output and Monte Carlo, confidence intervals. Includes exercises and a substantial list of references. CEC

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